Potential Analysis Revised proofs
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Corrigendum to "Asymptotics for the Unconstrained Polarization (Chebyshev) Problem"

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The purpose of this note is to address two needed modifications in the recently published article [3]; namely, revisions to Theorems 1.11 and 1.12 and their proofs.

In the revised Theorem 1.11, we add the condition in case (i) that A be strongly d-rectifiable and, in case (ii) we now require $s > p - 2$ and replace the condition on the regularity of the energy equilibrium measure μ_s^A by the requirement that $\mathrm{supp}(\mu_{s,A}) = A$ and that the potential $\mathcal{U}_s(\mu_{s,A}, x)$ is constant on A, where for a positive measure ν on $\mathbb{R}^p$ and $s > 0$:

$$\mathcal{U}_s(\nu, x) := \int \frac{1}{|x - y|^s} d\nu(y). \tag{1}$$

When ω is a finite set in $\mathbb{R}^p$, we also identify ω with its counting measure and write

$$\mathcal{U}_s(\omega, x) := \sum_{y \in \omega} \frac{1}{|x - y|^s}.$$

Theorem 1 (Replacement of [3, Thm. 1.11]) *For integers p, d such that $p \geq 2$, $1 \leq d \leq p$ and $A \subset \mathbb{R}^p$ compact, suppose that one of the following conditions holds:*

(i) $s > \max\{d, p - 2\}$ *and A is strongly d-rectifiable with $\mathcal{H}_d(A) > 0$.*

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(ii) $p-2<s<d$, *the set* A *is* d*-regular and the equilibrium measure* $\mu_{s,A}$ *satisfies* $\operatorname{supp}(\mu_{s,A})=A$ *and* $\mathcal{U}_s(\mu_{s,A},x)$ *is constant for* $x\in A$.

Then both limits $h_{s,d}(A)$ *and* $h^*_{s,d}(A)$ *exist, are finite, and*

$$h_{s,d}(A)=h^*_{s,d}(A). \tag{2}$$

The revision of Theorem 1.12 requires in the case $s=p$ that A be Jordan measurable, but no change for the $s>p$ case.

Theorem 2 (Replacement of [3, Thm. 1.12]) *If* $A\subset\mathbb{R}^p$ *is a compact set and* $s>p$, *or if* $s=p$ *and* $\mathcal{L}^p(\partial A)=0$, *then*

$$h^*_{s,p}(A)=h_{s,p}(A)=\frac{\sigma_{s,p}}{\mathcal{L}_p(A)^{s/p}}. \tag{3}$$

Moreover, if $\mathcal{L}_p(A)>0$, *then for any asymptotically extremal sequence* $\Omega=\{\omega_N\}_{N\geq 1}$ *(for either the constrained or unconstrained polarization problem) we have the weak-$*$ convergence*

$$\frac{1}{N}\sum_{x_i\in\omega_N}\delta_{x_i}\xrightarrow{*}\frac{\mathcal{L}_p\lfloor A}{\mathcal{L}_p(A)}\quad as\quad N\to\infty, \tag{4}$$

where $\mathcal{L}_p\lfloor A:=\mathcal{L}_p(\cdot\cap A)$ *is the restriction to* A *of* $\mathcal{L}_p$.

We first address Theorem 1 and its proof. We thank Alexander Reznikov who brought to our attention that the derivation of the inequality (4.20) in the proof of [3, Thm. 1.11] requires modifications in the case $p>d$. Under the assumptions of Theorem 1, we now conclude that the limit $h^*_{s,d}(A)$ exists and is finite (in contrast, this was only established in part (ii) of [3, Thm. 1.11]).

Proof of Theorem 1 We first discuss the subcase $p=d$ of case (i). We note that the proof of [3, Thm. 1.11] remains valid for this subcase; that is, if $\mathcal{L}_p(A)>0$, $s>p=d$ and $h^*_{s,p}(A)$ exists, then $h_{s,p}(A)$ also exists and equals $h^*_{s,p}(A)$. In the corrected proof given below for Theorem 2 we establish that in this subcase $h^*_{s,p}(A)$ exists and is equal to the rightmost expression in (3). In particular it is finite if $\mathcal{L}_p(A)>0$. Hence (2) holds for this subcase.

To complete the proof of case (i) for $p>d$, we remark that the proof of [3, Thm. 1.14] relies on the $s>p$ case of [3, Thm. 1.12], which coincides with the $s>p$ case of Theorem 2. Thus the proof of [3, Thm. 1.14] remains valid. Then (2) follows from [3, Thm. 1.14].

It remains to consider the case (ii). We will use the following notation. For $0<s<p$, $\omega\subset\mathbb{R}^p$ finite and ν a positive finite measure over $\mathbb{R}^p$, let $P_s(A,\nu):=\min_{y\in A}\mathcal{U}_s(\nu,y)$.

Our first observation is that under the assumptions that $0<s<d$ and A d-regular, the proof of [5, Thm. 1.8] extends under weaker hypotheses as stated in the following result.

Theorem 3 (Extension of [5, Thm. 1.8]) *Let* f *be a* d*-Riesz-like function and* $A\subset\mathbb{R}^p$ *be a* d*-regular compact set. If* ω_N *is a sequence of* N*-point configurations in* $\mathbb{R}^p$ *such that*

$$\nu_N:=\frac{1}{N}\sum_{x\in\omega_N}\delta_x\xrightarrow{*}\nu$$

for a probability measure ν *supported on* A, *then* $P_f(A,\nu_N)\to P_f(A,\nu)$.

For the definitions of d*-Riesz-like potentials* f (which include Riesz potentials with power $0 < s < d$) and the f*-polarization* $P_f(A, \mu)$ of a measure μ over the set A, see [5, Defs. 1.1 and 1.7]. Note that $P_s(A, \mu) = P_f(A, \mu)$ with $f(|x|) = |x|^{-s}$ for $s > 0$.

Proof of Theorem 3 The original version of [5, Thm. 1.8] required that $\omega_N \subset A$, however this hypothesis is not used since the main potential theoretic result needed in the proof is the principle of descent [5, Thm. 2.3], which works in general. On the other hand, the fact that ν is supported on A is required in order to use the minimum principle for Riesz-like f, see [5, Thm. 2.5]. □

Proof of Theorem 1 (ii) continued. Suppose now that the hypotheses of (ii) hold. For $N \geq 1$, let ω_N^A be such that

$$P_s(A, \omega_N^A) = \mathcal{P}_s^*(A, N). \tag{5}$$

By [3, Cor. 1.9] if the empirical measures of a subsequence ω_N^A, $N \in \mathcal{N}$, as in (5) converge weakly to a measure ν, then ν is supported on A. By Theorem 3, it follows that

$$\lim_{N \in \mathcal{N}} \frac{\mathcal{P}_s^*(A, N)}{N} = P_s(A, \nu). \tag{6}$$

Furthermore, by [4, Thm. 2], ν must maximize the continuous (integral) unconstrained polarization, which therefore coincides with the maximal continuous constrained polarization (since ν is supported on A):

$$P_s(A, \nu) = \max_{\mu \in \mathcal{P}(\mathbb{R}^p)} P_s(A, \mu) = \max_{\mu \in \mathcal{P}(A)} P_s(A, \mu). \tag{7}$$

We can then apply the result [6, Thm. 1.2], valid for constrained polarization, which implies $\nu = \mu_{s,A}$. In particular, ν has no atoms since it has finite s-energy.

Let $\mathrm{proj}_A : \mathbb{R}^p \to A$ be a measurable map which assigns to any point $x \in \mathbb{R}^p$ a point $y \in A$ such that $|x - y| = \mathrm{dist}(x, A)$. Then for fixed $\epsilon > 0$ we set

$$\omega_{N,\epsilon}^A := \mathrm{proj}_A(\omega_N^A \cap A_\epsilon), \qquad \nu_{N,\epsilon} := \frac{1}{N} \sum_{x \in \omega_{N,\epsilon}^A} \delta_x.$$

Let $y_{N,\epsilon} \in A$ be a point at which $\mathcal{U}_s(\nu_{N,\epsilon}, y)$ achieves its minimum over A. We then fix a sequence $\mathcal{E} = \{\epsilon_k\}$ with $\epsilon_k \to 0$ and an increasing sequence $\mathcal{N}^0 \subset \mathbb{N}$. Up to taking a subsequence, we may assume that

$$\nu_N := \frac{1}{N} \sum_{x \in \omega_N^A} \delta_x \xrightarrow{*} \nu.$$

By the discussion in the preceding paragraph we see that (6) and (7) hold for such ν, and we have $\nu = \mu_{s,A}$.

By the compactness of A, via a diagonalization procedure we can find an increasing subsequence $\mathcal{N} \subset \mathcal{N}^0$ such that for each $\epsilon \in \mathcal{E}$ there holds $\lim_{N \in \mathcal{N}} y_{N,\epsilon} = y_{\infty,\epsilon}$. Up to taking a subsequence of $\mathcal{E}$ we can further assume that $\lim_{\epsilon \in \mathcal{E}} y_{\infty,\epsilon} = y_\infty$.

Now fix $\delta > 0$. If we let

$$\omega_{N,\epsilon,\delta}^A := \mathrm{proj}_A(\omega_N^A \cap A_\epsilon \setminus B(y_\infty, \delta)), \quad \nu_{N,\epsilon,\delta} := \frac{1}{N} \sum_{x \in \omega_{N,\epsilon,\delta}^A} \delta_x,$$

then we can write

$$\mathcal{U}_s(\nu_{N,\epsilon,\delta}, y_{N,\epsilon}) - \mathcal{U}_s(\nu_N \lfloor (A_\epsilon \setminus B(y_\infty,\delta)), y_{N,\epsilon})$$
$$= \frac{1}{N} \sum_{x\in\omega_{N,\epsilon}^A \cap A_\epsilon \setminus B(y_\infty,\delta)} \left(\frac{1}{|y_{N,\epsilon} - \mathrm{proj}_A(x)|^s} - \frac{1}{|y_{N,\epsilon} - x|^s} \right).$$

Now note that for all but finitely many indices of the subsequences $\mathcal{N}$ and $\mathcal{E}$ there holds $|y_{N,\epsilon} - y_\infty| < \delta/3$ and for any $x \in \omega_N^A \cap A_\epsilon \setminus B(y_\infty,\delta)$ there holds $|\mathrm{proj}_A(x) - x| \le \epsilon < \delta/6$. By triangular inequality, for such N, ϵ, x we get

$$|y_{N,\epsilon} - \mathrm{proj}_A(x)| > \delta/2, \quad |y_{N,\epsilon} - x| > \delta/2, \quad \left| \frac{1}{|y_{N,\epsilon} - \mathrm{proj}_A(x)|^s} - \frac{1}{|y_{N,\epsilon} - x|^s} \right| < C\frac{\epsilon}{\delta^{s+1}}, \tag{8}$$

where for the last above estimate we have used the fact that $|x-y|^{-s}$ is regular over the region $|x-y| > \delta/2$. From the above it follows that for each fixed $\delta > 0$ there holds

$$\lim_{\epsilon\in\mathcal{E}} \lim_{N\in\mathcal{N}} \left| \mathcal{U}_s(\nu_{N,\epsilon,\delta}, y_{N,\epsilon}) - \mathcal{U}_s(\nu_N \lfloor (A_\epsilon \setminus B(y_\infty,\delta)), y_{N,\epsilon}) \right| = 0. \tag{9}$$

By using the positivity of our kernel and [3, Cor. 1.9], for all $\epsilon > 0$ there holds

$$0 \le \lim_{N\to\infty} \mathcal{U}_s(\nu_N \lfloor (\mathbb{R}^p \setminus (A_\epsilon \cup B(y_\infty,\delta))), y_{N,\epsilon})$$
$$\le \lim_{N\to\infty} \mathcal{U}_s(\nu_N \lfloor (\mathbb{R}^p \setminus A_\epsilon), y_{N,\epsilon}) \le \epsilon^{-s} \lim_{N\to\infty} \frac{\#(\omega_N^A \setminus A_\epsilon)}{N} = 0.$$

It follows that all the above terms are zero. Observing that $\mathbb{R}^p \setminus B(y_\infty,\delta) = (\mathbb{R}^p \setminus (A_\epsilon \cup B(y_\infty,\delta))) \cup (A_\epsilon \setminus B(y_\infty,\delta))$, it follows by superposition that

$$\lim_{\epsilon\in\mathcal{E}} \lim_{N\in\mathcal{N}} \mathcal{U}_s(\nu_N \lfloor (A_\epsilon \setminus B(y_\infty,\delta)), y_{N,\epsilon}) = \lim_{\epsilon\in\mathcal{E}} \lim_{N\in\mathcal{N}} \mathcal{U}_s(\nu_N \lfloor (\mathbb{R}^p \setminus B(y_\infty,\delta)), y_{N,\epsilon}). \tag{10}$$

By the principle of descent, applied to the measures $\nu_N \lfloor (\mathbb{R}^p \setminus B(y_\infty,\delta))$ at the points $y_{N,\epsilon}$, and then to $\nu \lfloor (\mathbb{R}^p \setminus B(y_\infty,\delta))$ and points $y_{\infty,\epsilon}$, we obtain

$$\liminf_{\epsilon\in\mathcal{E}} \liminf_{N\in\mathcal{N}} \mathcal{U}_s(\nu_N \lfloor (\mathbb{R}^p \setminus B(y_\infty,\delta)), y_{N,\epsilon})$$
$$\ge \liminf_{\epsilon\in\mathcal{E}} \mathcal{U}_s(\nu \lfloor (\mathbb{R}^p \setminus B(y_\infty,\delta)), y_{\infty,\epsilon}) \ge \mathcal{U}_s(\nu \lfloor (\mathbb{R}^p, \setminus B(y_\infty,\delta)), y_\infty). \tag{11}$$

After replacing the sequences $\mathcal{E}, \mathcal{N}$ by the subsequences $\mathcal{E}', \mathcal{N}'$ along which the liminf is achieved, we now take the limit over $\delta \to 0$. Again by the principle of descent, from (11) we get

$$\lim_{\delta\to 0} \lim_{\epsilon\in\mathcal{E}'} \lim_{N\in\mathcal{N}'} \mathcal{U}_s(\nu_N \lfloor (\mathbb{R}^p \setminus B(y_\infty,\delta)), y_{N,\epsilon})$$
$$\overset{(11)}{\ge} \lim_{\delta\to 0} \mathcal{U}_s(\nu \lfloor (\mathbb{R}^p \setminus B(y_\infty,\delta)), y_\infty) = \mathcal{U}_s(\nu, y_\infty) \ge P_s(A,\nu). \tag{12}$$

To justify the above equality, note that the potentials $\mathcal{U}_s(\nu \lfloor (\mathbb{R}^p \setminus B(y_\infty,\delta)), y)$ are decreasing in δ and that ν has no atoms.

From (9), (10) and (12) and since for any $\delta > 0$ there holds $\mathcal{U}_s(\nu_{N,\epsilon}, y) \ge \mathcal{U}_s(\nu_{N,\epsilon,\delta}, y)$ at all $y \in \mathbb{R}^d$, we get that

$$\lim_{\epsilon\in\mathcal{E}'} \lim_{N\in\mathcal{N}'} P_s(A, \nu_{N,\epsilon}) = \lim_{\epsilon\in\mathcal{E}'} \lim_{N\in\mathcal{N}'} \mathcal{U}_s(\nu_{N,\epsilon}, y_{N,\epsilon})$$
$$\ge \lim_{\delta\to 0} \lim_{\epsilon\in\mathcal{E}'} \lim_{N\in\mathcal{N}'} \mathcal{U}_s(\nu_{N,\epsilon,\delta}, y_{N,\epsilon}) \ge P_s(A,\nu). \tag{13}$$

We now add $\#(\omega_N^A \setminus A_\epsilon)$ points from A to the $\omega_{N,\epsilon}^A$, obtaining new configurations with N points. This has the only effect of increasing the generated potential. Since we can apply all the above reasoning to any initial subsequence $\mathcal{N}^0 \subset \mathbb{N}$, we get

$$\liminf_{N\to\infty} \frac{\mathcal{P}_s(A,N)}{N} \geq P_s(A,\nu). \tag{14}$$

On the other hand, the unconstrained optimum polarization is always larger than the constrained optimum polarization, and by using (6) we have

$$P_s(A,\nu) = \lim_{N\to\infty} \frac{\mathcal{P}_s^*(A,N)}{N} \geq \limsup_{N\to\infty} \frac{P_s(A,N)}{N}. \tag{15}$$

From (14) and (15) it follows that

$$\lim_{N\to\infty} \frac{\mathcal{P}_s^*(A,N)}{N} = \lim_{N\to\infty} \frac{\mathcal{P}_s(A,N)}{N}, \tag{16}$$

as desired. □

We now address Theorem 2 and its proof. The authors thank Alex Vlasiuk for pointing out that the derivation of equation (5.23) from equation (5.22) of [3] in the proof of [3, Thm. 1.12] did not take into account the measure of the boundary of A. We provide in Proposition 4 below, a substitute for this derivation valid in the case $s > p$, whereas for $s = p$ we add to [3, Thm. 1.12] the additional hypothesis $\mathcal{L}^p(\partial A) = 0$.

Proof of Theorem 2. The proof of Theorem 2 follows exactly like the one of [3, Thm. 1.12], except for the following changes:

- For the case $s = p$, with the further hypothesis $\mathcal{L}^p(\partial A) = 0$ in Theorem 2, we can take the sets $G_i := A \cap B_i$ in the paragraph following (5.21) of [3], and the proof holds verbatim.
- For the case $s > p$, Proposition 4 below, applied to the sets $A \cap B_i$ and B_i from eq. (5.22) of [3] allows to replace eq. (5.23) therein. Observing that B is convex and thus $h_{s,p}^*(B) = h_{s,p}(B)$ by [3, Prop. 1.7], and that by [1] we have $h_{s,p}(B)^{-p/s} = (\sigma_{s,p})^{-p/s}\mathcal{L}_p(B)$, it follows that $\bar{h}_{s,p}^*(B) = (\sigma_{s,p})^{-p/s}\mathcal{L}_p(B)$. The factor $1 - 2\epsilon$ in eq. (5.23), (5.24) of [3] should be replaced by $(1 - c_{s,p}\epsilon^{\frac{1}{p+1}})^{p/s}$ with notations as in Proposition 4 below. Noting that both these quantities tend to 1 as $\epsilon \to 0$, the proof of Theorem 1.12 of [3] follows with no further modifications.

The new result needed for the case $s > p$ in Theorem 2 is the following:

Proposition 4 *For $s > p \geq 1$, there exists a constant $c_{s,p} > 0$ with the following properties. Let $\epsilon \in (0, 1/2^{p+1})$ and $B \subset \mathbb{R}^p$ be a ball and $A \subset B$ a closed set such that $\mathcal{L}^p(B \setminus A) < \epsilon\mathcal{L}^p(B)$. Then there holds*

$$\overline{h}_{s,p}^*(A) \leq (1 - c_{s,p}\epsilon^{\frac{1}{p+1}})^{-1}\bar{h}_{s,p}^*(B). \tag{17}$$

In what follows we set $A, B \subset \mathbb{R}^p$ to be as in the statement of Proposition 4. Furthermore, let $\mathcal{P}_s^*(A,N)$ be the optimum N-point polarization of set A, and let ω_N^A be an N-point configuration be as in (5). The proof of Proposition 4 is based on two lemmas.

Revised proofs

Lemma 5 *Let* $0 < \delta < \min\{1, s/2\}$ *and let* N *be a positive integer. If* $y \in \mathbb{R}^p$ *is such that*

$$\mathrm{dist}(y, A) < \frac{\delta}{s} \left(\mathcal{P}_s^*(A, N)\right)^{-\frac{1}{s}} \tag{18}$$

then

$$\mathcal{U}_s(\omega_N^A, y) \geq (1 - 2\delta)\mathcal{P}_s^*(A, N). \tag{19}$$

Proof First note that if $\mathrm{dist}(y, \omega_N^A) < \left(\mathcal{P}_s^*(A, N)\right)^{-\frac{1}{s}}$ then

$$\mathcal{U}_s(\omega_N^A, y) \geq \mathcal{P}_s^*(A, N),$$

and thus (19) holds *a fortiori*. Therefore from now on we consider points $y \in \mathbb{R}^p$ such that (18) and $\mathrm{dist}(y, \omega_N^A) \geq \left(\mathcal{P}_s^*(A, N)\right)^{-\frac{1}{s}}$ hold, and our goal is to prove (19) for such y. Note that as a consequence of this assumption and of (18), we also get

$$\mathrm{dist}(y, A) \leq \frac{\delta}{s} \mathrm{dist}(y, \omega_N^A). \tag{20}$$

Let $y_1 \in A$ be such that $|y_1 - y| = \mathrm{dist}(y, A)$ and let

$$y_2 \in \mathrm{argmax}_{y' \in [y, y_1]} \mathcal{U}_s(\omega_N^A, y'). \tag{21}$$

Note that if $y' \in [y, y_1]$ and x_j is the point from ω_N^A closest to y', we have, using (20),

$$\mathrm{dist}(y', \omega_N^A) = |y' - x_j| \geq |y - x_j| - |y' - y| \geq \mathrm{dist}(y, \omega_N^A) - \mathrm{dist}(y, A) \geq \left(1 - \frac{\delta}{s}\right) \mathrm{dist}(y, \omega_N^A). \tag{22}$$

We now claim that the following chain of inequalities holds:

$$\begin{aligned}
\mathcal{U}_s(\omega_N^A, y) &\geq \mathcal{U}_s(\omega_N^A, y_2) - \int_0^1 \left| \nabla \mathcal{U}_s(\omega_N^A, y + t(y_2 - y)) \cdot (y_2 - y) \right| dt && (23)\\
&\geq \mathcal{U}_s(\omega_N^A, y_2) - s|y_2 - y| \left[\min_{y' \in [y, y_2]} \mathrm{dist}(y', \omega_N^A) \right]^{-1} \max_{y' \in [y, y_2]} \mathcal{U}_s(\omega_N^A, y') && (24)\\
&\geq (1 - 2\delta) \mathcal{U}_s(\omega_N^A, y_2) \geq (1 - 2\delta) \mathcal{P}_s^*(A, N). && (25)
\end{aligned}$$

We now prove the above. The bound (23) follows by Taylor expansion. Inequality (24) follows by noting that whenever $x \neq y'$ we have $|\nabla_y |x - y'|^{-s}| = s|x - y'|^{-s-1}$, therefore for $y' \in [y, y_2]$ we have

$$\left| \nabla \mathcal{U}_s(\omega_N^A, y') \right| \leq s \sum_{x \in \omega_N^A} |x - y'|^{-s-1} \leq s \left[\min_{y' \in [y, y_2]} \mathrm{dist}(y', \omega_N^A) \right]^{-1} \mathcal{U}_s(\omega_N^A, y').$$

For the first inequality in (25), note that from our hypotheses on y we get $|y_2 - y| \leq |y_1 - y| = \mathrm{dist}(y, A) \leq \frac{\delta}{s} (\mathcal{P}_s^*(A, N))^{-\frac{1}{s}}$ and from (22) we get $\min_{y' \in [y, y_2]} \mathrm{dist}(y', \omega_N^A) \geq \left(1 - \frac{\delta}{s}\right) (\mathcal{P}_s^*(A, N))^{-\frac{1}{s}}$. This gives, using the hypothesis $\delta < s/2$,

$$s|y_2 - y| \left[\min_{y' \in [y, y_2]} \mathrm{dist}(y', \omega_N^A) \right]^{-1} \leq \frac{\delta s}{s - \delta} < 2\delta.$$

To get the first inequality in (25) it now suffices to recall definition (21) of y_2. The second inequality in (25) follows by the fact that $y_1 \in A$ and the definition of y_2, and of ω_N^A:

$$\mathcal{P}_s^*(A, N) = \min_{x \in A} \mathcal{U}_s(\omega_N^A, x) \leq \mathcal{U}_s(\omega_N^A, y_1) \leq \mathcal{U}_s(\omega_N^A, y_2)$$

□

Recall the notation, for the r-neigborhood of a closed set $K \subset \mathbb{R}^d$, for $r > 0$:

$$(K)_r := \{x \in \mathbb{R}^d : \operatorname{dist}(x, K) < r\}.$$

Lemma 6 *There exists a constant $c_p > 0$ with the following properties. Let $B \subset \mathbb{R}^p$ be a ball and let $A \subset B$ be a closed set, such that for some $\epsilon \in (0, 1)$ there holds $\mathcal{L}^p(B \setminus A) < \epsilon \mathcal{L}^p(B)$. Then for each $r \in (0, \epsilon \operatorname{diam}(B))$ we can cover all of $B \setminus (A)_r$ by at most $c_p \frac{\epsilon \mathcal{L}^p(B)}{r^p}$ balls of radius r.*

Proof Let $r' := r/(2\sqrt{p})$ and we show that we may take as the set of ball centers the following:

$$W := \{r'k : k \in \mathbb{Z}^p, \ (r'k + [-r', r']^p) \cap (B \setminus (A)_r) \neq \emptyset\}.$$

We note that the balls with centers in W and radius $\frac{r'}{2}$ are disjoint and contained in the r-neighborhood of $B \setminus (A)_r$. Furthermore, we have the inclusion

$$(B \setminus (A)_r)_r \subset (B)_r \setminus A = ((B)_r \setminus B) \cup (B \setminus A),$$

from which it follows that, denoting by $\#W$ the cardinality of W,

$$\#W \cdot \mathcal{L}^p(B_{\frac{r'}{2}}) \le \mathcal{L}^p((B)_r \setminus B) + \mathcal{L}^p(B \setminus A) \le \left(C_p \frac{r}{\operatorname{diam}(B)} + \epsilon\right) \mathcal{L}^p(B) \le (C_p + 1)\, \epsilon\, \mathcal{L}^p(B).$$

This implies that for $c_p := 2^p(C_p + 1)/(p^{\frac{p}{2}} \beta_p)$, in which β_p is the volume of the unit ball in $\mathbb{R}^p$, there holds

$$\#W \le c_p \epsilon \frac{\mathcal{L}^p(B)}{r^p}.$$

It remains to show that radius-r balls with centers in W cover $B \setminus (A)_r$. Indeed, note that if the cube $r'k + [-r', r']^p$ with $k \in \mathbb{Z}^p$ meets $B \setminus (A)_r$, then it is contained in the ball $B(r'k, \sqrt{p}\, r') \subset B(r'k, r)$, and thus

$$B \setminus (A)_r \subset \bigcup_{r'k \in W} (r'k + [-r', r']^p) \subset \bigcup_{r'k \in W} B(r'k, r),$$

as desired. □

Proof of Proposition 4: Observe that by the same proof as in [2, Thm. 2.4], which applies also without the restriction that the optimum polarization points belong to A, with $\mathcal{L}^p$ used as the measure μ in the proof, there exists a constant $C_{s,p} > 0$ independent of N, A, such that $\mathcal{P}_s^*(A, N) \le C_{s,p} N^{s/p}/(\mathcal{L}^p(A))^{s/p}$. Now, applying Lemma 5 with $\delta := \epsilon^{\frac{1}{p+1}}$ we find that for the optimum configuration ω_N^A like in (5), with

$$r_N := \frac{\epsilon^{\frac{1}{p+1}}}{s} \left(\mathcal{P}_s^*(A, N)\right)^{-\frac{1}{s}} \ge \frac{\epsilon^{\frac{1}{p+1}} (\mathcal{L}^p(A))^{\frac{1}{p}}}{s (C_{s,p})^{\frac{1}{s}}} N^{-\frac{1}{p}}, \tag{26}$$

we have

$$\forall y \in (A)_{r_N}, \quad \mathcal{U}_s(\omega_N^A, y) \ge \left(1 - 2\epsilon^{\frac{1}{p+1}}\right) \mathcal{P}_s^*(A, N). \tag{27}$$

Next, for N large enough so that $r_N < \epsilon \operatorname{diam}(B)$ we apply Lemma 6 with r_N playing the role of r, and we find a set of centers W such that, using also (26) in the second below inequality:

$$B\backslash(A)_{r_N} \subset \bigcup_{x\in W} B(x,r_N), \qquad \#W \le c_p\,\epsilon\frac{\mathcal{L}^p(B)}{r_N^p} \le \widetilde{C}_{s,p}\,\epsilon^{\frac{1}{p+1}}\frac{\mathcal{L}^p(B)}{\mathcal{L}^p(A)}N \le \widetilde{C}_{s,p}\frac{\epsilon^{\frac{1}{p+1}}}{1-\epsilon}N. \tag{28}$$

By considering the new configuration $\omega_{\widetilde{N}} := \omega_N^A \cup W$ whose cardinality is denoted $\widetilde{N}$, we claim that

$$\widetilde{N}\in\left[N, N\left(1+\widetilde{C}_{s,p}\frac{\epsilon^{\frac{1}{p+1}}}{1-\epsilon}\right)\right], \quad \mathcal{U}_s(\omega_{\widetilde{N}}, y) \ge \begin{cases}\left(1-2\epsilon^{\frac{1}{p+1}}\right)\mathcal{P}_s^*(A,N) & \text{for } y\in(A)_{r_N},\\ \mathcal{P}_s^*(A,N) & \text{for } y\in B\setminus(A)_{r_N}.\end{cases} \tag{29}$$

The bound in the first line of (29) follows from (27). For the second bound in (29), note that since $\epsilon < 1 \le p < s$, using the first part of (28) the following holds for $x\in W$, $y\in B(x,r_N)$:

$$\mathcal{U}_s(\omega_{\widetilde{N}}, y) \ge |x-y|^{-s} \ge r_N^{-s} = \left(\frac{\epsilon^{\frac{1}{p+1}}}{s}\right)^{-s}\mathcal{P}_s^*(A,N) \ge \mathcal{P}_s^*(A,N).$$

We then find that, using also the fact that $\epsilon\in(0,1/2^{p+1})$,

$$\frac{\mathcal{P}_s^*(B,\widetilde{N})}{\widetilde{N}^{s/p}} \ge \frac{1-2\epsilon^{\frac{1}{p+1}}}{\left(1+\widetilde{C}_{s,p}\frac{\epsilon^{\frac{1}{p+1}}}{1-\epsilon}\right)^{s/p}}\frac{\mathcal{P}_s^*(A,N)}{N^{s/p}} \ge (1-2\epsilon^{\frac{1}{p+1}})\left(1-\frac{s}{p}\widetilde{C}_{s,p}\frac{\epsilon^{\frac{1}{p+1}}}{1-\epsilon}\right)\frac{\mathcal{P}_s^*(A,N)}{N^{s/p}}, \tag{30}$$

from which the bound (17) follows, with $c_{s,p} = 2 + \frac{2^{p+1}}{2^{p+1}-1}\frac{s}{p}\widetilde{C}_{s,p}$. □

Now the proof of Theorem 2 is complete. □

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