Constr. Approx. (1989) 5: 357–370

CONSTRUCTIVE APPROXIMATION

The Distribution of Extreme Points in Best Complex Polynomial Approximation

H.-P. Blatt, E. B. Saff, and V. Totik

Abstract. Let K be a compact point set in the complex plane having positive logarithmic capacity and connected complement. For any f continuous on K and analytic in the interior of K we investigate the distribution of the extreme points for the error in best uniform approximation to f on K by polynomials. More precisely, if

$$A_n(f) := \{z \in K : |f(z) - p_n^*(f; z)| = \|f - p_n^*(f)\|_K\},$$

where $p_n^*(f)$ is the polynomial of degree $\leq n$ of best uniform approximation to f on K, we show that there is a subsequence $\{n_k\}$ with the property that the sequence of (n_k+2)-point Fekete subsets of A_{n_k} has limiting distribution (as $k \to \infty$) equal to the equilibrium distribution for K. Analogues for weighted approximation are also given.

1. Introduction

Let K be a compact set in the complex plane $\mathbf{C}$ and let $C_A(K)$ be the set of continuous complex-valued functions on K that are analytic in the interior of K endowed with the supremum norm $\|\cdot\|_K$. For $f \in C_A(K)$ we denote by $p_n^*(f)$ its best approximant out of Π_n, the set of polynomials of degree at most n, and we set

$$A_n(f) := \{z \in K : |f(z) - p_n^*(f; z)| = \|f - p_n^*(f)\|_K\}. \tag{1.1}$$

We investigate the distribution of the points in $A_n(f)$ as $n \to \infty$.

Kadec [4] established that if $f \in C[-1, 1]$ is real-valued on $K = [-1, 1]$, then any (n_k+2)-point alternating subset of $A_{n_k}(f)$ has the arcsine distribution for a suitable subsequence $\{n_k\}$ of the natural numbers, but Lorentz [8] showed that this is not necessarily true for every subsequence. Kadec even proved that the rate of this convergence is $O(n_k^{-1/2+\varepsilon})$ for every $\varepsilon > 0$, which was subsequently sharpened by Blatt and Lorentz [1] to $O(\sqrt{(\log n_k)/n_k})$. All these results are based on the fact that consecutive points in a maximal alternation set and the

Date received: December 2, 1987. Date revised: May 27, 1988. Communicated by Dieter Gaier.
AMS classification: 41A50, 30E10.
Key words and phrases: Polynomial approximation, Best polynomial approximants, Extreme points, Equilibrium distribution, Weighted approximation.

zeros of $p_n^*(f) - p_{n-1}^*(f)$ interlace; a fact that has no analogue for complex-valued functions. Thus, the asymptotic behavior of some subset of $A_n(f)$ (in the spirit of Kadec's result) has been open even for $K = [-1, 1]$ when f is complex-valued. The only known result for the general complex case belongs to Kroó and Saff [6] who verified, for regular sets K, the denseness of $\{A_{n_k}(f)\}$ on the boundary ∂K of K for some subsequence $\{n_k\}$.

In this paper we show that the Fekete points of $A_{n_k}(f)$ are distributed, for a suitable $\{n_k\}$, like the equilibrium distribution of K. The same is true for the "maximal value" points of the Chebyshev polynomials for K. Furthermore, for the case when K is bounded by a smooth Jordan curve Γ, we establish that for some $\{n_k\}$ the relative density of the Fekete points of $A_{n_k}(f)$ in any subarc of Γ converges to the equilibrium measure of the subarc with rate $O((\log n)/\sqrt{n})$. Finally, we indicate how our results can be extended to weighted approximation. A surprising corollary is that if w is a nonnegative continuous weight on the closed unit disk K, with w positive on the open unit disk, then every point of the boundary $|z| = 1$ is a limit point of extreme points for best polynomial approximation of any $f \in C_A(K)$ with weight w, even if w vanishes identically on $|z| = 1$.

2. Results

To state our results we introduce some concepts and notations from potential theory. If $K \subseteq \mathbf{C}$ is compact, then $\operatorname{cap}(K)$ denotes the *logarithmic capacity* of K (see [10]). We will always assume that the complement of K is connected and $\operatorname{cap}(K) > 0$. It is well known that

$$\operatorname{cap}(K) = \exp(-I(\mu_K)),$$

where $I(\mu_K)$ is the smallest value of the energy integral

$$I(\mu) := \int\int \log|z - t|^{-1}\, d\mu(t)\, d\mu(z),$$

where μ runs through all unit measures supported on K. There is a unique measure μ_K, called the *equilibrium measure* of K, for which this minimum is attained.

If E is any compact set, we denote by $\mathscr{F}_n(E)$ any n-point subset S of E for which the Vandermonde expression

$$V(S) := \left[\prod_{\substack{z,t \in S \\ z \neq t}} |z - t| \right]^{1/2} \tag{2.1}$$

is as large as possible. The points in $\mathscr{F}_n(E)$ are called *Fekete points* of E. Their relation to the capacity is given by

$$\lim_{n \to \infty} V(\mathscr{F}_n(K))^{2/n(n-1)} = \operatorname{cap}(K).$$

With any finite point set S we associate ν_S, the counting measure on S; that is,

$$\nu_S(E) := \text{number of points of } S \cap E.$$

Finally, the weak-star convergence of measures will be denoted by $\xrightarrow{*}$.

Our main result is

Theorem 1. *Let K be a compact subset of* $\mathbf{C}$ *with connected complement and positive capacity, $f \in C_A(K)$, and let $A_n(f)$ be the set of extreme points defined in* (1.1). *Then, for some subsequence $\mathcal{N}$ of the natural numbers,*

$$\text{(2.2)} \qquad \frac{1}{n+1} \nu_{\mathcal{F}_{n+1}(A_{n-1}(f))} \xrightarrow{*} \mu_K, \qquad n \in \mathcal{N}, \quad n \to \infty,$$

where μ_K is the equilibrium measure for K.

We note that the counterexample given in [6] implies that (2.2) need not be true for every $\mathcal{N}$. Note also that if $A_{n-1}(f)$ happens to be an $(n+1)$-point set for $n \in \mathcal{N}$, then (2.2) gives the limiting distribution of these sets $A_{n-1}(f)$. However, $A_{n-1}(f)$ can actually coincide with ∂K for every n; as a nontrivial example, consider $K := \{z: |z| \le 1\} \cup \{2\}$ and

$$f(z) := \begin{cases} 0 & \text{if } |z| \le 1, \\ 1 & \text{if } z = 2. \end{cases}$$

Our second result concerns the points where the Chebyshev polynomials assume their maximum absolute value. For any compact set $K \subset \mathbf{C}$, we set

$$\text{(2.3)} \qquad t_n(K) := \inf_{p_{n-1} \in \Pi_{n-1}} \|z^n - p_{n-1}\|_K = \inf_{z^n + \cdots \in \Pi_n} \|z^n + \cdots\|_K.$$

As is well known (see [10]), the constants $t_n(K)$ are related to the capacity of K via the formula

$$\text{(2.4)} \qquad \lim_{n \to \infty} t_n(K)^{1/n} = \operatorname{cap}(K).$$

If K contains at least n points, there is a unique monic polynomial $T_n^* = T_n^*(K; z) = z^n + \cdots \in \Pi_n$ such that

$$t_n(K) = \|T_n^*\|_K.$$

We call T_n^* the *Chebyshev polynomial* of degree n associated with K.

Our next result concerns the extremal point sets

$$A_n^*(K) := \{z \in K: |T_n^*(K; z)| = t_n(K)\}.$$

We shall prove

Theorem 2. *Let K be compact with* $\operatorname{cap}(K) > 0$. *Then*

$$\frac{1}{n+1} \nu_{\mathcal{F}_{n+1}(A_n^*(K))} \xrightarrow{*} \mu_K \qquad \textit{as} \quad n \to \infty.$$

We emphasize that, unlike Theorem 1, the weak-star convergence in Theorem 2 holds for the full sequence of nonnegative integers.

For the case when the boundary of K is sufficiently smooth, we shall prove a result concerning the rate of convergence: let K be a closed region bounded by a Jordan curve Γ, where Γ has a continuous tangent. Moreover, assume that the angle of inclination $\theta(s)$ of the tangent to Γ as a function of the arclength s along Γ satisfies a Hölder condition; that is,

$$|\theta(s)-\theta(s')|\le L|s-s'|^{\alpha}, \tag{2.5}$$

where L, α are constants and $0<\alpha\le 1$. Then we say that Γ belongs to the class $C^{1,H}$.

As in [5], we define, for any signed measure σ on Γ, the number

$$[\sigma]:=\sup|\sigma(\gamma)|, \tag{2.6}$$

where the supremum is taken over all subarcs γ of the boundary Γ. Then the rate of convergence of

$$\mu_n:=\frac{1}{n+1}\nu_{\mathcal{F}_{n+1}(A_{n-1}(f))} \tag{2.7}$$

to the equilibrium measure μ_K can be estimated as follows.

Theorem 3. *Let K be a closed Jordan region with boundary $\Gamma\in C^{1,H}$, and let $f\in C_A(K)$. Then there is a constant c (depending on Γ) and a subsequence $\mathcal{N}$ of natural numbers such that*

$$[\mu_K-\mu_n]\le c\frac{\log n}{\sqrt{n}}, \tag{2.8}$$

whenever $n\in\mathcal{N}$.

This means that when K is a sufficiently smooth Jordan region, the weak-star convergence established in Theorem 1 can be quantified and the rate of convergence is at least $(\log n)/\sqrt{n}$. It remains open to determine the best possible rate of convergence.

If K is a compact subset of the real line, a quantified version of Theorem 1 can also be given. For example, it is possible to show that when K is the finite union of compact intervals, there exists a constant c and a subsequence $\mathcal{N}$ of natural numbers such that, for any interval $[a, b]$,

$$|(\mu_K-\mu_n)[a,b]|\le c\left(\frac{\log n}{n}\right)^{1/3}, \tag{2.9}$$

whenever $n\in\mathcal{N}$. Since the proof of (2.9) depends substantially on generalizations of the Erdös-Turán inequalities [3], we leave the verification of (2.9) for another occasion.

Finally, we mention that our results hold for weighted approximation. Let $w\ge 0$ be a continuous weight function on K and set

$$A_n(f,w):=\{z\in K\colon w(z)|f(z)-p_n^*(f;z)_w|=\|w(f-p_n^*(f)_w)\|_K\},$$

where $p_n^*(f)_w$ denotes the best weighted approximation of f out of Π_n with weight w. If E is a compact subset of K, then $\mathcal{F}_n(E)_w$ denotes an n-point subset $\{z_0, \ldots, z_{n-1}\}$ of E maximizing the absolute value of the determinant

$$|w(z_j)z_j^i|_{0\le i\le n-1, 0\le j\le n-1}.$$

With these notations we can prove the following generalization of Theorem 1.

Theorem 4. *Let K be compact with* $\operatorname{cap}(K)>0$ *and connected complement, and assume that $f\in C_A(K)$. Furthermore, let w be a nonnegative continuous weight function on K such that the (inner logarithmic) capacity of the set $\{z\in K\colon w(z)>0\}$ equals the capacity of K. Then, for some subsequence $\mathcal{N}$ of the natural numbers,*

$$\frac{1}{n+1}\nu_{S_{n+1}} \xrightarrow{*} \mu_K \qquad \text{as} \quad n\to\infty, \quad n\in\mathcal{N}, \tag{2.10}$$

where $S_{n+1}:=\mathcal{F}_{n+1}(A_{n-1}(f, w))_w$. In particular, this holds if w vanishes only on a set of capacity zero.

We also mention that the weighted analogue of Theorem 2 holds as well.

Corollary. *With the assumptions of Theorem* 4, *every point in the support of μ_K is a limit point of the sets $A_n(f, w)$.*

We remark that the support of μ_K lies on the boundary ∂K of K, and so the corollary means that most of the points of ∂K (i.e., except for a set of capacity zero) attract extremal points. This is fairly surprising since w can vanish on a large part of ∂K; consider, for example,

$$w(z)=1-|z|$$

on the unit disk $\{z\colon |z|\le 1\}$.

3. Proofs

Our proofs are based on two simple lemmas.

Lemma 1. *Suppose $f\in C_A(K)$ and the complement of K is connected. Then there exists a subsequence $\mathcal{N}$ of the natural numbers with the property*

$$t_n(A_{n-1}(f))\ge \frac{1}{n^2}t_n(K) \qquad \text{for} \quad n\in\mathcal{N},$$

where t_n is defined in (2.3).

For the proof see Lemma 2.3 and (2.18) of [6].

Lemma 2. *For any compact set E containing at least $n+1$ points,*

$$t_n(\mathcal{F}_{n+1}(E))\ge \frac{1}{n+1}t_n(E).$$

For the proof see Section 7.9, Lemma, p. 177, of [11].

Combining Lemmas 1 and 2 we see that

$$(3.1) \qquad t_n(\mathcal{F}_{n+1}(A_{n-1}(f))) \geq (n+1)^{-3} t_n(K), \qquad n \in \mathcal{N}.$$

Let $\mathcal{F}_{n+1}(A_{n-1}(f)) = \{z_{0,n}, \ldots, z_{n,n}\}$. Since this is an $(n+1)$-point set, we can easily compute its nth Chebyshev polynomial T_n^* and Chebyshev number $t_n(\mathcal{F}_{n+1}(A_{n-1}(f))) =: t_n$. In fact, T_n^* must have the same absolute value, t_n, at every $z_{k,n}$, and must have leading coefficient 1, which easily yields

$$T_n^*(z) = \left(\sum_{k=0}^{n} \frac{\omega_{n+1}(z) \operatorname{sgn} \omega'_{n+1}(z_{k,n})}{\omega'_{n+1}(z_{k,n})(z - z_{k,n})} \right) \left(\sum_{k=0}^{n} \frac{1}{|\omega'_{n+1}(z_{k,n})|} \right)^{-1},$$

where

$$(3.2) \qquad \omega_{n+1}(z) := \prod_{k=0}^{n} (z - z_{k,n})$$

and

$$t_n = \left(\sum_{k=0}^{n} \frac{1}{|\omega'_{n+1}(z_{k,n})|} \right)^{-1}.$$

Indeed, if T_n is any nth degree polynomial with

$$T_n(z_{k,n}) = \varepsilon_{k,n} t, \qquad 0 \leq k \leq n,$$

with $t > 0$ and $|\varepsilon_{k,n}| = 1$ that has leading coefficient 1, then, by the Lagrange interpolation formula,

$$T_n(z) = \sum_{k=0}^{n} \frac{\omega_{n+1}(z)}{\omega'_{n+1}(z_{k,n})(z - z_{k,n})} \varepsilon_{k,n} t$$

and

$$t \sum_{k=0}^{n} \frac{\varepsilon_{k,n}}{\omega'_{n+1}(z_{k,n})} = 1,$$

which implies

$$t \geq \left(\sum_{k=0}^{n} \frac{1}{|\omega'_{n+1}(z_{k,n})|} \right)^{-1}.$$

The expression for t_n and (3.1) show that

$$(3.3) \qquad |\omega'_{n+1}(z_{k,n})| = \prod_{\substack{j=0 \\ j \neq k}}^{n} |z_{k,n} - z_{j,n}| \geq (n+1)^{-3} t_n(K)$$

for every $0 \leq k \leq n$ and $n \in \mathcal{N}$.

Proof of Theorem 1. We keep the above notations and give two proofs, a complex function theoretical one and a real variable argument.

Let $\mathcal{F}_{n+1}(K) = \{y_{0,n+1}, \ldots, y_{n,n+1}\}$ be a set of $(n+1)$th Fekete points of K. The polynomials

$$\tau_{n+1}(z) = \tau_{n+1}(K, z) := \prod_{k=0}^{n} (z - y_{k,n+1})$$

are called *Fekete polynomials* of K. By the Lagrange interpolation formula, we have for the polynomials ω_{n+1} of (3.2)

$$\tau_{n+1}(z)-\omega_{n+1}(z)=\sum_{k=0}^{n}(\tau_{n+1}(z_{k,n})-\omega_{n+1}(z_{k,n}))\frac{\omega_{n+1}(z)}{\omega'_{n+1}(z_{k,n})(z-z_{k,n})}$$

(note that $\tau_{n+1}-\omega_{n+1}$ is of degree at most n). Since $\omega_{n+1}(z_{k,n})=0$ for every k and

$$|\tau_{n+1}(z_{k,n})|\le\|\tau_{n+1}\|_K=:d_{n+1}(K),$$

we see that for the n's satisfying (3.3), i.e., for $n\in\mathcal{N}$,

$$|\tau_{n+1}(z)-\omega_{n+1}(z)|\le(n+1)^3\frac{d_{n+1}(K)}{t_n(K)}|\omega_{n+1}(z)|\sum_{k=0}^{n}\frac{1}{|z-z_{k,n}|},$$

$$\left|\frac{\tau_{n+1}(z)}{\omega_{n+1}(z)}\right|\le(n+1)^3\frac{d_{n+1}(K)}{t_n(K)}\sum_{k=0}^{n}\frac{1}{|z-z_{k,n}|}+1,$$

which implies that, for $n\in\mathcal{N}$,

$$\frac{1}{n+1}\log\left|\frac{\tau_{n+1}(z)}{\omega_{n+1}(z)}\right|\le C_H\varepsilon_{n+1},\qquad z\in H, \tag{3.4}$$

where

$$\varepsilon_{n+1}:=\frac{\log(n+1)}{n+1}+\frac{1}{n+1}\log^+\frac{d_{n+1}(K)}{t_n(K)},$$

uniformly on every closed subset H of $\bar{\mathbf{C}}\backslash K$, the complement of K with respect to the closed Riemann sphere. By Lemma 11.2 of [9]

$$\lim_{n\to\infty}d_n(K)^{1/n}=\operatorname{cap}(K)$$

which, together with (2.4), imply that $\varepsilon_n\to 0$ as $n\to\infty$. It is known (see Theorem 11.1 of [9] and Chapters I and III of [10]) that, for $z\in\mathbf{C}\backslash K$,

$$\lim_{n\to\infty}\frac{1}{n+1}\log|\tau_{n+1}(z)|=g(z)+\log[\operatorname{cap}(K)], \tag{3.5}$$

where g is Green's function for $\bar{\mathbf{C}}\backslash K$ with the pole at infinity.

Since all the zeros of ω_{n+1} and τ_{n+1} lie in K, the functions

$$\frac{1}{n+1}\log\left|\frac{\tau_{n+1}(z)}{\omega_{n+1}(z)}\right|,\qquad n\in\mathcal{N},$$

are harmonic in $\bar{\mathbf{C}}\backslash K$, vanish at ∞, and have the upper bound $C_H\varepsilon_{n+1}$ on every closed subset H of $\bar{\mathbf{C}}\backslash K$ (cf. (3.4)). Thus we can apply Harnack's inequality and get that, uniformly on closed subsets H' of the interior of H,

$$\left|\frac{1}{n+1}\log\left|\frac{\tau_{n+1}(z)}{\omega_{n+1}(z)}\right|\right|\le C'_{H'}\varepsilon_{n+1},\qquad n\in\mathcal{N}.$$

Making use of (3.5) we finally obtain that, uniformly on compact subsets of $\mathbf{C}\backslash K$,

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{1}{n+1}\log|\omega_{n+1}(z)| = g(z)+\log[\operatorname{cap}(K)]. \tag{3.6}$$

Denoting the potential corresponding to a measure μ by $U(\mu; z)$; that is

$$U(\mu; z) := \int \log|z-t|^{-1}\, d\mu(t),$$

then (3.6) can be written in the equivalent form

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} U(\mu_n; z) = U(\mu_K; z), \qquad z\in\mathbf{C}\backslash K, \tag{3.7}$$

where

$$\mu_n := \frac{1}{n+1}\nu_{\mathcal{F}_{n+1}(A_{n-1}(f))}.$$

Proceeding as in the proof of Theorem 2.1 of [2], we let μ be a weak-star limit of $\{\mu_n\}_{n\in\mathcal{N}}$; that is, there exists $\mathcal{N}_1\subseteq\mathcal{N}$ such that

$$\mu_n \xrightarrow{*} \mu \qquad \text{as} \quad n\to\infty, \qquad n\in\mathcal{N}_1.$$

Note that since the Fekete points of $A_{n-1}(f)$ lie on the boundary ∂K of K, then $\operatorname{supp}(\mu)\subseteq\partial K$ and $\mu(\partial K)=1$. From (3.7) we get

$$U(\mu; z) = U(\mu_K; z), \qquad z\in\mathbf{C}\backslash K. \tag{3.8}$$

Further, since $U(\mu_K; z)\le I(\mu_K)$ for all $z\in\mathbf{C}$ (see Theorem III.12 of [10]) and since $U(\mu; z)$ is lower semicontinuous, we obtain from (3.8) that

$$U(\mu; z)\le I(\mu_K), \qquad z\in\partial K.$$

Integrating this inequality with respect to $d\mu$ over ∂K we see that the energy $I(\mu)$ satisfies $I(\mu)\le I(\mu_K)$. Thus, by the unicity of μ_K, we get $\mu=\mu_K$ and so the whole sequence $\{\mu_n\}_{n\in\mathcal{N}}$ converges to μ_K in the weak-star topology.

Our second proof also starts with (3.3). The inequality implies that

$$\begin{aligned} V(\mathcal{F}_{n+1}(A_{n-1}(f))) &= \left(\prod_{k=0}^{n}\prod_{\substack{j=0\\ j\ne k}}^{n} |z_{k,n}-z_{j,n}|\right)^{1/2} \\ &\ge (n+1)^{-3(n+1)/2}(t_n(K))^{(n+1)/2}, \end{aligned} \tag{3.9}$$

and so taking into account (2.4), all we have to prove is the following proposition which is interesting in itself.

Proposition. *Assume K is compact with* $\operatorname{cap}(K)>0$ *and, for every $n\in\mathcal{N}$, where $\mathcal{N}$ is a subsequence of the natural numbers, let*

$$Z_n := \{z_{1,n}, \dots, z_{n,n}\}$$

be an n-point subset of K. If

$$\liminf_{\substack{n\to\infty \\ n\in N}} V(Z_n)^{2/n(n-1)} \ge \operatorname{cap}(K), \tag{3.10}$$

then

$$\frac{1}{n}\nu_{Z_n} \xrightarrow{*} \mu_K \qquad as \quad n\to\infty, \quad n\in\mathcal{N}.$$

Recall that $V(Z_n)$ was defined in (2.1).

Proof. Let $\lambda_n := (1/n)\nu_{Z_n}$ and let λ be a weak-star limit point of $\{\lambda_n\}_{n\in\mathcal{N}}$, say

$$\lambda_n \xrightarrow{*} \lambda \qquad \text{as} \quad n\to\infty, \quad n\in\mathcal{N}_1\subseteq\mathcal{N}. \tag{3.11}$$

We have to show that $\lambda = \mu_K$. First we mention that λ does not have point masses. In fact, if we have $\lambda(\{z_0\}) = c > 0$, then for every $\delta > 0$ and large $n\in\mathcal{N}_1$ there would be at least $(c/2)n$ points from Z_n in a δ neighborhood of z_0 and so we would have

$$V(Z_n) \le \delta^{((c/2)n-1)^2/2}(1+\operatorname{diam} K)^{n^2}.$$

But, for small δ, this contradicts (3.10).

Thus, by Tonelli's theorem, we get that

$$I(\lambda) = \iint \log\frac{1}{|z-t|}\,d\lambda(z)\,d\lambda(t) = \iint_{z\neq t} \log\frac{1}{|z-t|}\,d\lambda(z)\,d\lambda(t).$$

Set

$$g_\varepsilon(t) := \begin{cases} \log\dfrac{1}{t} & \text{if } t > 2\varepsilon, \\ 0 & \text{if } t < \varepsilon, \\ \dfrac{1}{\varepsilon}(t-\varepsilon)\log\dfrac{1}{2\varepsilon} & \text{if } \varepsilon\le t\le 2\varepsilon. \end{cases}$$

Then g_ε is continuous and, as $\varepsilon\to 0+$, tends monotonically to $\log(1/t)$ for $t>0$. Thus, by the monotone convergence theorem, (3.10), and (3.11), we have for the energy integral

$$\begin{aligned} I(\lambda) &= \lim_{\varepsilon\to 0+} \iint_{z\neq t} g_\varepsilon(|z-t|)\,d\lambda(z)\,d\lambda(t) \\ &= \lim_{\varepsilon\to 0+}\lim_{n\to\infty} \iint_{z\neq t} g_\varepsilon(|z-t|)\,d\lambda_n(z)\,d\lambda_n(t), \qquad n\in\mathcal{N}_1, \\ &\le \lim_{\varepsilon\to 0+}\lim_{n\to\infty} \iint_{z\neq t} \log\frac{1}{|z-t|}\,d\lambda_n(z)\,d\lambda_n(t) \\ &= \lim_{\varepsilon\to 0+}\lim_{n\to\infty} \frac{1}{n^2}\log\frac{1}{V(Z_n)^2} \le \log\frac{1}{\operatorname{cap}(K)} = I(\mu_K). \end{aligned}$$

Each $\lambda_n, n \in \mathcal{N}_1$, is supported on K and $\lambda_n(K)=1$. Therefore, the weak-star limit λ also has these properties, and so $\lambda = \mu_K$ follows from the unicity of μ_K. ∎

Proof of Theorem 2. It is well known and easy to see that $t_n(A_n^*(K)) = t_n(K)$. Thus, by Lemma 2,

$$t_n(\mathcal{F}_{n+1}(A_n^*(K))) \geq (n+1)^{-1} t_n(K), \qquad n = 0, 1, \ldots.$$

Since the preceding proofs used only the estimate (3.1), they can be applied in the present situation as well and we are done. ∎

Proof of Theorem 3. Let $z_{0,n}, z_{1,n}, \ldots, z_{n,n}$ be an $(n+1)$-point Fekete set of $A_{n-1}(f)$ and denote by μ_n the discrete unit measure with mass $1/(n+1)$ at each point $z_{i,n}$ (cf. (2.7)).

We follow a technique of Kleiner [5] and consider the circles $K_{j,n} = K(z_{j,n}, 1/(n+1)^3)$ with center at $z_{j,n}$ and radius $1/(n+1)^3$. We denote by μ_n^* the unit measure that has constant density $(n+1)^2/2\pi$ on $\bigcup_{j=0}^n K_{j,n}$.

Furthermore, we measure the angle between two lines in the absolute sense; that is, such an angle is between 0 and $\pi/2$. We say that a subarc γ of Γ is inclined to the line l with angle $\geq \varepsilon$ if all tangents to γ have this property. Now, if $z_0 \in \Gamma$, there exists a real number $r = r(z_0) > 0$ such that the subarc

$$\Gamma \cap \{z: |z - z_0| \leq r\}$$

is inclined to the normal line l_0 of Γ in z_0 with an angle $\geq \pi/3$.

Let γ be a subarc of Γ with endpoints a, b. Define A (resp. B) as the set of points ξ on the normal line in a (resp. b) such that $|\xi - a| \leq r(a)$ (resp. $|\xi - b| \leq r(b)$). Then, for n large, the point set

$$\Gamma^* := \Gamma \cup K_{0,n} \cup K_{1,n} \cup \cdots \cup K_{n,n} \tag{3.12}$$

is separated by $A \cup B$ into two connected components, one of which, say γ^*, contains the subarc γ.

Now, for any (bounded) signed measure σ on Γ^* we define

$$[\sigma] := \sup_{\gamma \in \Gamma} |\sigma(\gamma^*)|, \tag{3.13}$$

which coincides with the definition in (2.6) if $\text{supp}(\sigma) \subset \Gamma$.

To avoid minor difficulties, we assume hereafter that the length of Γ is less than one. Then, again following Kleiner [5, p. 133], we obtain

$$|[\mu_K - \mu_n] - [\mu_K - \mu_n^*]| \leq c_1/n^3, \tag{3.14}$$

for some constant c_1, and

$$\|\mu_K\|^2 \geq \|\mu^*\|^2 \geq \|\mu_K\|^2 - \frac{1}{(n+1)^2 \operatorname{cap}(K)}, \tag{3.15}$$

where μ^* is the equilibrium measure for Γ^* (cf. (3.12)), and where, for a signed measure on Γ^*, we use the energy norm

$$\|\sigma\| := \left[\iint \log|z-t|^{-1}\, d\sigma(z)\, d\sigma(t)\right]^{1/2}$$

(see p. 82 of [7]). Now since

$$\langle \mu^*, \mu^* - \mu_n^*\rangle := \iint \log|z-t|^{-1}\, d\mu^*(z)\, d(\mu^* - \mu_n^*)(t)$$
$$= \int U(\mu^*; t)\, d(\mu^* - \mu_n^*)(t),$$

where $U(\mu^*; t)$ is the logarithmic potential of μ^* (so that $U(\mu^*; t) = \|\mu^*\|^2$ for all $t \in \Gamma^*$), we obtain

$$\langle \mu^*, \mu^* - \mu_n^*\rangle = 0$$

and consequently

$$\|\mu^* - \mu_n^*\|^2 = \|\mu_n^*\|^2 - \|\mu^*\|^2. \tag{3.16}$$

Similarly, we obtain

$$\|\mu_K - \mu^*\|^2 = \|\mu_K\|^2 - \|\mu^*\|^2,$$

and by (3.15) we get

$$\|\mu_K - \mu^*\|^2 \le \frac{1}{(n+1)^2 \operatorname{cap}(K)}. \tag{3.17}$$

Now,

$$\|\mu_n^*\|^2 = \iint \log|z-t|^{-1}\, d\mu_n^*(z)\, d\mu_n^*(t)$$
$$= \sum_{j=0}^{n} \int_{K_{j,n}} \left(\sum_{i=0}^{n} \int_{K_{i,n}} \log|z-t|^{-1}\, d\mu_n^*(z)\right) d\mu_n^*(t).$$

Since, for fixed t, the function $\log|z-t|^{-1}$ is superharmonic in z, we have

$$\int_{K_{i,n}} \log|z-t|^{-1}\, d\mu_n^*(z) = \frac{(n+1)^2}{2\pi} \int_{K_{i,n}} \log|z-t|^{-1} |dz|$$
$$\le \frac{1}{n+1} \log|t - z_{i,n}|^{-1}.$$

Hence

$$\|\mu_n^*\|^2 \le \frac{1}{n+1} \sum_{i,j=0}^{n} \int_{K_{j,n}} \log|t - z_{i,n}|^{-1}\, d\mu_n^*(t),$$

and since $\log|t - z_{i,n}|^{-1}$ is superharmonic in t it follows that

$$\int_{K_{j,n}} \log|t - z_{i,n}|^{-1}\, d\mu_n^*(t) \le \frac{1}{n+1} \log|z_{j,n} - z_{i,n}|^{-1}, \qquad \text{for } i \ne j,$$

3. P. Erdös, P. Turán (1950): *On the distribution of roots of polynomials.* Ann. of Math., **51**:105–119.
4. M. I. Kadec (1963): *On the distribution of points of maximum deviation in the approximation of continuous functions by polynomials.* Tran. Amer. Math. Soc., **26**:231–234.
5. W. Kleiner (1964): *Sur l'approximation de la représentation conforme par la méthode des points extremaux de M. Lega.* Ann. Polon. Math., **XIV**:131–140.
6. A. Kroó, E. B. Saff (1988): *The density of extreme points in complex polynomial approximation.* Proc. Amer. Math. Soc., **103**:203–209.
7. N. S. Landkof (1972): Foundations of Modern Potential Theory. Heidelberg: Springer-Verlag.
8. G. G. Lorentz (1984): *Distribution of alternation points in uniform polynomial approximation.* Proc. Amer. Math. Soc., **92**:401–403.
9. Chr. Pommerenke (1975): Univalent Functions. Göttingen: Vandenhoeck and Ruprecht.
10. M. Tsuji (1958): Potential Theory in Modern Function Theory, 2nd edn., New York: Chelsea.
11. J. L. Walsh (1935): Interpolation and Approximation by Rational Functions in the Complex Domain. American Mathematical Society Colloquium Publications, vol. 20. (1969): 5th edn.

H.-P. Blatt
Lehrstuhl fur Mathematik-Angewandte Mathematik
Katholische Universität Eichstätt
Ostenstrasse 18
D-8078 Eichstätt
F.R.G.

E. B. Saff
Institute for Constructive Mathematics
Department of Mathematics
Tampa
Florida 33620
U.S.A.

V. Totik
Bolyai Institute
Aradi V. tere 1
6720 Szeged
Hungary