POLYNOMIAL APPROXIMATION OF PIECEWISE ANALYTIC FUNCTIONS

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ABSTRACT

For a function f that is piecewise analytic on $[-1,1]$, we construct a sequence of polynomial approximants that converges to f at an exponential rate at each point of analyticity of f. For the uniform norm on $[-1,1]$, these polynomials approximate f to within a constant times the least possible error while, locally, the approximants give a 'best possible' rate of convergence. Moreover, unlike the best uniform approximants, the polynomials that we construct overconverge to an analytic continuation of f. Also, we prove a conjecture of Grothmann and Saff concerning the rate of polynomial approximation in a region of the plane to a complex extension of the absolute value function. As the starting point for our proofs, we obtain 'best possible' polynomial approximations to the sign function.

1. *Introduction*

Let f be a *piecewise analytic function* on $[-1,1]$ belonging to $C^k[-1,1]$, $k \geqslant 0$, by which we mean that $f \in C^k[-1,1]$ and there exist points

$$-1 < x_1 < x_2 < \dots < x_m < 1$$

such that, on each of the *closed* intervals $[-1, x_1], [x_1, x_2], \dots, [x_m, 1]$, the restriction of f is analytic, but f itself is not analytic at each point $x_1, \dots, x_m$. We call the x_i *points of singularity* of f and always assume that f has at least one such singularity ($m \geqslant 1$).

We let Π_n denote the collection of all algebraic polynomials of degree at most n, and let $p_n^* = p_n^*(f)$ be the polynomial in Π_n of best uniform approximation to f on $[-1,1]$; that is

$$\|f-p_n^*\|_{[-1,1]} = \inf_{p\in\Pi_n} \|f-p\|_{[-1,1]},$$

where $\|\cdot\|_{[-1,1]}$ denotes the supremum norm on $[-1,1]$. If $f \notin C^{k+1}[-1,1]$, then it is well known that

$$e_n(f) := \|f-p_n^*\|_{[-1,1]}$$

cannot tend to zero faster than $1/n^{k+1}$; in fact, there exist constants $c_1, c_2 > 0$ such that

$$c_1/n^{k+1} \leqslant e_n(f) \leqslant c_2/n^{k+1}, \quad n = 1,2,\dots \tag{1.1}$$

(cf. [**10**, Chapter 7]). It seems reasonable to expect, however, that the rate of convergence should be faster (say, geometric) on subintervals of $[-1,1]$ where f is analytic (that is, subintervals not containing the x_i). Unfortunately the polynomials

Received 3 February 1988.

1980 *Mathematics Subject Classification* (1985 *Revision*) 41A10.

The research of the first author was supported, in part, by the National Science Foundation, under grant DMS-862-0098.

The research of the second author was partially supported by the Hungarian National Science Foundation for Research, grant no. 1157. This work was done while he visited the Institute for Constructive Mathematics, University of South Florida, Tampa.

$p_n^*(f)$ of best approximation do not enjoy this property because, as first shown by Kadec [6], there is a subsequence $\{n_k\}$ for which the errors $f-p_{n_k}^*(f)$ have extreme points that are dense in $[-1,1]$. This shortcoming lends support to the *principle of contamination* in best polynomial approximation, which was introduced by the first author in [9]. Roughly speaking, this principle asserts that the existence of one or more singularities of f adversely affects the behavior *over the whole interval* of some subsequence of the best polynomial approximants.

Thus it is natural to seek 'near best' polynomial approximants $p_n \in \Pi_n$, $n = 1, 2, \ldots$, that have the following desirable properties:

(i) $\|f-p_n\|_{[-1,1]} \leqslant C/n^{k+1}$, $n = 1, 2, \ldots$;
(ii) at each point $x \in [-1,1] \backslash \{x_1, \ldots, x_m\}$,

$$\limsup_{n\to\infty} |f(x)-p_n(x)|^{1/n} < 1. \tag{1.2}$$

The purpose of this paper is to construct such a sequence of approximants p_n and to determine the best possible rate of convergence in (1.2). In fact, for our construction, the inequality (1.2) holds uniformly on every subinterval $[a,b] \subset [-1,1] \backslash \{x_1, \ldots, x_m\}$, from which we obtain another desirable property:

(iii) each point $x \in [-1,1] \backslash \{x_1, \ldots, x_m\}$ is the center of a disk in which the sequence $\{p_n(z)\}_1^\infty$ converges uniformly to the analytic continuation of f.

In contrast, Blatt and Saff [3] have shown (as further evidence of the principle of contamination) that *every* point of $[-1,1]$ is a limit point of the (complex) zeros of the best approximants $\{p_n^*(f)\}_1^\infty$, which means that this sequence cannot be used for the purpose of analytic continuation to a (2-dimensional) neighborhood of any point on $[-1,1]$.

As an application of our results we shall prove a conjecture of R. Grothmann and E. B. Saff [5] concerning polynomial approximation to the complex extension of the absolute value function given by

$$g(z) := \begin{cases} z, & \text{if } \operatorname{Re}(z) \geqslant 0, \\ -z, & \text{if } \operatorname{Re}(z) < 0. \end{cases} \tag{1.3}$$

The outline of the paper is as follows. In Section 2, we discuss the best possible rate of polynomial approximation to the sign function. In Section 3, we use the approximations to the sign function to construct a sequence of polynomials p_n having properties (i) and (ii) above. We also show that the rate of convergence obtained is best possible. Finally, in Section 4, we consider polynomial approximants in regions of the complex plane to the function $g(z)$ given in (1.3).

2. *Polynomial approximation of the sign function*

The following question arises in connection with several problems of approximation theory (cf. [8] and the subsequent sections). For what non-negative numbers α, β do there exist constants $C, c > 0$ and polynomials $P_n \in \Pi_n$ such that for all $n \geqslant 1$ and all $x \in [-1,1]$,

$$|\operatorname{sign} x - P_n(x)| \leqslant C \exp(-cn^\alpha |x|^\beta)? \tag{2.1}$$

The answer is given by the following.

THEOREM 1. *For $\alpha,\beta \geqslant 0$, there exist constants C, $c > 0$ and polynomials $P_n \in \Pi_n$ satisfying* (2.1) *for all $n \geqslant 1$ and all $x \in [-1, 1]$ if and only if*

$$\alpha < 1 \text{ and } \beta \geqslant \alpha \quad \text{or} \quad \alpha = 1 \text{ and } \beta > 1. \tag{2.2}$$

The case $\alpha = \beta < 1$ was considered by Nevai and Totik in [8, Corollary 2] and exactly as there our proof of Theorem 1 is based upon the existence of fast decreasing polynomials. More precisely, we shall prove the following.

THEOREM 2. *Let $\alpha,\beta \geqslant 0$. Then there exist C', $c' > 0$ and $Q_n \in \Pi_n$ such that*

$$Q_n(0) = 1, \quad |Q_n(x)| \leqslant C' \exp(-c'n^\alpha |x|^\beta), \tag{2.3}$$

for all $n = 1, 2, \ldots$ and all $x \in [-1, 1]$ if and only if (2.2) *holds.*

Assuming, for the moment, that Theorem 2 is true, we shall give the following.

Proof of Theorem 1. It suffices to show that the existence of C, c and $\{P_n\}$ with property (2.1) is equivalent to the existence of C', c' and $\{Q_n\}$ with property (2.3).

Suppose first that (2.1) holds for all $n \geqslant 1$ and $x \in [-1, 1]$. Then (2.1) also holds for a sequence of odd polynomials $P_n(x)$, $n = 1, 2, \ldots$. Setting

$$Q_n(x) := 1 - (P_{[n/2]}(x))^2, \quad n = 1, 2, \ldots,$$

gives a sequence satisfying (2.3).

Suppose now that (2.3) holds. We assume, as we may, that the polynomials Q_n satisfying (2.3) are even. In [8, Theorem 1] it was shown that there are even polynomials $R_{[n/4]} \in \Pi_{[n/4]}$ with the property that

$$R_{[n/4]}(0) = 1, \quad |R_{[n/4]}(x)| \leqslant C_1 \exp(-\sqrt{(n|x|)}), \tag{2.4}$$

for all $n \geqslant 1$, all $x \in [-1, 1]$. Define

$$P_n(x) := \frac{2}{\gamma_n} \int_{-1}^{x} (R_{[n/4]}(u)\, Q_{[n/4]-1}(u))^2 \, du - 1, \tag{2.5}$$

where

$$\gamma_n := \int_{-1}^{1} (R_{[n/4]}(u)\, Q_{[n/4]-1}(u))^2 \, du. \tag{2.6}$$

Obviously $P_n \in \Pi_n$ and $-1 \leqslant P_n(x) \leqslant 1$ for $x \in [-1, 1]$. Moreover, for $x \in [-1, 0]$ and $n \geqslant 4$ we have from (2.3) and (2.4),

$$\begin{aligned}
|P_n(x) + 1| &\leqslant \frac{2}{\gamma_n} \int_{-1}^{x} (C'C_1)^2 \exp(-2\sqrt{(n|u|)} - 2c'([n/4]-1)^\alpha |u|^\beta) \, du \\
&\leqslant \frac{2}{\gamma_n} (C'C_1)^2 \exp(-c_1 n^\alpha |x|^\beta) \int_{-\infty}^{x} \exp(-2\sqrt{(n|u|)}) \, du \\
&\leqslant \frac{C_2}{\gamma_n n} \exp(-c_1 n^\alpha |x|^\beta),
\end{aligned} \tag{2.7}$$

for suitable constants C_2, $c_1 > 0$. Next we note that

$$\gamma_n \geqslant c_3/n, \quad n = 1, 2, \ldots, \tag{2.8}$$

for some constant $c_3 > 0$; this follows from the fact that the sequences $\{R_{[n/4]}\}$, $\{Q_n\}$ are uniformly bounded on $[-1, 1]$ so that the derivative of $(R_{[n/4]}\, Q_{[n/4]-1})^2$ is $O(n)$ on $[-1/2, 1/2]$. Thus, if c_3 is sufficiently small, then

$$(R_{[n/4]}(u)\, Q_{[n/4]-1}(u))^2 > 1/2 \quad \text{for } u \in [-c_3/n, c_3/n],$$

which, in view of (2.6), yields (2.8).

From (2.7) and (2.8) we see that (2.1) is true on $[-1, 0]$. Since $P_n(x)$ is odd, inequality (2.1) also holds on $[0, 1]$.

REMARK. From the method of proof of Theorem 1, it follows more generally that, given finitely many pairs $\{(\alpha_j, \beta_j)\}_1^\nu$ satisfying (2.2), there exist constants $C, c > 0$ and polynomials $P_n \in \Pi_n$ such that for all $x \in [-1, 1]$

$$|\operatorname{sign} x - P_n(x)| \leqslant C \exp\left(-c \sum_{j=1}^{\nu} n^{\alpha_j} |x|^{\beta_j}\right), \quad n = 1, 2, \ldots.$$

It remains to establish Theorem 2.

Proof of Theorem 2. We first establish the necessity of (2.2). That $\beta \geqslant \alpha$ must hold follows from the fact that a polynomial Q_n having property (2.3) must satisfy $Q_n(c_1/n) > 1/2$, for $c_1 > 0$ sufficiently small (see the reasoning above). Furthermore, if we had $\alpha > 1$, then on $[1/2, 1]$ the polynomial Q_n would satisfy

$$|Q_n(x)| < \exp(-An), \tag{2.9}$$

for every $A > 0$ and large n ($\geqslant n_A$). But, by the well-known inequality of S. N. Bernstein (cf. [**10**, 2.13.27]), (2.9) implies that $Q_n(0) \to 0$ as $n \to \infty$, which contradicts (2.3). Hence $\alpha \leqslant 1$. Finally, if $\alpha = 1$ then $\beta = 1$ is impossible because of [**8**, Theorem 1].

We now establish the sufficiency of (2.2). For $\alpha = \beta < 1$, the construction was given in [**8**] and, of course, the polynomials obtained for $\alpha = \beta$ are suitable for any $\beta > \alpha$ as well. So let $\alpha = 1$ and $\beta > 1$. The substitution $y = n^{1/\beta} x$ shows that it is enough to construct polynomials $Q_n^* \in \Pi_n$ such that, for some $C, c > 0$,

$$1/C \leqslant Q_n^*(x) \exp(|x|^\beta) \leqslant C \quad \text{for } x \in [-cn^{1/\beta}, cn^{1/\beta}].$$

This construction is well known in the theory of orthogonal polynomials associated with Freud (exponential) weights (cf. [**4, 7**]). Hence our proof is complete.

3. *Approximation of piecewise analytic functions*

Our goal is to construct a sequence of polynomial approximants having the properties (i) to (iii) described in the introduction. We shall prove the following.

THEOREM 3. *Let the non-negative numbers α, β satisfy $\alpha < 1$ and $\beta \geqslant \alpha$ or $\alpha = 1$ and $\beta > 1$, and suppose that $f \in C^k[-1, 1]$ is piecewise analytic on $[-1, 1]$. Then there exist constants c, $C > 0$ and polynomials $p_n \in \Pi_n$, $n = 1, 2, \ldots$, such that for every $x \in [-1, 1]$*

$$|f(x) - p_n(x)| \leqslant \frac{C}{n^{k+1}} \exp(-cn^\alpha d(x)^\beta), \tag{3.1}$$

where $d(x)$ denotes the distance from x to the nearest singularity of f in $[-1, 1]$.

To be more precise,

$$d(x) := \min_{1 \leqslant i \leqslant m} |x - x_i|,$$

where $\{x_i\}_1^m$ are the points of singularity of f defined in the introduction. Of course, if f has no singularities on $[-1, 1]$, then f is analytic on $[-1, 1]$ and the rate of best polynomial approximation to f on $[-1, 1]$ is geometric. Thus the case of interest is when f has singular points on $(-1, 1)$; that is, $m \geqslant 1$.

Proof. Let $-1 < x_1 < \ldots < x_m < 1$ be the points of singularity of f and set $x_0 := -1$, $x_{m+1} := 1$. By subtracting a suitable polynomial from f, we can assume, without loss of generality, that

$$f^{(j)}(x_i) = 0 \quad \text{for } 0 \leqslant i \leqslant m+1, 0 \leqslant j \leqslant k. \tag{3.2}$$

Set

$$f_i(x) := \begin{cases} f(x), & \text{if } x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Then each f_i is piecewise analytic on $[-1, 1]$ and belongs to the class $C^k[-1, 1]$. Furthermore,

$$f = \sum_{i=0}^{m} f_i,$$

and so it is enough to verify the theorem for every f_i.

Thus we may assume that f has at most two points of singularity. We consider the case when f has singularities at the points a, b, with $-1 < a < b < 1$, and f vanishes to the left of a and to the right of b. (The case when f has only one singularity is similar.) By assumption, there is an $\varepsilon > 0$ such that on $[a, b]$ the function f agrees with a function f^* defined and analytic on $(a-\varepsilon, b+\varepsilon)$. Since

$$g(x) := f^*(x)/[(x-a)(x-b)]^{k+1} \tag{3.4}$$

is analytic on $(a-\varepsilon, b+\varepsilon)$, there are polynomials $r_\nu \in \Pi_\nu$, $\nu = 1, 2, \ldots$, such that for some c_1, $C_1 > 0$

$$|g(x) - r_\nu(x)| \leqslant C_1 \exp(-c_1 \nu) \quad \text{for } x \in [a - \varepsilon/2, b + \varepsilon/2]. \tag{3.5}$$

These r_ν are uniformly bounded on $[a-\varepsilon/2, b+\varepsilon/2]$ and so, by Bernstein's inequality (cf. [**10**, 2.13.27]), we have

$$|r_\nu(x)| \leqslant C_2 \exp(c_2 \nu) \quad \text{for } x \in [-1, 1], \tag{3.6}$$

where c_2, C_2 are suitable positive constants.

From the remark following the proof of Theorem 1 we know that there are constants, c_3, $C_3 > 0$ and polynomials $q_l \in \Pi_l$, $l = 1, 2, \ldots$, such that on $[-1, 1]$

$$|\tfrac{1}{2}[1 + \operatorname{sign}(x-a)] - q_l(x)| \leqslant C_3 \exp(-c_3 h_l(x)), \tag{3.7}$$

where

$$h_l(x) := l^\alpha |x - \alpha|^\beta + l|x-a|^2 + \sqrt{(l|x-a|)}. \tag{3.8}$$

Notice that $[1 + \operatorname{sign}(x-a)]/2$ is, except at $x = a$, the same as the characteristic function $\chi_{[a,\infty)}(x)$ of the interval $[a, \infty)$. Integrating (3.7) we get

$$\int_{-1}^{1} |\chi_{[a,1]}(u) - q_l(u)|\, du \leqslant C_3 \int_{-1}^{1} \exp(-c_3 \sqrt{(l|u-a|)})\, du \leqslant C_4/l;$$

hence defining

$$q_{l,1}(x) := \int_{-1}^{x} q_l(u)\,du + \left(\int_{-1}^{1} (\chi_{[a,1]}(u) - q_l(u))\,du\right) q_l(x),$$

we have, for $-1 \leqslant x < a$,

$$|q_{l,1}(x)| \leqslant C_3 \exp(-\tfrac{1}{2}c_3 h_l(x)) \int_{-1}^{x} \exp(-\tfrac{1}{2}c_3 \sqrt{(l|u-a|)})\,du + \frac{1}{l} C_4 |q_l(x)|$$

$$\leqslant \frac{1}{l} C_5 \exp(-c_4 h_l(x)). \tag{3.9}$$

Furthermore, for $a < x \leqslant 1$,

$$\begin{aligned}
|(x-a) - q_{l,1}(x)| &= \left| \int_{-1}^{x} \chi_{[a,1]}(u)\,du - q_{l,1}(x) \right| \\
&= \left| \int_{-1}^{x} (\chi_{[a,1]}(u) - q_l(u))\,du - \int_{-1}^{1} (\chi_{[a,1]}(u) - q_l(u))\,du \right. \\
&\quad \left. + \left(\int_{-1}^{1} (\chi_{[a,1]}(u) - q_l(u))\,du\right)(\chi_{[a,1]}(x) - q_l(x)) \right| \\
&\leqslant \int_{x}^{1} |\chi_{[a,1]}(u) - q_l(u)|\,du + \left(\int_{-1}^{1} |\chi_{[a,1]}(u) - q_l(u)|\,du\right) |\chi_{[a,1]}(x) - q_l(x)| \\
&\leqslant C_3 \exp(-\tfrac{1}{2}c_3 h_l(x)) \int_{x}^{1} \exp(-\tfrac{1}{2}c_3 \sqrt{(l|u-a|)})\,du + \frac{1}{l} C_6 \exp(-c_3 h_l(x)) \\
&\leqslant \frac{1}{l} C_7 \exp(-c_5 h_l(x)).
\end{aligned} \tag{3.10}$$

Thus, from (3.9) and (3.10), we get for $x \in [-1, 1]$ and suitable constants C_8, $c_8 > 0$

$$|\chi_{[a,1]}(x)(x-a) - q_{l,1}(x)| \leqslant \frac{1}{l} C_8 \exp(-c_8 h_l(x)). \tag{3.11}$$

Notice that (3.11) implies that

$$\int_{-1}^{1} |\chi_{[a,1]}(u)(u-a) - q_{l,1}(u)|\,du \leqslant C_9/l^2,$$

and so by reasoning as above it is easy to show that

$$q_{l,2}(x) := \int_{-1}^{x} q_{l,1}(u)\,du + \left(\int_{-1}^{1} (\chi_{[a,1]}(u)(u-a) - q_{l,1}(u))\,du\right) q_l(x)$$

satisfies

$$\begin{aligned}
\left| \chi_{[a,1]}(x) \frac{(x-a)^2}{2} - q_{l,2}(x) \right| &= \left| \int_{-1}^{x} \chi_{[a,1]}(u)(u-a)\,du - q_{l,2}(x) \right| \\
&\leqslant \frac{1}{l^2} C_9 \exp(-c_9 h_l(x)).
\end{aligned}$$

Iterating this process $(k+1)$ times we get polynomials $R_{l+k+1} \in \Pi_{l+k+1}$ satisfying for all $x \in [-1, 1]$

$$|\chi_{[a,1]}(x)(x-a)^{k+1} - R_{l+k+1}(x)| \leqslant \frac{1}{l^{k+1}} C_{10} \exp(-c_{10} h_l(x)). \tag{3.12}$$

Similarly we can construct polynomials $\hat{R}_{l+k+1} \in \Pi_{l+k+1}$ for which

$$|\chi_{[-1,b]}(x-b)^{k+1} - \hat{R}_{l+k+1}(x)| \leqslant \frac{1}{l^{k+1}} C_{11} \exp(-c_{11} H_l(x)), \tag{3.13}$$

where

$$H_l(x) := l^\alpha |x-b|^\beta + l|x-b|^2 + \sqrt{(l|x-b|)}.$$

Thus, if $d(x) := \min(|x-a|, |x-b|)$, then

$$|(x-a)^{k+1}(x-b)^{k+1}\chi_{[a,b]}(x) - R_{l+k+1}(x)\hat{R}_{l+k+1}(x)| \leqslant \frac{1}{l^{k+1}} C_{12} \exp(-c_{12}(l^\alpha d(x)^\beta + ld(x)^2)). \tag{3.14}$$

Now let $l = [n/4]$ in (3.14) and $\nu = [cn]$ in (3.5) and (3.6), where $0 < c < 1/4$ is sufficiently small so that

$$c_2 \nu < c_{12} l \left(\frac{\varepsilon}{2}\right)^2; \quad \text{that is, } [cn] < \frac{\varepsilon^2 c_{12}}{4c_2}[n/4].$$

With this choice we get from (3.5), (3.6) and (3.14) that on $[-1,1]$

$$p_n(x) := r_\nu(x) R_{l+k+1}(x) \hat{R}_{l+k+1}(x) \in \Pi_n$$

satisfies

$$|f^*(x)\chi_{[a,b]}(x) - p_n(x)| \leqslant \frac{1}{n^{k+1}} C_{13} \exp(-c_{13} n^\alpha d(x)^\beta).$$

Since $f(x) \equiv f^*(x)\chi_{[a,b]}(x)$, the proof is complete.

We have actually proved the following more general result.

THEOREM 4. *Let α, β satisfy condition (2.2) and suppose that $f \in C[-1,1]$ is a piecewise analytic function on $[-1,1]$ with points of singularity $\{x_i\}_{i=1}^m \subset (-1,1)$. If k_i denotes the number of times f is differentiable at x_i, $1 \leqslant i \leqslant m$, then there are constants c, $C > 0$ and polynomials $p_n \in \Pi_n$, $n = 1, 2, \ldots$, such that for all $x \in [-1,1]$*

$$|f(x) - p_n(x)| \leqslant \frac{1}{n^{k_i+1}} C \exp(-cn^\alpha |x-x_i|^\beta), \tag{3.15}$$

where x_i is a closest point of singularity to x.

From the proof of Theorem 3 it is also clear that if α_j, β_j, $1 \leqslant j \leqslant s$, are any s pairs of numbers satisfying condition (2.2), then on the right-hand side of (3.15) we can have, more generally,

$$\frac{1}{n^{k_i+1}} C \exp\left(-c \sum_{j=1}^{s} n^{\alpha_j} |x-x_i|^{\beta_j}\right).$$

Next we show that condition (2.2) imposed in the theorems of this section is necessary.

THEOREM 5. *Let $\alpha, \beta > 0$ and k be a non-negative integer. Set*

$$f_k(x) := \begin{cases} x^{k+1}, & \text{if } 1 \geqslant x \geqslant 0, \\ 0, & \text{if } -1 \leqslant x \leqslant 0, \end{cases}$$

and assume that there are constants C, $c > 0$ and a sequence of polynomials $\{p_n\}_1^\infty$, $p_n \in \Pi_n$, $n = 1, 2, \ldots$, such that for all $x \in [-1, 1]$

$$|f_k(x) - p_n(x)| \leqslant \frac{C}{n^{k+1}} \exp(-cn^\alpha |x|^\beta). \tag{3.16}$$

Then, either $\alpha < 1$ and $\beta \geqslant \alpha$ or $\alpha = 1$ and $\beta > 1$.

Proof. That we cannot have $\alpha > 1$ follows exactly as in the proof of Theorem 2. Now suppose that $\beta < \alpha$. Then on any interval $[c_1/n, 1]$, with $c_1 > 0$, we have

$$|x^{k+1} - p_n(x)| \leqslant \frac{C}{n^{k+1}} \exp(-cc_1^\beta n^{\alpha-\beta}),$$

and so, by Markoff's inequality,

$$|(k+1)! - p_n^{(k+1)}(x)| \leqslant C_1 n^{k+1} \exp(-cc_1^\beta n^{\alpha-\beta}), \quad x \in [c_1/n, 1]. \tag{3.17}$$

Similarly,

$$|p_n^{(k+1)}(x)| \leqslant C_1 n^{k+1} \exp(-cc_1^\beta n^{\alpha-\beta}), \quad x \in [-1, -c_1/n]. \tag{3.18}$$

Hence, for any fixed $c_1 > 0$, if n is large enough, then

$$p_n^{(k+1)}\left(\frac{c_1}{n}\right) > \frac{2}{3} \quad \text{and} \quad p_n^{(k+1)}\left(\frac{-c_1}{n}\right) < \frac{1}{3}.$$

Thus, for some $y \in (-c_1/n, c_1/n)$, we have

$$p_n^{(k+2)}(y) > \frac{n}{6c_1}. \tag{3.19}$$

From (3.17) and (3.18) we also see that, for n large,

$$|p_n^{(k+1)}(x)| \leqslant 2(k+1)!, \quad |x| \in [c_1/n, 1]. \tag{3.20}$$

Next we use the fact that, for any polynomial $Q_n \in \Pi_n$,

$$\|Q_n\|_{[-1,1]} \leqslant 2 \max\{|Q_n(u)|; 1/(4n) \leqslant |u| \leqslant 1\}, \tag{3.21}$$

which is an easy consequence of the Bernstein inequality for the derivative of Q_n (see [**10**, 4.8.7]). Thus, for $c_1 < 1/4$, inequality (3.20) implies that $\|p_n^{(k+1)}\|_{[-1,1]} \leqslant 4(k+1)!$ and so, by Bernstein's inequality

$$|p_n^{(k+2)}(y)| \leqslant 8(k+1)!n, \quad y \in [-1/2, 1/2]. \tag{3.22}$$

However, for c_1 sufficiently small, (3.22) and (3.19) contradict each other. Thus we must have $\beta \geqslant \alpha$.

All that remains to show is that $\alpha = \beta = 1$ is impossible. For simplicity, we carry out the proof only for $k = 0$; the case in which $k > 0$ can be similarly handled. By subtracting $x/2$ from both $f_0(x)$ and $p_n(x)$ in (3.16), we have to show that it is impossible to have

$$||x| - P_n(x)| \leqslant \frac{C}{n} \exp(-cn|x|) \tag{3.23}$$

uniformly in $n = 1, 2, \ldots$ and $x \in [-1, 1]$, where $P_n \in \Pi_n$ and C, $c > 0$.

Suppose to the contrary that (3.23) is possible. Then

$$|P_n(0)| \leqslant C/n,$$

and so there exist a subsequence $\mathcal{N}$ of the natural numbers and a constant A such that

$$nP_n(0) \to A \quad \text{as} \quad n \to \infty, \quad n \in \mathcal{N}. \tag{3.24}$$

It follows from (3.23) that for $q_n(x) := nP_n(x/n)$ we have on $[-n, n]$

$$||x| - q_n(x)| \leqslant C \exp(-c|x|), \tag{3.25}$$

and hence, for $1/4 \leqslant |x| \leqslant n$,

$$\left| \frac{q_n(x) - q_n(0)}{x} \right| \leqslant 8C + 1.$$

Thus we can apply the general inequality of (3.21) to conclude that

$$\left| \frac{q_n(x) - q_n(0)}{x} \right| \leqslant 16C + 2, \quad x \in [-n, n]. \tag{3.26}$$

Next, we apply the technique of [**8**, Theorem 1]. Writing

$$\frac{q_n(x) - q_n(0)}{x} = \sum_{j=0}^{n-1} c_{nj} x^j,$$

it follows from (3.26) and an inequality of S. N. Bernstein (see, for example, [**1**, Appendix 42, p. 323]) that

$$|c_{nj}| \leqslant (16C + 2)/(2[j/2])!.$$

The set $\mathscr{U}$ of entire functions

$$g(z) = \sum_{j=0}^{\infty} c_j z^j$$

satisfying

$$|c_j| \leqslant (16C + 2)/(2[j/2])!, \quad j = 0, 1, \ldots,$$

is compact in the topology of uniform convergence on compact subsets of $\mathbb{C}$; hence there is a subsequence $\mathcal{N}' \subseteq \mathcal{N}$ and a $g \in \mathscr{U}$ such that $(q_n(x) - q_n(0))/x$ converges to g in that topology as $n \to \infty, n \in \mathcal{N}'$. From (3.24) and (3.25) we get that

$$\left| \operatorname{sign} x - g(x) - \frac{A}{x} \right| \leqslant C \exp(-c|x|), \quad |x| \geqslant 1, \tag{3.27}$$

and, from (3.26) we see that

$$|g(x)| \leqslant 16C + 2, \quad -\infty < x < \infty.$$

Thus, g is an entire function of exponential type 1 which is bounded on the real line; hence [**1**, Chapter 4]

$$|g(z)| \leqslant (16C + 2) \exp(|\operatorname{Im} z|), \quad z \in \mathbb{C}.$$

This and (3.27) imply that

$$|x^2 - (xg(x) + A)^2| \leqslant C(C + 2)\, x^2 \exp(-c|x|), \quad |x| \geqslant 1 \tag{3.28}$$

and, for a suitable constant C_1,

$$|z^2 - (zg(z) + A)^2| \leqslant C_1(|z| + 1)^2 \exp(2|\operatorname{Im} z|), \quad z \in \mathbb{C}. \tag{3.29}$$

If we set

$$g^*(z) := z^2 - (zg(z) + A)^2$$

and

$$h(w) := g^*\left(i\frac{1+w}{1-w}\right),$$

then exactly as in [8, Theorem 1], (3.29) implies that h is in the Nevanlinna class

$$\sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |h(re^{it})|\, dt < \infty,$$

and so, since $g^* \not\equiv 0$ (cf. (3.27)),

$$\int_{-\pi}^{\pi} \log |h(e^{it})|\, dt > -\infty.$$

But the boundary value at e^{it} of h is $g^*(-\cot(t/2))$ and so the preceding inequality is equivalent to

$$\int_{-\infty}^{\infty} \frac{\log |g^*(x)|}{1+x^2}\, dx > -\infty,$$

which contradicts (3.28). This contradiction shows that (3.23) is indeed impossible.

4. *A problem in complex approximation*

In connection with the distribution of the zeros of certain 'near best' approximating polynomials, R. Grothmann and E. B. Saff [5] stated the following conjecture: if

$$E := \{z \in \mathbb{C} : |\mathrm{Re}\, z| \leqslant 1, |\mathrm{Im}\, z| \leqslant |\mathrm{Re}\, z|^2\}$$

(that is, E is a parabolic region with $z = 0$ a double point of ∂E), and

$$g(z) := \begin{cases} z & \text{for } \mathrm{Re}\, z \geqslant 0, \\ -z & \text{for } \mathrm{Re}\, z < 0, \end{cases}$$

then there are polynomials $p_n \in \Pi_n$, $n = 1, 2, \ldots$, such that uniformly on E

$$|g(z) - p_n(z)| \leqslant \frac{C}{n}. \tag{4.1}$$

Notice that for $x \in [-1, 1]$, $g(x) = |x|$; hence the rate of approximation in (4.1) cannot be better than $O(1/n)$.

Using our constructions we can verify the above conjecture as follows.†

Let $1 < \beta < 3/2$. By Theorem 2 there are constants C_1, $c_1 > 0$ and polynomials $R_n \in \Pi_n$ such that

$$R_n(0) = 1$$

and

$$|R_n(u)| \leqslant C_1 \exp(-c_1(n|u|^\beta + \surd(2n|u|))), \quad u \in [-1, 1].$$

Let $z = x + iy$, $|y| \leqslant x^2$, $x > 0$. By the preceding estimate,

$$|R_n(u)| \leqslant C_1 \exp(-c_1(n(x/2)^\beta + \surd(nx)))$$

† We remark that this was the original form of the conjecture (published as an ICM report). In print it appears with $|\mathrm{Re}\, z| \leqslant 2$ in the definition of E and to prove this one has to adjust the reasoning of this section. We also mention that V. V. Andrievskii claims that the conjecture follows from the general results of [2].

for every $u\in[x/2,1]$. Hence by the well-known inequality of S. N. Bernstein ([**10**, 2.13.27])

$$|R_n(z)| \leqslant C_1 \exp(-c_1(n(x/2)^\beta+\sqrt{(}(nx)))\,\gamma^n,$$

where $\gamma\geqslant 1$ is defined by

$$\gamma+\gamma^{-1}=\frac{2}{1-x/2}(|z-x/2|+|z-1|).$$

Elementary computation shows that if x is sufficiently small, say $0<x\leqslant x_0$, then

$$\gamma\leqslant 1+16|y|/\sqrt{x}.$$

Hence

$$|R_n(z)|\leqslant C_1\exp(-c_1 n(x/2)^\beta+16nx^{3/2}-c_1\sqrt{(nx)})\leqslant C_1\exp(-c_1\sqrt{(nx)}), \quad (4.2)$$

provided $0<x\leqslant x_0$, $|y|\leqslant x^2$ and x_0 is sufficiently small, but fixed. Since we can assume that R_n is an even polynomial, we have the same estimate on

$$E(x_0):=\{z\,|\,|\mathrm{Re}\,z|\leqslant x_0, |\mathrm{Im}\,z|\leqslant|\mathrm{Re}\,z|^2\}.$$

Using again Bernstein's inequality we get for some C_2 and every $z\in E\backslash E(x_0)$

$$|R_n(z)|\leqslant C_2\exp(C_2 n). \quad (4.3)$$

Let $N=C_3 n$, where C_3 will be chosen later, and set

$$R_n^*(z):=\sum_{k=0}^{50N}\frac{(Nz^2(z^2-2))^k}{k!}. \quad (4.4)$$

By examining the remainder term for the Maclaurin expansion of $\exp(Nz^2(z^2-2))$, it is easy to see that, for $z\in E$,

$$|\exp(Nz^2(z^2-2))-R_n^*(z)|<e^{-N}.$$

The real part of $z^2(z^2-2)$ for $z=x+iy$ is

$$(x^2-y^2)^2-2(x^2-y^2)-4x^2y^2,$$

which is easily seen to be at most $-x^2/4$ for every $z\in E$. Hence the above R_n^* has the properties $R_n^*\in\Pi_{200N}$, $R_n^*(0)=1$ and

$$|R_n^*(x+iy)|<\exp\left(-\frac{N}{4}x^2\right)$$

for any $x+iy\in E$. Thus

$$Q_n(z):=R_n(z)\,R_n^*(z)$$

belongs to $\Pi_{200N+n}=\Pi_{(200C_3+1)n}$ and satisfies the conditions

$$Q_n(0)=1$$

and

$$|Q_n(z)|\leqslant C_1\exp(-c_1\sqrt{(}n|\mathrm{Re}\,z|)),\quad z\in E, n=1,2,\ldots,$$

if we choose C_3 so large that on $E\backslash E(x_0)$

$$C_2\exp\left(C_2 n-\frac{C_3 n}{4}x^2\right)<C_1\exp(-c_1\sqrt{(}n|x|))$$

is satisfied (cf. (4.2), (4.3) and (4.5)). Setting

$$P_n^*(z) := \frac{2}{\gamma_n} \int_{-1}^{z} Q_n^2(u)\, du - 1,$$

with

$$\gamma_n := \int_{-1}^{1} Q_n^2(u)\, du,$$

we get exactly as in the proof of Theorem 1 that, for $z \in E$, $\operatorname{Re} z < 0$,

$$|-1 - P_n^*(z)| \leqslant C_4 \exp(-c_1 \surd(n|\operatorname{Re} z|))$$

and for $z \in E$, $\operatorname{Re} z > 0$,

$$|1 - P_n^*(z)| \leqslant C_4 \exp(-c_1 \surd(n|\operatorname{Re} z|))$$

with some constant C_4. Multiplying these estimates by $|z|$ and using that, for $z \in E$,

$$|z| \exp(-c_1 \surd(n|\operatorname{Re} z|)) \leqslant \frac{2}{n} n|\operatorname{Re} z| \exp(-c_1 \surd(n|\operatorname{Re} z|)) \leqslant \frac{C_5}{2(200C_3+1)n+2},$$

we get (4.1) with $p_m(z) = zP_n^*(z)$ for every m of the form $m = 2(200C_3+1)n+2$ and hence for every m.

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