A PRINCIPLE OF CONTAMINATION IN BEST POLYNOMIAL APPROXIMATION

by

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Abstract. We discuss the qualitative behavior of the polynomials $p_n^*(z)$ of best uniform approximation to a function f that is continuous on a compact set E of the z-plane, analytic in the interior of E, but not analytic at some point of the boundary of E. Particularly, we survey results on the asymptotic behavior of the zeros of the $p_n^*(z)$ and the extreme points for the error $f(z) - p_n^*(z)$. The theorems and examples presented support a "principle of contamination," which roughly states that the existence of one or more singularities of f on the boundary of E adversely affects the behavior over the *whole* boundary of E of a subsequence of the best approximants $p_n^*(z)$.

§1. Introduction.

Let E be a compact set in the complex plane $\mathbb{C}$ whose complement $\overline{\mathbb{C}} \setminus E$ with respect to the extended plane $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is connected. By the classical theorem of Mergelyan (cf. [18]), every function f that is continuous on E and analytic in the (2-dimensional) interior $\mathring{E}$ of E can be uniformly approximated on E, as closely as desired, by algebraic polynomials. To measure the rate of this approximation we consider the sequence of polynomials of best approximation.

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Let $\mathcal{P}_n$ denote the collection of all algebraic polynomials of degree $\leq n$, and $\|\cdot\|_E$ denote the sup norm over E. Then for each $n = 0,1,2,\ldots$, there exists a $p_n^* = p_n^*(f) \in \mathcal{P}_n$ satisfying

$$(1.1) \qquad \|f - p_n^*\|_E = \inf_{p\in\mathcal{P}_n} \|f - p\|_E ,$$

and p_n^* is uniquely determined by (1.1) provided $\text{card}(E) \geq n + 1$. We set

$$(1.2) \qquad e_n = e_n(f,E) := \|f - p_n^*\|_E ,$$

and note that, with the previously mentioned assumptions on E and f, Mergelyan's theorem asserts that

$$(1.3) \qquad e_n \downarrow 0.$$

As is well-known, the rate of decay in (1.3) is intimately related to the smoothness properties of f. For example, if $A(E)$ denotes the collection of all functions f analytic on the compact set E, then under mild geometric assumptions on E the theorem of Bernstein-Walsh (cf. [18]) asserts that

$$(1.4) \qquad f \in A(E) \Leftrightarrow \limsup_{n\to\infty} [e_n]^{1/n} < 1.$$

That is, the error in best polynomial approximation decays exponentially (geometrically) if and only if f has an analytic continuation to some open set containing E. Furthermore, there exist, in this complex setting, more refined theorems of the Bernstein-Jackson type that relate the modulus of continuity of f over E to the rate of decay of e_n (cf. [1], [2]). The following two examples of this slower than geometric convergence are probably familiar to the reader:

If $f_1(x) = |x|$ on $E := [-1,1]$, then

$$(1.5) \qquad \frac{c_2}{n} \leq e_n(f_1) \leq \frac{c_1}{n} , \qquad n = 1,2,\ldots .$$

Here and below c_1, c_2 denote positive constants independent of n.

If $f_2(z) = \sqrt{z}$ on the disk $E := \{z : |z - 1| \le 1\}$, then

$$\frac{c_2}{\sqrt{n}} \le e_n(f_2) \le \frac{c_1}{\sqrt{n}}, \qquad n = 1, 2, \ldots . \tag{1.6}$$

It is the purpose of this paper to study the *qualitative* behavior of the polynomials of best approximation to f in the case when f has one or more singularities on the boundary of E (such as the functions f_1, f_2 defined above). We shall present theorems that support the following general property:

<u>*Principle of Contamination*</u>. *Let f be continuous on E and analytic in $\mathring{E}$, where E is a compact set with connected and regular complement. If f has one or more singularities on the boundary of E (i.e., $f \notin A(E)$), then these singularities adversely affect the behavior over the whole boundary of E of a subsequence of the best polynomial approximants p_n^* to f on E.*

By the assumption that $\overline{\mathbb{C}} \setminus E$ is *regular* we mean that this set possesses a classical Green's function with pole at infinity. In particular, regularity holds if $\overline{\mathbb{C}} \setminus E$ is simply connected; that is, if E is a continuum (not a single point).

Of course the principle of contamination is not a mathematical theorem. Rather it is a rough summary of rigorous theorems to be discussed below. It is hoped that the statement of this principle will lead to further supporting theorems as well as to comparisons with other methods of approximation for which "the contamination" is nonexistent or less severe.

The outline of this paper is as follows. In Section 2 we discuss theorems and examples concerning the asymptotic behavior of the zeros of the best approximants $p_n^*(f)$. Such results are intimately related to the possibility of using these approximants to obtain analytic continuations of f. In Section 3 we consider the behavior of the extreme points for the error $f - p_n^*(f)$. The latter results are significant for purposes of comparing e_n with the rate of convergence on a subset of the boundary of E.

Before embarking on the theorems that support the principle of contamination, we wish to emphasize three important limitations of the principle.

(i) The principle refers only to best *polynomial* approximation. As we shall see below, best *rational* approximants may not exhibit the same ill effects.

(ii) The principle applies specifically to *best* polynomial approximants. For example, a sequence of "near-best" polynomials $q_n \in \mathcal{P}_n$, $n = 0,1,\ldots,$ satisfying

$$\|f - q_n\|_E \leq 2\,\|f - p_n^*(f)\|_E\,, \qquad n = 0,1,2,\ldots,$$

may behave qualitatively better than the $p_n^*(f)$ at those points of the boundary of E where f is analytic.

(iii) The principle refers only to some *subsequences* of the polynomials $\{p_n^*\}_1^\infty$. It is possible that there are other subsequences with less contaminated behavior.

Although we shall deal mainly with best *uniform* approximants, there do exist theorems that support the principle of contamination for sequences of best L^p polynomial approximants to f.

§2. Zeros of Best Polynomial Approximants

We assume here and throughout that E is a compact subset of $\mathbb{C}$ with connected and regular complement. For $f \in C(E) \cap A(\mathring{E})$; that is, f is continuous on E and analytic in the (possibly empty) interior $\mathring{E}$ of E, we ask the following question. What can be said about the locations (in the complex plane) of the zeros of the polynomials $p_n^*(f)$ of best uniform approximation to f on E? This question was studied by J. L. Walsh [19], [20], for the case when f is analytic *on* E $(f \in A(E))$ but not entire. Walsh's results are analogues of the classical theorem of Jentzsch [11] which states that the partial sums s_n of a power series (about $z = 0$) with finite radius of convergence $r > 0$ have the property that every point on the circle $|z| = r$ is a limit point of the set of zeros of the polynomials s_n, $n = 0,1,2,\ldots$.

More recently Blatt and Saff [5] investigated the zeros of $p_n^*(f)$ for the more delicate case when f has one or more singularities on the boundary ∂E of E. They proved the following.

THEOREM 2.1 ([5]). *Suppose* $f \in C(E) \cap A(\mathring{E})$, *but* f *is not analytic on* E. *Assume further that* f *does not vanish identically on any component of the interior* $\mathring{E}$. *Then every boundary point of* E *is a limit point of the set of zeros of the sequence of best uniform approximants* $\{p_n^*(f)\}_1^\infty$ *to* f *on* E.

In particular, if f is continuous on the interval $I := [a,b]$ of the real axis and if f is not the restriction to I of a function analytic in a neighborhood of I, then every point of I attracts zeros of the sequence of best approximants $\{p_n^*(f)\}_1^\infty$ to f on I. In Figure 2.1 we illustrate this fact for $f_1(x) = |x|$ on $[-1,1]$ by plotting the zeros of $p_{26}^*(f_1)$.

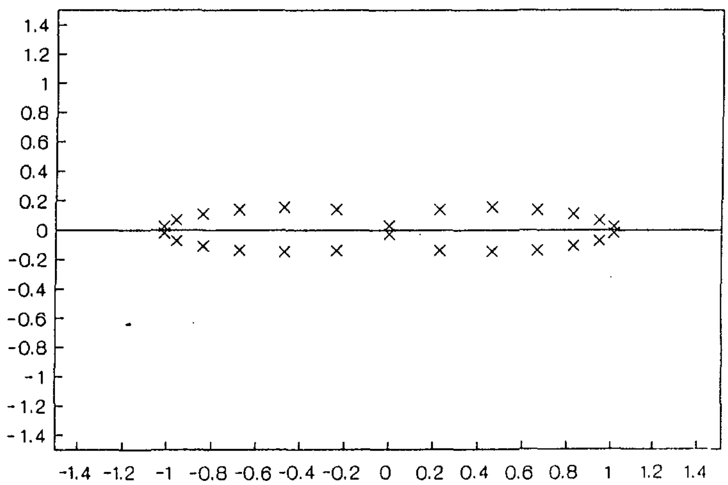

Figure 2.1 Zeros of $p_{26}^*(f_1)$, where $f_1(x) = |x|$ on $E = [-1,1]$

Notice that in Theorem 2.1 we assume $f \notin A(E)$ and so, from (1.4), the errors $e_n = e_n(f,E)$ of (1.2) satisfy

$$\limsup_{n\to\infty} [e_n]^{1/n} = 1. \tag{2.1}$$

Thus there is a subsequence $\Lambda = \Lambda(f,E)$ of integers for which

$$(2.2) \qquad \lim_{n\to\infty} [e_{n-1} - e_n]^{1/n} = 1, \quad n \in \Lambda.$$

In the proof of Theorem 2.1 it is shown, more generally, that, for any subsequence Λ satisfying (2.2), the Jentzsch-type behavior holds for the zeros of $\{p_n^*(f)\}_{n\in\Lambda}$. We further remark that Theorem 2.1 applies not only to the zeros of the $p_n^*(f)$ but also to their α-values; that is, to the roots of $p_n^*(f,z) = \alpha$, where α is any complex constant. Indeed, the function $f_\alpha(z) := f(z) - \alpha$ satisfies $f_\alpha \in C(E) \cap A(\mathring{E})$, $f_\alpha \notin A(E)$, and the polynomials of best uniform approximaton to f_α are just $p_n^*(f) - \alpha$. Thus we have

COROLLARY 2.2. *Suppose* $f \in C(E) \cap A(\mathring{E})$, $f \notin A(E)$, *and* f *is not identically constant on any component of* $\mathring{E}$. *Let* z_0 *be any boundary point of* E, $U(z_0)$ *a neighborhood (open disk) about* z_0, *and* Λ *a subsequence of integers for which* (2.2) *holds. Then, for every constant* α *and every sufficiently large* $n \in \Lambda$, *the equation* $p_n^*(f,z) = \alpha$ *has a root in* $U(z_0)$.

In Figures 2.2 and 2.3 we illustrate this corollary for the case of $f_1(x) = |x|$ on $E = [-1,1]$ by plotting the roots of $p_{26}^*(f_1,z) = -5$ and $p_{26}^*(f_1,z) = -i$, respectively. The reader should note the strong resemblance in Figures 2.1, 2.2, and 2.3.

Recalling the classical theorem of Picard concerning the behavior of an analytic function in a neighborhood of an isolated essential singularity, we can summarize Corollary 2.2 by saying that the sequence $\{p_n^*(f)\}_{n\in\Lambda}$ has an "asymptotic essential singularity" at each point of ∂E. In the context of *normal families* of analytic functions, we see that no point of ∂E is a normal point for the sequence $\{p_n^*(f)\}_1^\infty$. As shown in [5], this fact holds even when f is constant on some component of $\mathring{E}$; that is, we have

THEOREM 2.3. *Suppose* $f \in C(E) \cap A(\mathring{E})$, *but* $f \notin A(E)$. *Then the sequence* $\{p_n^*(f)\}_1^\infty$ *does not converge uniformly in any neighborhood of a boundary point of* E.

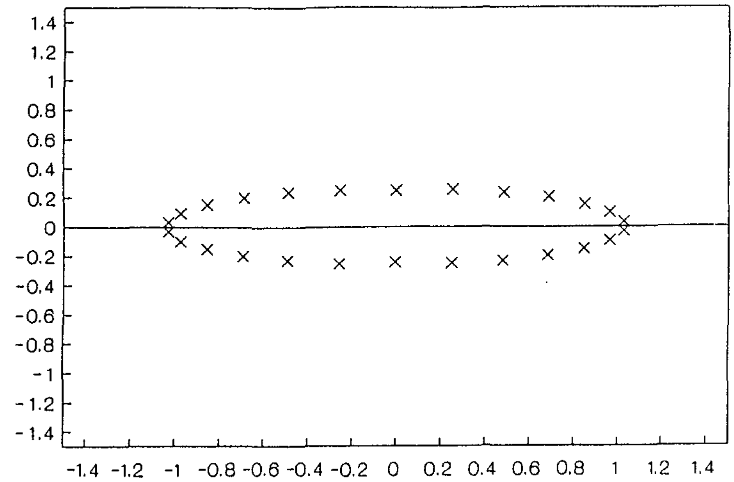

Figure 2.2 Roots of $p^*_{26}(f_1, z) = -5$, where $f_1(x) = |x|$ on $E = [-1, 1]$

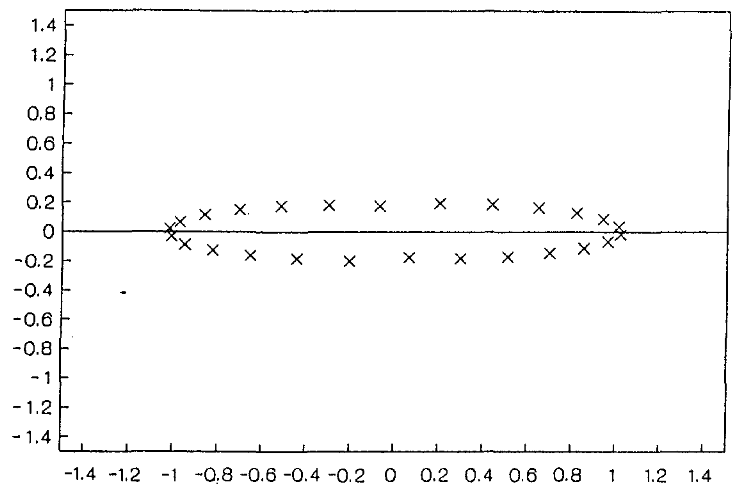

Figure 2.3 Roots of $p^*_{26}(f_1, z) = -i$, where $f_1(x) = |x|$ on $E = [-1, 1]$.

The above results show that polynomials of best uniform approximation have an undesirable property: Consider a function

$f \in C(E) \cap A(\mathring{E})$ that is analytic at some boundary points of E, but is not analytic at all the boundary points (e.g. $f_1(z)$ and $f_2(z)$ in the Introduction). Then, in any neighborhood of an analytic boundary point, the sequence $\{p_n^*(f)\}_1^\infty$ fails to converge to the analytic continuation of f. We remark, however, that it may be possible for some subsequence of $\{p_n^*(f)\}$ to converge in a neighborhood of an analytic boundary point.

It is somewhat surprising that the Jentzsch-type behavior of zeros and α-values described in Theorem 2.1 and Corollary 2.2 need not hold for polynomials of "near-best" approximation; that is, for polynomials $\{q_n\}_1^\infty$, $q_n \in \mathcal{P}_n$, that satisfy for some fixed constant $K > 1$,

$$\|f - q_n\|_E \leq K\|f - p_n^*(f)\|_E = Ke_n, \quad n = 1,2,\dots . \tag{2.3}$$

Examples of this type were constructed by Grothmann and Saff [10] and Saff and Totik [15] where the only boundary points of E that attract zeros of the q_n are the singular points of f. To be more specific we describe the results of [15] dealing with the absolute value function.

Let $f(x) = |x|$ on $E = [-1/2,1/2]$ and let $g(z)$ be the analytic extension of f defined by

$$g(z) := \begin{cases} z, & \text{for } \operatorname{Re} z \geq 0 , \\ -z, & \text{for } \operatorname{Re} z < 0 . \end{cases} \tag{2.4}$$

In [15] a sequence $\{q_n\}_1^\infty$, $q_n \in \mathcal{P}_n$, is constructed such that

$$\|g - q_n\|_{E_1} \leq \frac{C}{n}, \quad n = 1,2,\dots, \tag{2.5}$$

where

$$E_1 := \{z \in \mathbb{C} : |\operatorname{Re} z| \leq 1, \quad |\operatorname{Im} z| \leq |\operatorname{Re} z|^2\}$$

(i.e., E_1 is a parabolic region with $z = 0$ a double point of ∂E_1). Since $E \subset E_1$ and f is the restriction to E of g, we have

$$\|f - q_n\|_E \le \frac{C}{n} \le Ke_n(f,E), \qquad n = 1,2,\ldots,$$

where the last inequality follows easily from (1.5). Also, from (2.5), we see that the q_n converge uniformly to g on the region E_1. Thus, since $g(z) \neq 0$ for $z(\neq 0) \in E_1$, it follows that *only one point* of E (namely, $z = 0$) can be a limit point of zeros of the sequence $\{q_n\}_1^\infty$.

The above example illustrates that "near-best" polynomial approximants may behave qualitatively better than the best polynomial approximants. Indeed, in [15], Saff and Totik have shown that for the class of *piecewise-analytic* functions f on $[-1,1]$ it is possible to construct polynomials q_n satisfying (2.3) that converge uniformly and geometrically in some open disk about each point of $[-1,1]$ where f is analytic (see Theorem 4.1).

In comparison to best polynomial approximants, best *rational* approximants can also have qualitatively much better behavior. To illustrate this statement we return to the approximation of the absolute value function $f(x) = |x|$ on $E = [-1,1]$. In [3], Blatt, Iserles and Saff proved the following.

PROPOSITION 2.4. Let $R_n^* = P_n/Q_n$ *denote the unique real rational function of degree at most* n *of best uniform approximation to* $|x|$ *on* $[-1,1]$. *Then for each* $n = 1,2,\ldots$, *all the zeros and all the poles of* R_n^* *lie on the imaginary axis. Moreover,*

$$\lim_{n\to\infty} R_n^*(z) = \begin{cases} z, & \text{for } \operatorname{Re} z > 0, \\ -z, & \text{for } \operatorname{Re} z < 0. \end{cases}$$

Thus, unlike the behavior of the zeros of the best polynomial approximants to $|x|$ on $[-1,1]$, only one point of $[-1,1]$ attracts the zeros of the sequence of best rational approximants. Furthermore, the best rational approximants converge to the analytic continuation of $|x|$ in the right-hand and left-hand planes.

Thus far we have discussed only Jentzsch-type theorems concerning the limit points of zeros of best polynomial approximants. Much more information is provided by Szegö-type theorems that concern the *limiting distributions* of these zeros. In order to state such results we need to introduce some terminology from potential theory (cf. [17]).

Let $\mathcal{M}(E)$ denote the collection of all unit measures that are supported on the compact set E where, as above, we assume that $\overline{\mathbb{C}} \setminus E$ is connected and regular. Then it is known that there exists a unique measure $\mu_E \in \mathcal{M}(E)$ that minimizes the energy integral

$$I[\mu] := \iint \log |z-t|^{-1} d\mu(t) d\mu(z)$$

over all $\mu \in \mathcal{M}(E)$. The measure μ_E is called the *equilibrium distribution* for E and (since $\overline{\mathbb{C}} \setminus E$ is regular) we have $\operatorname{supp}(\mu_E) = \partial E$. For example, if E is the closed disk $|z| \le r$, then $d\mu_E = ds/2\pi r$, where ds is arclength measured on the circumference $|z| = r$. If $E = [-1,1]$, then $d\mu_E$ is the arcsine distribution, $d\mu_E = (1/\pi)dx/\sqrt{1 - x^2}$.

Next, to each nonconstant best polynomial approximant $p_n^*(f)$ we associate a discrete unit measure ν_n^*, called the *zero distribution* of $p_n^*(f)$, by

$$(2.6) \qquad \nu_n^*(B) := \frac{\text{number of zeros of } p_n^*(f) \text{ in } B}{\deg p_n^*(f)} ,$$

for Borel sets $B \subset \mathbb{C}$, where we count the zeros according to their multiplicity.

We can now state the theorem of Blatt, Saff, and Simkani concerning the limiting distribution of the zeros of best polynomial approximants.

THEOREM 2.5 ([6]). *Suppose* $f \in C(E) \cap A(\mathring{E})$, *but* $f \notin A(E)$. *Assume further that* f *does not vanish identically on any component of* $\mathring{E}$. *Then* ν_n^* *converges in the weak-star topology to* μ_E *as* $n \to \infty$ *through a sequence* $\Lambda = \Lambda(f,E)$ *of positive integers.*

Consequently, for any Borel set $B \subset \mathbb{C}$,

$$(2.7) \qquad \mu_E(\mathring{B}) \le \liminf_{n\to\infty} \upsilon_n^*(B) \le \limsup_{n\to\infty} \upsilon_n^*(B) \le \mu_E(\overline{B}), \quad n \in \Lambda .$$

In the above theorem, Λ is any sequence of integers for which (2.2) holds. We remark that since $\mathrm{supp}(\mu_E) = \partial E$, the assertion of Theorem 2.1 that each point of ∂E is a limit point of the sets of zeros of $\{p_n^*(f)\}_1^\infty$ is an immediate consequence of (2.7).

In Theorem 2.5 the convergence of the zero distributions to the equilibrium distribution of E holds only for suitable *subsequences*. In [10], Grothmann and Saff have shown that for any admissible compact set E, there exists a function f satisfying the hypotheses of Theorem 2.5 and a subsequence $\left\{\upsilon_{n_k}^*\right\}$ of the measures (2.6) such that

$$\lim_{k\to\infty} \upsilon_{n_k}^*(B) = 0 \quad \text{for all bounded sets } B \subset \mathbb{C}.$$

Although the results of this section have been concerned only with best *uniform* polynomial approximants, theorems analogous to Theorem 2.5 have been proved by Simkani [16] and Blatt, Saff and Simkani [6] that apply to best L^P polynomial approximants (as measured by a line integral over a rectifiable Jordan curve) and to best uniform rational approximants having a bounded number of free poles.

§3. Behavior of Extreme Points.

In this section we present further evidence for the principle of contamination by examining the behavior of extreme points in best polynomial and best rational approximation. We first recall the classical result that, for a real-valued f continuous on the interval $[-1,1]$, there exist $n + 2$ points

$$-1 \le x_1^{(n)} < x_2^{(n)} < \dots < x_{n+2}^{(n)} \le 1$$

such that

$$(3.1) \qquad |(f - p_n^*)(x_k^{(n)})| = \|f - p_n^*\|_{[-1,1]}, \quad k = 1,\dots,n+2$$

(3.2) $$(f - p_n^*)(x_k^{(n)}) = -(f - p_n^*)(x_{k+1}^{(n)}) , \quad k = 1,\dots,n+1,$$

where $p_n^* = p_n^*(f,x)$ is the best uniform approximation to f on $[-1,1]$. A set of points $\{x_k^{(n)}\}_{k=1}^{n+2}$ satisfying (3.1) and (3.2) is called an *alternation set* for the error $f - p_n^*$. We remark that such an alternation set need not be unique. For the special case when $f(x) = x^{n+1}$, the error $x^{n+1} - p_n^*(x)$ is just the Chebyshev polynomial of the first kind and the corresponding alternation set is

(3.3) $$\left\{\cos\left(\frac{k\pi}{n+1}\right)\right\}_{k=0}^{n+1} .$$

It turns out that for an arbitrary real-valued $f \in C[-1,1]$, there is always a *subsequence* of the errors $f - p_n^*$ for which the corresponding alternation sets have the same limiting distribution as the points (3.3); that is, the arcsine distribution. This fact was first proved by Kadec [12], who also provides estimates for the rate of this convergence. Further improvements in Kadec's result were obtained by Fuchs [9] and Blatt and Lorentz [4]. Unfortunately, the method of proof of Kadec's theorem (as well as its generalizations) relies heavily on the fact that consecutive points in an alternation set and the zeros of $p_n^*(f) - p_{n-1}^*(f)$ interlace; a fact that has no analogue for the approximation of complex-valued functions. To circumvent this difficulty, Kroó and Saff [13] used, instead, a potential theoretic argument to establish a Jentzsch-type result concerning the *denseness* of extreme points. To state their result we need to introduce some notation.

Let $E \subset \mathbb{C}$ be a compact set with connected and regular complement. Given $f \in C(E) \cap A(\mathring{E})$, we set

(3.4) $$A_n(f) := \{z \in E : |f(z) - p_n^*(f,z)| = \|f - p_n^*(f)\|_E\} ,$$

and refer to $A_n(f)$ as the set of extreme points for $f - p_n^*$. We remark that each set $A_n(f)$ consists of at least $n + 2$ points. To measure the denseness of a set A in a set B $(A,B \subset \mathbb{C})$ we define

(3.5) $$\rho(A,B) := \sup_{z\in B} \inf_{\xi\in A} |z - \xi|.$$

The main result of Kroó and Saff asserts that there is a subsequence of integers n for which the sets $A_n(f)$ become dense in the boundary ∂E. More precisely, we have

THEOREM 3.1([13]). *If* $f \in C(E) \cap A(\mathring{E})$, *then*

(3.6) $$\liminf_{n\to\infty} \rho(A_n(f), \partial E) = 0.$$

Furthermore, there exists an entire function g *such that*

(3.7) $$\limsup_{n\to\infty} \rho(A_n(g), \partial E) > 0.$$

In the proof of this theorem it is shown, moreover, that if Λ is any increasing sequence of integers for which the errors e_n of (1.2) satisfy

(3.8) $$\lim_{n\to\infty} \left[\frac{e_{n-1} - e_n}{e_{n-1} + e_n}\right]^{1/n} = 1 , \quad n \in \Lambda,$$

then

(3.9) $$\lim_{n\to\infty} \rho(A_n(f), \partial E) = 0 , \quad n \in \Lambda.$$

The second part of Theorem 3.1 asserts that denseness need not hold for all subsequences - a fact that was first proved by G.G. Lorentz [14] for the Kadec case of a real function on a real interval.

Theorem 3.1 is the analogue of Theorem 2.1 concerning the denseness of zeros of best polynomial approximants. However, there is an important difference - namely, in the latter theorem we require that f have a singularity on ∂E ($f \notin A(E)$), while Theorem 3.1 applies as well to functions f that are analytic on E.

Theorem 3.1 shows that the polynomials $p_n^*(f)$ of best uniform approximation have another undesirable feature. Suppose, for example, that E is a Jordan arc or a Jordan region and consider a function

$f \in C(E) \cap A(\mathring{E})$ that is analytic at some but not all points of the boundary ∂E (such as the functions $f_1(z)$ and $f_2(z)$ in the Introduction). Since $f \notin A(E)$, the rate of decay of the errors e_n is slow (not geometric). However, it would seem reasonable to expect that on those subarcs of ∂E where f is analytic (say, a subarc Γ), the rate of convergence of $\{p_n^*(f)\}_1^\infty$ to f is faster. Unfortunately this is not the case since, by Theorem 3.1, for infinitely many n this subarc must contain points of $A_n(f)$; that is,

$$\|f - p_n^*(f)\|_\Gamma = \|f - p_n^*(f)\|_E = e_n$$

for infinitely many n.

What can be said about the denseness of extreme points for the case of best *rational* approximation? While this question has not yet been answered in full generality, there is an important case for which results are known - namely, for best real rational approximation to a real-valued function f on $[-1,1]$. We now describe these results in the context of the Walsh array.

Let $\mathcal{R}_{m,n}$ be the collection of all real rational functions with numerator in $\mathcal{P}_m$ and denominator in $\mathcal{P}_n$. For real-valued $f \in C[-1,1]$, we let $r^*_{m,n} = r^*_{m,n}(f)$ denote the best uniform approximation to f on $[-1,1]$ out of $\mathcal{R}_{m,n}$. These best approximants are typically displayed in the following doubly-infinite table called the *Walsh array*.

r^*_{00}	r^*_{10}	r^*_{20}	. . .
r^*_{01}	r^*_{11}	r^*_{21}	. . .
r^*_{02}	r^*_{12}	r^*_{22}	. . .
.	.	.	
.	.	.	
.	.	.	

Notice that the first row of this array consists of the polynomials of best uniform approximation $(r^*_{m,0} = p^*_m)$. We know, therefore, that extreme points are dense for the approximants in the first row. But what about other sequences formed from the array such as the diagonal, other rows, or a "ray sequence" that consists of rationals $\{r^*_{m,n}\}$ for which the ratio m/n has a finite limit as $m,n \to \infty$?

To state the main result [8] of Borwein, Grothmann, Kroó and Saff concerning the denseness of such extreme points, we first recall that, for each pair of nonnegative integers (m,n), the best approximant $r^*_{m,n} = p^*_{m,n}/q^*_{m,n}$ is characterized by the following equioscillation property: There are $m + n + 2 - d$ points

$$(3.10) \qquad -1 \le x_1^{(m,n)} < \dots < x_{m+n+2-d}^{(m,n)} \le 1 ,$$

where

$$d := d(m,n) := \min\{m - \deg p^*_{m,n} , n - \deg q^*_{m,n}\}$$

such that

$$(3.11) \qquad |(f-r^*_{m,n})(x_k^{(m,n)})| = \|f-r^*_{m,n}\|_{[-1,1]}, \qquad k = 1,\dots,m+n+2-d,$$

$$(3.12) \qquad (f-r^*_{m,n})(x_k^{(m,n)}) = -(f-r^*_{m,n})(x_{k+1}^{(m,n)}), \quad k = 1,\dots,m+n+1-d.$$

Next, we measure the denseness of the extreme points $\{x_k^{(m,n)}\}_1^{m+n+2-d}$ in $[-1,1]$ by defining

$$(3.13) \qquad \rho_{m,n}(f) := \sup_{x\in[-1,1]} \min_k |x - x_k^{(m,n)}| .$$

The main result of [8] is the following.

THEOREM 3.2. Let the sequence of nonnegative integers $n = n(m)$ *satisfy*

$$(3.14) \qquad n(m) \le n(m + 1) \le n(m) + 1 , \qquad n(m) \le m ,$$

for $m = 0,1,\dots$. *If* $f \in C[-1,1]$, $f \notin \mathcal{R}_{m,n(m)}$, $m = 0,1,\dots$, then

$$\text{(3.15)} \qquad \liminf_{m\to\infty} \left[\frac{m - n(m)}{\log m}\right] \rho_{m,n(m)}(f) < \infty .$$

Notice that Theorem 3.2 applies in the case of ray sequences of the form $n(m) = [cm]$ for any constant $c \le 1$, where $[\cdot]$ denotes the greatest integer function. If $c < 1$, we deduce from (3.15) that

$$\liminf_{m\to\infty} \rho_{m,[cm]}(f) = 0$$

which means that the extreme points are dense in [-1,1]. In other words, if we proceed down the Walsh array at an angle of less than $\pi/4$ with the first row, then the Jentzsch-type (denseness) result holds for the extreme points.

Of course, Theorem 3.2 gives no information about the important case of diagonal sequences; that is, when we proceed down the table at an angle of $\pi/4$. In [8], it is shown that denseness need no longer hold for such sequences. In fact, for a diagonal sequence, all alternation points can occur in an arbitrarily small subinterval of [-1,1]. More precisely we have

THEOREM 3.3. *For every* $2 > \epsilon > 0$, *there is a function* $f \in C[-1,1]$ *such that for each* $n = 1,2,\ldots$, *the error* $f - r^*_{n-1,n}(f)$ *has no extreme points in* $(-1 + \epsilon, 1]$.

Returning to the behavior of extreme points for best polynomial approximation, a Szegö-type *limiting distribution* result has recently been obtained by Blatt, Saff and Totik [7]. To describe this result we must first associate a unit measure with each extremal set $A_n(f)$ in (3.4). For the Kadec case of real approximation on an interval, one can use the discrete unit measure that is supported in an $(n + 2)$ - point alternation set $\{x^*_k\}_{k=1}^{n+2}$ (cf. (3.1) and (3.2)). For complex approximation, the notion of alternation set is replaced by that of an *extremal signature*, which is a subset of $A_n(f)$ consisting of at most $2n + 3$ points. However, unlike the case of real approximation, the cardinality of an extremal signature can vary from $n + 2$ to $2n + 3$ points. Thus it is not immediately clear how to select $n + 2$ points from $A_n(f)$ in order to define a discrete measure. This difficulty was resolved in [7] by selecting as the subset of $A_n(f)$ an $(n+2)$-point *Fekete subset* which we denote by $\mathcal{F}_{n+2}$. To be precise, $\mathcal{F}_{n+2}$ is an $(n+2)$-point subset S of $A_n(f)$ for which the Vandermonde expression

$$V(S) := \prod_{\substack{z,t \in S \\ z \neq t}} |z - t|$$

is as large as possible.

Next, as in (2.6), we associate a unit measure λ_n with $\mathcal{F}_{n+2}$ by defining

$$(3.16) \qquad \lambda_n(B) := \frac{\text{number of points of } \mathcal{F}_{n+2} \text{ in } B}{n+2},$$

for any Borel set $B \subset \mathbb{C}$.

With the above notation we can now state

THEOREM 3.4([7]). *Suppose* $f \in C(E) \cap A(\mathring{E})$ *and let* $\mathcal{F}_{n+2}$ *be an* $(n+2)$*-point Fekete subset of the set of extreme points* $A_n(f)$ *defined in (3.4). Then the measures* λ_n *of (3.16) converge in the weak-star topology to the equilibrium distribution* μ_E *as* $n \to \infty$ *through a sequence* $\Lambda = \Lambda(f,E)$ *of positive integers.*

§4. Concluding Remarks

The results of the preceding sections have hopefully convinced the reader that best polynomial approximants have significant drawbacks. Moreover, in a sense, "near-best may be better than best." In [15] Saff and Totik construct such near-best polynomial approximants for the case of piecewise-analytic functions f on $[-1,1]$ (such as $f(x) = |x|$ on $[-1,1]$). The following is a sample of their results.

THEOREM 4.1 *Suppose* $f \in C^k[-1,1]$ *is piecewise analytic on* $[-1,1]$ *and* $\beta > 1$ *is given. Then there exist constants* $c, C > 0$ *and polynomials* $q_n \in \mathcal{P}_n$, $n = 1,2,\ldots$, *such that for every* $x \in [-1,1]$

$$(4.1) \qquad |f(x) - q_n(x)| \le \frac{C}{n^{k+1}} \exp(-cn[d(x)]^{\beta}) \; ,$$

where $d(x)$ *denotes the distance from* x *to the nearest singularity of* f *in* $[-1,1]$.

Notice from (4.1) that the q_n converge geometrically in an open interval and hence in an open disk about each point of $[-1,1]$ where f is analytic.

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