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On Approximation in the L^p-Norm by Reciprocals of Polynomials

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1. INTRODUCTION

Recently the second and third authors have considered the question of approximating a real-valued continuous function f on $[-1, 1]$ by reciprocals of polynomials having real or complex coefficients. While no restrictions on f are necessary for the approximation by reciprocals of complex polynomials, it is obvious that if we limit ourselves to reciprocals of real polynomials we must assume that f does not change sign in the interval. Under this assumption it was shown in [3] that one can approximate $f(\neq 0)$ by reciprocals of real polynomials at the rate $\omega(f, 1/n)$, where $\omega(f, \cdot)$ is the usual modulus of continuity of f. The purpose of this note is to improve the above estimates by replacing $\omega(f, 1/n)$ by the Ditzian–Totik modulus of continuity $\omega_\varphi(f, 1/n)$ and also to obtain estimates on the rate of approximation by reciprocals of polynomials in the L^p-norm, $1 \leqslant p < \infty$.

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Here, unfortunately, we have to assume $f \in L^{p+1}[-1, 1]$ and give the estimates in terms of $\omega_\varphi(f, 1/n)_{p+1}$. The last section is devoted to some estimates on shape-preserving approximation by reciprocals of polynomials in the various norms.

2. APPROXIMATION IN $C[-1, 1]$

Let $\varphi(x) := \sqrt{1-x^2}$ and set

$$\Delta_{h\varphi} f(x) := \begin{cases} f(x+(h/2)\,\varphi(x)) - f(x-(h/2)\,\varphi(x)), & x \pm (h/2)\,\varphi(x) \in [-1, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Following Ditzian and Totik [2], define

$$\omega_\varphi(f, t) := \sup_{0<h\leqslant t} \|\Delta_{h\varphi} f\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the sup norm over $[-1, 1]$. Then it is readily seen that, for any $f \in C[-1, 1]$,

$$\omega_\varphi(f, t) \leqslant \omega(f, t)$$

while, for instance, for $f(x) = \sqrt{1+x}$ we have

$$\omega_\varphi(f, t) = O(t) \qquad \text{and} \qquad \omega(f, t) \sim t^{1/2}.$$

Ditzian and Totik [2] proved that $\omega_\varphi(f, t)$ is equivalent to the modified Peetre kernel

$$K_\varphi(f, t) := \inf\{\|f-g\|_\infty + t\,\|\varphi g'\|_\infty + t^2\,\|g'\|_\infty\}.$$

where the infimum is taken over all g that are absolutely continuous in $[-1, 1]$ and such that $g' \in L^\infty[-1, 1]$. Our first main result is

THEOREM 1. *Let $f \in C[-1, 1]$ be nonconstant and nonnegative. Then there exists a sequence of polynomials $\{p_n\}_1^\infty$, with $p_n \in \mathscr{P}_n$, such that*

$$\left\| f - \frac{1}{p_n} \right\|_\infty \leqslant C\omega_\varphi\left(f, \frac{1}{n}\right), \qquad n = 1, 2, \ldots. \tag{1}$$

Here and throughout this paper, C is an absolute constant independent of f and n whose value may be different from line to line and $\mathscr{P}_n$ denotes the collection of all real polynomials of degree at most n.

Remark. Obviously, a nonzero constant function f is approximable at the rate (1), while $f \equiv 0$ is not.

In the proof of Theorem 1 we shall need the following.

LEMMA 2. *Let $f \in C[-1, 1]$ and define $g \in C[-\pi, \pi]$ by $g(\theta) := f(\cos\theta)$. Let $K_n(t)$ be the Jackson kernel that satisfies*

$$\int_{-\pi}^{\pi} K_n(t)\,dt = 1, \qquad \int_{-\pi}^{\pi} |t|^k K_n(t)\,dt \sim n^{-k}, \qquad k = 1, 2, 3, 4. \tag{2}$$

Then, for $-\pi \leqslant \theta \leqslant \pi$,

$$\int_{-\pi}^{\pi} |g(\theta+t) - g(\theta)|^k K_n(t)\,dt \leqslant C\left[\omega_\varphi\left(f, \frac{1}{n}\right)\right]^k, \qquad k = 1, 2.$$

Proof. By virtue of the equivalence between $\omega_\varphi(f, t)$ and $K_\varphi(f, t)$, given $f \in C[-1, 1]$, for each $n = 1, 2, \ldots$, there exists an f_n absolutely continuous on $[-1, 1]$ such that

$$\begin{aligned} \|f - f_n\|_\infty &\leqslant C\omega_\varphi\left(f, \frac{1}{n}\right), \\ \|\varphi f_n'\|_\infty &\leqslant Cn\omega_\varphi\left(f, \frac{1}{n}\right), \\ \|f_n'\|_\infty &\leqslant Cn^2\omega_\varphi\left(f, \frac{1}{n}\right). \end{aligned} \tag{3}$$

Setting $g_n(\theta) := f_n(\cos\theta)$, then by (2) and (3) we have for $k = 1, 2$,

$$\begin{aligned} &\int_{-\pi}^{\pi} |g(\theta+t) - g(\theta)|^k K_n(t)\,dt \\ &\qquad \leqslant C\left[\omega_\varphi\left(f, \frac{1}{n}\right)\right]^k + \int_{-\pi}^{\pi} |g_n(\theta+t) - g_n(\theta)|^k K_n(t)\,dt. \end{aligned} \tag{4}$$

Now, for each u between $\cos\theta$ and $\cos(\theta+t)$ we have

$$\begin{aligned} 1 &= -\frac{2\sin(\theta+t/2)\sin(t/2)}{\cos(\theta+t) - \cos\theta} \\ &= -\frac{2\varphi(u)\sin(t/2)}{\cos(\theta+t) - \cos\theta} + \frac{2[\varphi(u) - \sin(\theta+t/2)]\sin(t/2)}{\cos(\theta+t) - \cos\theta}. \end{aligned}$$

Hence

$$\int_{-\pi}^{\pi} |g_n(\theta+t)-g_n(\theta)|^k K_n(t)\,dt$$

$$=\int_{-\pi}^{\pi} \left|\int_{\cos\theta}^{\cos(\theta+t)} f_n'(u)\,du\right|^k K_n(t)\,dt$$

$$\leqslant C\int_{-\pi}^{\pi} \left|\frac{1}{\cos(\theta+t)-\cos\theta}\int_{\cos\theta}^{\cos(\theta+t)} |f_n'(u)|\,\varphi(u)\,du\right|^k |t|^k K_n(t)\,dt$$

$$+C\int_{-\pi}^{\pi} \left|\frac{1}{\cos(\theta+t)-\cos\theta}\int_{\cos\theta}^{\cos(\theta+t)} |f_n'(u)|\,du\right|^k |t|^{2k} K_n(t)\,dt$$

$$\leqslant C\,\|f_n'\varphi\|_\infty^k \int_{-\pi}^{\pi} |t|^k K_n(t)\,dt + C\,\|f_n'\|_\infty^k \int_{-\pi}^{\pi} |t|^{2k} K_n(t)\,dt. \qquad (5)$$

Thus, from (2) and (3), we conclude that for $k=1, 2$,

$$\int_{-\pi}^{\pi} |g_n(\theta+t)-g_n(\theta)|^k K_n(t)\,dt \leqslant C\{\|f_n'\varphi\|_\infty^k n^{-k}+\|f_n'\|_\infty^k n^{-2k}\}$$

$$\leqslant C\left[\omega_\varphi\left(f,\frac{1}{n}\right)\right]^k.$$

Combining this last inequality with (4) proves Lemma 2. ∎

We now turn to the proof of Theorem 1. Although we basically follow the ideas of the proof in [3] (except that Lemma 2 provides sharper estimates), there is one major difference. In preparation for the L^p case we do not wish to use the pointwise value of f in order to get a lower estimate on the product fp_n and thereby prove that p_n does not vanish in $[-1, 1]$. In fact, in the L^p case, it makes no sense to look for such a lower estimate. Nevertheless we prove that p_n does not vanish in $[-1, 1]$.

Proof of Theorem 1. Given a nonconstant $f\in C[-1, 1]$, $f\geqslant 0$, and $\varepsilon>0$, let $f_\varepsilon(x):=f(x)+\varepsilon$ and let $g_\varepsilon(\theta):=f_\varepsilon(\cos\theta)$, $\theta\in[-\pi, \pi]$. Then $|1/g_\varepsilon|\leqslant 1/\varepsilon$ and we can define the algebraic polynomials

$$p_n(x):=\int_{-\pi}^{\pi} \frac{1}{g_\varepsilon(\theta+t)} K_n(t)\,dt, \qquad n=1, 2, ..., \qquad (6)$$

where $x=\cos\theta$ and $K_n(t)$ is the Jackson kernel of Lemma 2.

By Hölder's inequality,

$$\begin{aligned} 1 &= \left(\int_{-\pi}^{\pi} K_n(t)\,dt\right)^2 \\ &\leqslant \int_{-\pi}^{\pi} g_\varepsilon(\theta+t)\,K_n(t)\,dt \cdot \int_{-\pi}^{\pi} \frac{1}{g_\varepsilon(\theta+t)}\,K_n(t)\,dt \\ &= p_n(x) \cdot \int_{-\pi}^{\pi} g_\varepsilon(\theta+t)\,K_n(t)\,dt. \end{aligned}$$

Thus p_n does not vanish in $[-1, 1]$ and

$$\frac{1}{p_n(x)} \leqslant \int_{-\pi}^{\pi} g_\varepsilon(\theta+t)\,K_n(t)\,dt. \tag{7}$$

Let $E := \{x: (1/p_n(x)) > f_\varepsilon(x)\}$. Then, by (7) and Lemma 2, we have for $x \in E$

$$\begin{aligned} 0 < \frac{1}{p_n(x)} - f_\varepsilon(x) &\leqslant \int_{-\pi}^{\pi} [g_\varepsilon(\theta+t) - g_\varepsilon(\theta)]\,K_n(t)\,dt \\ &\leqslant C\omega_\varphi\left(f_\varepsilon, \frac{1}{n}\right) = C\omega_\varphi\left(f, \frac{1}{n}\right). \end{aligned} \tag{8}$$

For x in the complement of E we have

$$\frac{1}{p_n(x)} \leqslant f_\varepsilon(x).$$

Hence

$$\begin{aligned} 0 \leqslant f_\varepsilon(x) - \frac{1}{p_n(x)} &= \int_{-\pi}^{\pi} \left[\frac{1}{g_\varepsilon(\theta+t)} - \frac{1}{g_\varepsilon(\theta)}\right] \frac{g_\varepsilon(\theta)}{p_n(\cos\theta)}\,K_n(t)\,dt \\ &\leqslant \int_{-\pi}^{\pi} \frac{|g_\varepsilon(\theta) - g_\varepsilon(\theta+t)|}{g_\varepsilon(\theta)\,g_\varepsilon(\theta+t)} \frac{g_\varepsilon(\theta)}{p_n(\cos\theta)}\,K_n(t)\,dt \\ &\leqslant \int_{-\pi}^{\pi} \frac{|g_\varepsilon(\theta) - g_\varepsilon(\theta+t)|}{g_\varepsilon(\theta+t)}\,g_\varepsilon(\theta)\,K_n(t)\,dt \\ &\leqslant \int_{-\pi}^{\pi} |g_\varepsilon(\theta) - g_\varepsilon(\theta+t)|\,K_n(t)\,dt + \frac{1}{\varepsilon}\int_{-\pi}^{\pi} |g_\varepsilon(\theta) - g_\varepsilon(\theta+t)|^2\,K_n(t)\,dt, \end{aligned}$$

where for the last inequality we used the fact that $1/g_\varepsilon \leqslant 1/\varepsilon$. By virtue of Lemma 2, we have for $x \notin E$

$$0 \leqslant f_\varepsilon(x) - \frac{1}{p_n(x)} \leqslant C\omega_\varphi\left(f, \frac{1}{n}\right) + C\,\frac{1}{\varepsilon}\left[\omega_\varphi\left(f, \frac{1}{n}\right)\right]^2$$

(since $\omega_\varphi(f_\varepsilon, t) \equiv \omega_\varphi(f, t)$). Choosing $\varepsilon = \omega_\varphi(f, 1/n)$, which is not zero since $f \not\equiv$ const, yields

$$0 \leqslant f_\varepsilon(x) - \frac{1}{p_n(x)} \leqslant C\omega_\varphi\left(f, \frac{1}{n}\right), \qquad \text{for } x \notin E.$$

Combining with (8) we have

$$\left\| f_\varepsilon - \frac{1}{p_n} \right\|_\infty \leqslant C\omega_\varphi\left(f, \frac{1}{n}\right).$$

Thus

$$\begin{aligned} \left\| f - \frac{1}{p_n} \right\|_\infty &\leqslant \|f - f_\varepsilon\|_\infty + \left\| f_\varepsilon - \frac{1}{p_n} \right\|_\infty \\ &\leqslant \varepsilon + C\omega_\varphi\left(f, \frac{1}{n}\right) \\ &\leqslant C\omega_\varphi\left(f, \frac{1}{n}\right). \end{aligned}$$

This completes the proof. ▮

Remark. If we work with $C[0, 1]$ instead of $C[-1, 1]$, then φ takes the form $\varphi(x) = \sqrt{x(1-x)}$ and for x^α, $0 < \alpha < 1$, we have $\omega_\varphi(x^\alpha, t) = O(t^{2\alpha})$. Hence the error in approximating x^α, $0 < \alpha < 1$, on $[0, 1]$ by reciprocals of polynomials can be estimated by $Cn^{-2\alpha}$, where C is an absolute constant. This fact was also proved in [3] where a special construction is used. Note, however, that our present proof is valid only for $0 < \alpha < 1$, while in [3] a similar estimate is established for all $\alpha > 0$ with $C = C(\alpha)$ increasing to infinity as $\alpha \to \infty$.

3. APPROXIMATION IN $L^p[-1, 1]$

Here again we follow Ditzian and Totik [2] as we denote

$$\omega_\varphi(f, t)_p := \sup_{0 \leqslant h \leqslant t} \|\Delta_{h\varphi} f\|_p.$$

It was shown in [2] that $\omega_\varphi(f, t)_p$ is equivalent to the Peetre kernel

$$K_\varphi(f, t)_p := \inf\{\|f - g\|_p + t\,\|\varphi g'\|_p + t^2\,\|g'\|_p\},$$

where the infimum is taken over all $g \in L^p[-1, 1]$ that are absolutely continuous in $[-1, 1]$ and such that $g' \in L^p[-1, 1]$.

Our result in this case is not as satisfactory as in $C[-1, 1]$. We will prove

THEOREM 3. *Let $f \in L^{p+1}[-1, 1]$, $1 \leqslant p < \infty$, be nonconstant and nonnegative. Then there exists a sequence of polynomials $\{p_n\}_1^\infty$, with $p_n \in \mathscr{P}_n$, such that*

$$\left\| f - \frac{1}{p_n} \right\|_p \leqslant C\omega_\varphi\left(f, \frac{1}{n}\right)_{p+1}, \qquad n = 1, 2, \ldots. \tag{9}$$

Remark. Obviously $L^{p+1}[-1, 1]$ is a proper subset of $L^p[-1, 1]$ and we have the inequality

$$\omega_\varphi(f, t)_p \leqslant \omega_\varphi(f, t)_{p+1},$$

but we are not able to replace the right-hand side of (9) by $C\omega_\varphi(f, 1/n)_p$. We do not know if this gap is indeed necessary or is due to the limitations of our method of proof.

Proof of Theorem 3. It follows from the equivalence of $\omega_\varphi(f, \cdot)_{p+1}$ and $K_\varphi(f, \cdot)_{p+1}$ that, for each n, there exists an absolutely continuous function $f_n \in L^{p+1}[-1, 1]$ such that

$$\|f - f_n\|_p \leqslant C\,\|f - f_n\|_{p+1} \leqslant C\omega_\varphi\left(f, \frac{1}{n}\right)_{p+1},$$

$$\|\varphi f_n'\|_{p+1} \leqslant Cn\omega_\varphi\left(f, \frac{1}{n}\right)_{p+1},$$

$$\|f_n'\|_{p+1} \leqslant Cn^2\omega_\varphi\left(f, \frac{1}{n}\right)_{p+1}. \tag{10}$$

Moreover, a close look at the proof of Ditzian and Totik [2, Sect. 3.1] reveals that f_n is nonnegative if $f \geqslant 0$. Thus it suffices to approximate f_n at the proper rate and this together with (10) will yield (9).

We proceed as in the proof of Theorem 1. Let $F_n(x) := f_n(x) + \varepsilon$ and let $g_\varepsilon(\theta) := g_{\varepsilon,n}(\theta) := F_n(\cos\theta)$, $-\pi \leqslant \theta \leqslant \pi$. Let $K_n(t)$ be a suitable Jackson kernel, i.e., such that

$$\int_{-\pi}^{\pi} K_n(t)\,dt = 1, \qquad \int_{-\pi}^{\pi} |t|^k K_n(t)\,dt \sim n^{-k}, \qquad k = 1, 2, \ldots, [2p+3]. \tag{11}$$

Then again $g_\varepsilon^{-1} \leqslant 1/\varepsilon$ and we can define the polynomial p_n by (6). We still have the estimate (7), although the right-hand side of (7) may be infinite for $f \in L^{p+1}[-1, 1]$. That this is not so for a differentiable f follows from (14) and (15) later in our proof.

Let

$$E_1 := \left\{x : \frac{1}{p_n(x)} > F_n(x)\right\}.$$

Then, by (7) and Minkowski's inequality,

$$\begin{aligned}\left[\int_{E_1} \left|\frac{1}{p_n(x)} - F_n(x)\right|^p dx\right]^{1/p} &\leqslant \left[\int_{E_1} \left|\int_{-\pi}^{\pi} [g_\varepsilon(\theta+t) - g_\varepsilon(\theta)]\, K_n(t)\, dt\right|^p dx\right]^{1/p} \\ &\leqslant \int_{-\pi}^{\pi} K_n(t) \left[\int_{E_1} |g_\varepsilon(\theta+t) - g_\varepsilon(\theta)|^p\, dx\right]^{1/p} dt. \qquad (12)\end{aligned}$$

Next, for any $x \in [-1, 1]$,

$$\begin{aligned}\left|\frac{1}{p_n(x)} - F_n(x)\right| &= \frac{|1 - p_n(x)\, F_n(x)|}{p_n(x)} \\ &\leqslant \frac{1}{p_n(x)} \int_{-\pi}^{\pi} \frac{|g_\varepsilon(\theta+t) - g_\varepsilon(\theta)|}{g_\varepsilon(\theta+t)}\, K_n(t)\, dt.\end{aligned}$$

and so using the integral representation (6) and Hölder's inequality we get

$$\left|\frac{1}{p_n(x)} - F_n(x)\right| \leqslant \left[\frac{1}{p_n(x)} \int_{-\pi}^{\pi} \frac{|g_\varepsilon(\theta+t) - g_\varepsilon(\theta)|^p}{g_\varepsilon(\theta+t)}\, K_n(t)\, dt\right]^{1/p}.$$

Now for $x \in E_2 := [-1, 1] \backslash E_1$, we have

$$\frac{1}{p_n(x)} \leqslant F_n(x),$$

and so it follows that for $x \in E_2$

$$\left|\frac{1}{p_n(x)} - F_n(x)\right|^p \leqslant \int_{-\pi}^{\pi} \frac{|g_\varepsilon(\theta+t) - g_\varepsilon(\theta)|^p}{g_\varepsilon(\theta+t)}\, g_\varepsilon(\theta)\, K_n(t)\, dt.$$

Hence

$$\begin{aligned}\int_{E_2} \left|\frac{1}{p_n(x)} - F_n(x)\right|^p dx &\leqslant \int_{E_2} \int_{-\pi}^{\pi} |g_\varepsilon(\theta+t) - g_\varepsilon(\theta)|^p\, K_n(t)\, dt\, dx \\ &\quad + \frac{1}{\varepsilon} \int_{E_2} \int_{-\pi}^{\pi} |g_\varepsilon(\theta+t) - g_\varepsilon(\theta)|^{p+1}\, K_n(t)\, dt\, dx, \qquad (13)\end{aligned}$$

where we used the inequality $g_\varepsilon^{-1} \leqslant 1/\varepsilon$.

It remains to estimate the integrals on the right-hand sides of (12) and (13). They are similar and we use the method of proof of Lemma 2 in order to estimate each of them. What we get is

$$\int_{-\pi}^{\pi} K_n(t)\left[\int_{E_1} |g_\varepsilon(\theta+t)-g_\varepsilon(\theta)|^p\,dx\right]^{1/p} dt \leqslant C\omega_\varphi\left(f,\frac{1}{n}\right)_{p+1}, \tag{14}$$

$$\int_{-\pi}^{\pi} K_n(t)\int_{E_2} |g_\varepsilon(\theta+t)-g_\varepsilon(\theta)|^q\,dx\,dt \leqslant C\left[\omega_\varphi\left(f,\frac{1}{n}\right)_{p+1}\right]^q, \tag{15}$$

for $q=p$ or $q=p+1$.

We shall only prove (15) ((14) being similar). Consider

$$\begin{aligned}
&\int_{-\pi}^{\pi} K_n(t)\int_{E_2} |g_\varepsilon(\theta+t)-g_\varepsilon(\theta)|^q\,dx\,dt \\
&\quad =\int_{-\pi}^{\pi} K_n(t)\int_{E_2} \left|\int_{\cos\theta}^{\cos(\theta+t)} F_n'(u)\,du\right|^q dx\,dt \\
&\quad \leqslant C\int_{-\pi}^{\pi}\left|\frac{1}{\cos(\theta+t)-\cos\theta}\int_{\cos\theta}^{\cos(\theta+t)} |F_n'(u)|\,\varphi(u)\,du\right|^q |t|^q K_n(t)\,dx\,dt \\
&\qquad +C\int_{-\pi}^{\pi}\left|\frac{1}{\cos(\theta+t)-\cos\theta}\int_{\cos\theta}^{\cos(\theta+t)} |F_n'(u)|\,du\right|^q |t|^{2q} K_n(t)\,dx\,dt
\end{aligned}$$

as in the proof of Lemma 2. Denote by $M_F(x)$ the Hardy maximal function of F, i.e.,

$$M_F(x) := \sup_{x\in I}\frac{1}{|I|}\left|\int_I F(s)\,ds\right|.$$

Then it follows that for $x=\cos\theta$

$$\begin{aligned}
&\int_{-\pi}^{\pi} K_n(t)\int_{E_2} |g_\varepsilon(\theta+t)-g_\varepsilon(\theta)|^q\,dx\,dt \\
&\quad \leqslant C\int_{-\pi}^{\pi} |t|^q K_n(t)\int_{E_2} |M_{|F_n'|\varphi}(x)|^q\,dx\,dt \\
&\qquad +C\int_{-\pi}^{\pi} |t|^{2q} K_n(t)\int_{E_2} |M_{|F_n'|}(x)|^q\,dx\,dt \\
&\quad \leqslant Cn^{-q}\,\|M_{|F_n'|\varphi}\|_q^q + Cn^{-2q}\,\|M_{|F_n'|}\|_q^q \\
&\quad \leqslant Cn^{-q}\,\|M_{|F_n'|\varphi}\|_{p+1}^q + Cn^{-2q}\,\|M_{|F_n'|}\|_{p+1}^q \\
&\quad \leqslant Cn^{-q}\,\|F_n'\varphi\|_{p+1}^q + Cn^{-2q}\,\|F_n'\|_{p+1}^q,
\end{aligned}$$

by virtue of the inequality (see [4, p. 58])

$$\|M_F\|_p \leqslant C_p \|F\|_p, \qquad 1 < p \leqslant \infty.$$

The proof of (15) now follows from (10).

Finally, we choose $\varepsilon = \omega_\varphi(f, 1/n)_{p+1}$. Then (12) through (15) yield

$$\left\| F_n - \frac{1}{p_n} \right\|_p \leqslant C\omega_\varphi \left(f, \frac{1}{n} \right)_{p+1},$$

which together with (10) proves (9). ▮

4. Shape-Preserving Approximation

Returning to continuous functions we will show that a monotone increasing $f \in C[-1, 1]$ is approximable by reciprocals of monotone decreasing polynomials p_n (so that $1/p_n$ is monotone increasing) at the same rate (1). To this end we observe that Beatson [1] proved the existence of a Jackson-type kernel satisfying (2) and such that it takes increasing functions into increasing functions. Using this kernel in the proof of Theorem 1, we see that whenever f is increasing so is f_ε and hence f_ε^{-1} is decreasing. Therefore the polynomials p_n defined by (6) are decreasing. We summarize these observations in

Theorem 4. *Let $f \in C[-1, 1]$ be nonnegative and increasing. Then for each n there is a decreasing $p_n \in \mathscr{P}_n$ such that* (1) *holds.*

References

1. Rick Beatson, Joint approximation of a function and it's derivatives, *in* "Approx. Theory III, Proc. of Conf. on Approx. Theory, Austin, TX, 1980 (E. W. Cheney, Ed.), pp. 199–206, Academic Press, New York, 1980.
2. Z. Ditzian and V. Totik, "Moduli of Smoothness," Springer-Verlag, Berlin, 1987.
3. A. L. Levin and E. B. Saff, Degree of approximation of real functions by reciprocals of real and complex polynomials, *SIAM J. Math. Anal.* **19** (1988), 233–245.
4. E. M. Stein, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, NJ, 1980.

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