CORRIGENDUM TO "ASYMPTOTICS FOR THE UNCONSTRAINED POLARIZATION (CHEBYSHEV) PROBLEM"

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The authors are grateful to Alex Vlasiuk for pointing out that the derivation of equation (5.23) from equation (5.22) of [2] in the proof of [2, Thm. 1.12] did not take into account the measure of the boundary of A. We provide here, in Proposition 2 below, a substitute for this derivation valid in the case $s > p$, whereas for $s = p$ we add to [2, Thm. 1.12] the additional hypothesis $\mathcal{L}^p(\partial A) = 0$; namely, that A is Jordan-measurable. The amended statement of [2, Thm. 1.12] is therefore as follows, with notation as in [2]:

Theorem 1 (Replacement of [2, Thm. 1.12])**.** *If $A \subset \mathbb{R}^p$ is a compact set and $s > p$, or if $s = p$ and $\mathcal{L}^p(\partial A) = 0$, then*

$$h^*_{s,p}(A) = h_{s,p}(A) = \frac{\sigma_{s,p}}{\mathcal{L}_p(A)^{s/p}}. \tag{0.1}$$

Moreover, if $\mathcal{L}_p(A) > 0$, then for any asymptotically extremal sequence $\Omega = \{\omega_N\}_{N\geqslant 1}$ (for either the constrained or unconstrained polarization problem) we have the weak-$*$ convergence

$$\frac{1}{N}\sum_{x_i\in\omega_N}\delta_{x_i} \overset{*}{\rightharpoonup} \frac{\mathcal{L}_p|_A}{\mathcal{L}_p(A)} \quad as \quad N\to\infty, \tag{0.2}$$

where $\mathcal{L}_p|_A := \mathcal{L}_p(\cdot\cap A)$ is the restriction to A of $\mathcal{L}_p$.

Note that there is no difference between the statement of Theorem 1 above and that of [2, Thm. 1.12] in the case $s > p$. As for the case $s = p$, the modified assumption has no impact for the remaining results of [2] as this case only arises in Thm. 1.12.

The proof of Theorem 1 follows exactly like the one of [2, Thm. 1.12], except for the following changes:

- For the case $s = p$, with the further hypothesis $\mathcal{L}^p(\partial A) = 0$ in Theorem 1, we can take the sets $G_i := A \cap B_i$ in the paragraph following (5.12) of [2], and the proof holds verbatim.
- For the case $s > p$, Proposition 2 below, applied to the sets $A \cap B_i$ and B_i from eq. (5.22) of [2] allows to replace eq. (5.23) therein. The new constant $c_{s,p}$ from Proposition 2 below replaces the constant 2 in eq. (5.23) of [2] after which the proof of Theorem 1.12 of [2] follows with no further modifications.

The new result needed for the case $s > p$ is the following:

Proposition 2. *For $s > p \geqslant 1$, there exists a constant $c_{s,p} > 0$ with the following properties. Let $\epsilon \in (0,1)$ and $B \subset \mathbb{R}^p$ be a ball and $A \subset B$ a closed set such that $\mathcal{L}^p(B\backslash A) < \epsilon\mathcal{L}^p(B)$. Then there holds*

$$\bar{h}^*_{s,p}(A) \leqslant (1 - c_{s,p}\epsilon^{\frac{1}{p+1}})\bar{h}^*_{s,p}(B). \tag{0.3}$$

The proof of Proposition 2 is based on two lemmas. For this section, we consider a closed ball $B \subset \mathbb{R}^p$ and a closed subset $A \subset B$, such that $\mathcal{L}^p(B\backslash A) < \epsilon$. Furthermore, let $\mathcal{P}^*_s(A,N)$ be the optimum N-point polarization of set A, and let ω_N^A be an N-point configuration such that

$$\mathcal{U}_{\omega_N^A}(y) := \sum_{x\in\omega_N^A}|x-y|^{-s} = \mathcal{P}^*_s(A,N). \tag{0.4}$$

Lemma 3. *Let $s > 0$, $\delta \in (0,1)$ and let N be a positive integer. If $y \in \mathbb{R}^p$ is such that*

$$\mathrm{dist}(y,A) < \frac{\delta}{s}\,(\mathcal{P}^*_s(A,N))^{-\frac{1}{s}} \tag{0.5}$$

then

$$\mathcal{U}_{\omega_N^A}(y) \geqslant (1-\delta)\mathcal{P}^*_s(A,N). \tag{0.6}$$

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Proof. First note that if $\mathrm{dist}(y,\omega_N^A) < (\mathcal{P}_s^*(A,N))^{-\frac{1}{s}}$ then

$$\mathcal{U}_{\omega_N^A}(y) \geqslant \mathcal{P}_s^*(A,N),$$

and thus (0.6) holds *a fortiori*. Therefore from now on we consider points $y \in \mathbb{R}^p$ such that (0.5) and $\mathrm{dist}(y,\omega_N^A) \geqslant (\mathcal{P}_s^*(A,N))^{-\frac{1}{s}}$ hold, and our goal is to prove (0.6) for such y.

Let $y_1 \in A$ be such that $|y_1 - y| = \mathrm{dist}(y,A)$ and let

$$y_2 \in \mathrm{argmax}_{y' \in [y,y_1]} \mathcal{U}_{\omega_N^A}(y'). \tag{0.7}$$

We claim that the following chain of inequalities holds:

$$\begin{aligned} \mathcal{U}_{\omega_N^A}(y) &\geqslant \mathcal{U}_{\omega_N^A}(y_2) - \int_0^1 \left|\nabla \mathcal{U}_{\omega_N^A}(y + t(y_2 - y)) \cdot (y_2 - y)\right| dt &(0.8)\\ &\geqslant \mathcal{U}_{\omega_N^A}(y_2) - s|y_2 - y| \left[\min_{y' \in [y,y_2]} \mathrm{dist}(y',\omega_N^A)\right]^{-1} \max_{y' \in [y,y_2]} \mathcal{U}_{\omega_N^A}(y') &(0.9)\\ &\geqslant (1-\delta)\,\mathcal{U}_{\omega_N^A}(y_2) \geqslant (1-\delta)\,\mathcal{P}_s^*(A,N). &(0.10) \end{aligned}$$

We now prove the above. The bound (0.8) follows by Taylor expansion. Inequality (0.9) follows by noting that whenever $x \neq y'$ we have $|\nabla_y |x - y'|^{-s}| = s|x-y'|^{-s-1}$, therefore for $y' \in [y,y_2]$ we have

$$\left|\nabla \mathcal{U}_{\omega_N^A}(y')\right| \leqslant s \sum_{x \in \omega_N^A} |x - y'|^{-s-1} \leqslant s \left[\min_{y' \in [y,y_2]} \mathrm{dist}(y',\omega_N^A)\right]^{-1} \mathcal{U}_{\omega_N^A}(y').$$

The first inequality in (0.10) follows by using definition (0.7) of y_2, and the bounds following from our hypotheses on y: $|y_2 - y| \leqslant |y_1 - y| = \mathrm{dist}(y,A) \leqslant \frac{\delta}{s}(\mathcal{P}_s^*(A,N))^{-\frac{1}{s}}$ and $\mathrm{dist}(y,\omega_N^A) \geqslant (\mathcal{P}_s^*(A,N))^{-\frac{1}{s}}$. The second inequality in (0.10) follows by the fact that $y_1 \in A$ and the definition of y_2, and of ω_N^A:

$$\mathcal{P}_s^*(A,N) = \min_{x \in A} \mathcal{U}_{\omega_N^A}(x) \leqslant \mathcal{U}_{\omega_N^A}(y_1) \leqslant \mathcal{U}_{\omega_N^A}(y_2)$$

□

Recall the notation, for the r-neigborhood of a closed set $K \subset \mathbb{R}^d$, for $r > 0$:

$$(K)_r := \{x \in \mathbb{R}^d : \ \mathrm{dist}(x,K) < r\}.$$

Lemma 4. *There exists a constant $c_p > 0$ with the following properties. Let $B \subset \mathbb{R}^p$ be a ball and let $A \subset B$ be a closed set, such that for some $\epsilon \in (0,1)$ there holds $\mathcal{L}^p(B \backslash A) < \epsilon \mathcal{L}^p(B)$. Then for each $r \in (0, \epsilon\ \mathrm{diam}(B))$ we can cover all of $B \backslash (A)_r$ by at most $c_p \frac{\epsilon \mathcal{L}^p(B)}{r^p}$ balls of radius r.*

Proof. Let $r' := r/\sqrt{p}$ and we show that we may take as the set of ball centers the following:

$$W := \{r'k : \ k \in \mathbb{Z}^p, \ (r'k + [-r',r']^p) \cap (B \backslash (A)_r) \neq \emptyset\}.$$

Equivalently, W is formed by those vertices of $(r'\mathbb{Z}^p)$-grid cubes of the form $r'(k + [0,1]^p)$ which meet $B \backslash (A)_r$.

We note that the balls with centers in W and radius $\frac{r'}{2}$ are disjoint and contained in the r'-neighborhood of $B \backslash (A)_r$. Furthermore, we have the inclusion

$$(B \backslash (A)_r)_{r'} \subset (B)_{r'} \backslash A = ((B)_{r'} \backslash B) \cup (B \backslash A),$$

from which it follows that, denoting by $\#W$ the cardinality of W,

$$\#W \cdot \mathcal{L}^p(B_{\frac{r'}{2}}) \leqslant \mathcal{L}^p((B)_{r'} \backslash B) + \mathcal{L}^p(B \backslash A) \leqslant \left(C_p \frac{r'}{\mathrm{diam}(B)} + \epsilon\right) \mathcal{L}^p(B) \leqslant (C_p + 1)\ \epsilon\ \mathcal{L}^p(B).$$

This implies that for $c_p := 2^p(C_p + 1)/(p^{\frac{p}{2}}\beta_p)$, in which β_p is the volume of the unit ball in $\mathbb{R}^p$, there holds

$$\#W \leqslant c_p \epsilon \frac{\mathcal{L}^p(B)}{r^p}.$$

It remains to show that radius-r balls with centers in W cover $B \backslash (A)_r$. Indeed, note that if the cube $r'k + [-r',r']^p$ with $k \in \mathbb{Z}^p$ meets $B \backslash (A)_r$, then it is contained in the ball $B(r'k, \sqrt{p}\ r') = B(r'k, r)$, and thus

$$B \backslash (A)_r \subset \bigcup_{r'k \in W} (r'k + [-r',r']^p) \subset \bigcup_{r'k \in W} B(r'k, r),$$

as desired. □

Proof of Proposition 2: Observe that by the same proof as in [1, Thm. 3.4], which applies also without the restriction that the optimum polarization points belong to A, with $\mathcal{L}^p$ used as the measure μ in the proof, there exists a constant $C_{s,p} > 0$ independent of N, A, such that $\mathcal{P}_s^*(A,N) \leqslant C_{s,p} N^{s/p}/(\mathcal{L}^p(A))^{s/p}$. Now, applying Lemma 3 with $\delta := \epsilon^{\frac{1}{p+1}}$ we find that for the optimum configuration ω_N^A like in (0.4), with

$$r_N := \frac{\epsilon^{\frac{1}{p+1}}}{s} \left(\mathcal{P}_s^*(A,N)\right)^{-\frac{1}{s}} \geqslant \frac{\epsilon^{\frac{1}{p+1}} \left(\mathcal{L}^p(A)\right)^{\frac{1}{p}}}{s(C_{s,p})^{\frac{1}{s}}} N^{-\frac{1}{p}}, \tag{0.11}$$

we have

$$\forall y \in (A)_{r_N}, \quad \mathcal{U}_{\omega_N^A}(y) \geqslant \left(1 - \epsilon^{\frac{1}{p+1}}\right) \mathcal{P}_s^*(A,N). \tag{0.12}$$

Next, for N large enough so that $r_N < \epsilon \operatorname{diam}(B)$ we apply Lemma 4 with r_N playing the role of r, and we find a set of centers W such that, using also (0.11) in the second inequality below:

$$B\backslash(A)_{r_N} \subset \bigcup_{x \in W} B(x, r_N), \qquad \#W \leqslant c_p\, \epsilon \frac{\mathcal{L}^p(B)}{r_N^p} \leqslant \widetilde{C}_{s,p}\, \epsilon^{\frac{1}{p+1}} \frac{\mathcal{L}^p(B)}{\mathcal{L}^p(A)} N \leqslant \widetilde{C}_{s,p}\, \epsilon^{\frac{1}{p+1}} (1-\epsilon) N. \tag{0.13}$$

By considering the new configuration $\omega_{\widetilde{N}} := \omega_N^A \cup W$ whose cardinality is denoted $\widetilde{N}$, we find that

$$\widetilde{N} \in \left[N, N\left(1 + \widetilde{C}_{s,p} \epsilon^{\frac{1}{p+1}} (1-\epsilon)\right)\right], \quad \mathcal{U}_{\omega_{\widetilde{N}}}(y) \geqslant \begin{cases} \left(1 - \epsilon^{\frac{1}{p+1}}\right) \mathcal{P}_s^*(A,N) & \text{for } y \in (A)_{r_N}, \\ \mathcal{P}_s^*(A,N) & \text{for } y \in B\backslash(A)_{r_N}, \end{cases} \tag{0.14}$$

in which for the first inequality we use (0.12) and for the second one we use the first part of (0.13) and the fact that for $x \in W, y \in B(x, r_N)$ there holds, since $\epsilon < 1 \leqslant p < s$,

$$\mathcal{U}_{\omega_{\widetilde{N}}}(y) \geqslant |x-y|^{-s} \geqslant r_N^{-s} \geqslant \left(\frac{\epsilon^{\frac{1}{p+1}}}{s}\right)^s \mathcal{P}_s^*(A,N) \geqslant \mathcal{P}_s^*(A,N).$$

We then find that, using also the fact that $\epsilon \in (0,1)$,

$$\frac{\mathcal{P}_s^*(B,\widetilde{N})}{\widetilde{N}^{s/p}} \geqslant \frac{1 - \epsilon^{\frac{1}{p+1}}}{\left(1 + \widetilde{C}_{s,p}\, \epsilon^{\frac{1}{p+1}} (1-\epsilon)\right)^{s/p}} \frac{\mathcal{P}_s^*(A,N)}{N^{s/p}} \geqslant (1 - \epsilon^{\frac{1}{p+1}}) \left(1 - \frac{s}{p} \widetilde{C}_{s,p} \epsilon^{\frac{1}{p+1}}\right) \frac{\mathcal{P}_s^*(A,N)}{N^{s/p}}, \tag{0.15}$$

from which the bound (0.3) follows, with $c_{s,p} = 1 + \frac{s}{p} \widetilde{C}_{s,p}$.

□

References

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