The Distribution of Extreme Points in Best Complex Polynomial Approximation

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Abstract. Let $K$ be a compact point set in the complex plane having positive logarithmic capacity and connected complement. For any $f$ continuous on $K$ and analytic in the interior of $K$ we investigate the distribution of the extreme points for the error in best uniform approximation to $f$ on $K$ by polynomials. More precisely, if

$$A_n(f) := \{ z \in K : |f(z) - p_n^*(f; z)| = \| f - p_n^*(f) \|_K \},$$

where $p_n^*(f)$ is the polynomial of degree $\leq n$ of best uniform approximation to $f$ on $K$, we show that there is a subsequence $\{n_k\}$ with the property that the sequence of $(n_k + 2)$-point Fejér subsets of $A_{n_k}$ has limiting distribution (as $k \to \infty$) equal to the equilibrium distribution for $K$. Analogues for weighted approximation are also given.

1. Introduction

Let $K$ be a compact set in the complex plane $C$ and let $C_A(K)$ be the set of continuous complex-valued functions on $K$ that are analytic in the interior of $K$ endowed with the supremum norm $\| \cdot \|_K$. For $f \in C_A(K)$ we denote by $p_n^*(f)$ its best approximant out of $\Pi_n$, the set of polynomials of degree at most $n$, and we set

$$A_n(f) := \{ z \in K : |f(z) - p_n^*(f; z)| = \| f - p_n^*(f) \|_K \}.$$

We investigate the distribution of the points in $A_n(f)$ as $n \to \infty$.

Kadec [4] established that if $f \in C[-1, 1]$ is real-valued on $K = [-1, 1]$, then any $(n_k + 2)$-point alternating subset of $A_{n_k}(f)$ has the arcsine distribution for a suitable subsequence $\{n_k\}$ of the natural numbers, but Lorentz [8] showed that this is not necessarily true for every subsequence. Kadec even proved that the rate of this convergence is $O(n_k^{-1/2+\epsilon})$ for every $\epsilon > 0$, which was subsequently sharpened by Blatt and Lorentz [1] to $O(\sqrt{\log n_k}/n_k)$. All these results are based on the fact that consecutive points in a maximal alternation set and the

AMS classification: 41A50, 30E10.
Key words and phrases: Polynomial approximation, Best polynomial approximants, Extreme points, Equilibrium distribution, Weighted approximation.

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zeros of $p^*_n(f) - p^*_{n-1}(f)$ interlace; a fact that has no analogue for complex-valued functions. Thus, the asymptotic behavior of some subset of $A_n(f)$ (in the spirit of Kadec's result) has been open even for $K = [-1, 1]$ when $f$ is complex-valued. The only known result for the general complex case belongs to Kőro and Saff [6] who verified, for regular sets $K$, the denseness of $\{A_n(f)\}$ on the boundary $\partial K$ of $K$ for some subsequence $\{n_k\}$.

In this paper we show that the Fekete points of $A_{n_k}(f)$ are distributed, for a suitable $\{n_k\}$, like the equilibrium distribution of $K$. The same is true for the "maximal value" points of the Chebyshev polynomials for $K$. Furthermore, for the case when $K$ is bounded by a smooth Jordan curve $\Gamma$, we establish that for some $\{n_k\}$ the relative density of the Fekete points of $A_{n_k}(f)$ in any subarc of $\Gamma$ converges to the equilibrium measure of the subarc with rate $O((\log n)/\sqrt{n})$. Finally, we indicate how our results can be extended to weighted approximation.

A surprising corollary is that if $w$ is a nonnegative continuous weight on the closed unit disk $K$, with $w$ positive on the open unit disk, then every point of the boundary $|z| = 1$ is a limit point of extreme points for best polynomial approximation of any $f \in C_A(K)$ with weight $w$, even if $w$ vanishes identically on $|z| = 1$.

2. Results

To state our results we introduce some concepts and notations from potential theory. If $K \subseteq \mathbb{C}$ is compact, then $\text{cap}(K)$ denotes the logarithmic capacity of $K$ (see [10]). We will always assume that the complement of $K$ is connected and $\text{cap}(K) > 0$. It is well known that

$$\text{cap}(K) = \exp(-I(\mu_K)),$$

where $I(\mu_K)$ is the smallest value of the energy integral

$$I(\mu) := \iint \log|z-t|^{-1} \, d\mu(t) \, d\mu(z),$$

where $\mu$ runs through all unit measures supported on $K$. There is a unique measure $\mu_K$, called the equilibrium measure of $K$, for which this minimum is attained.

If $E$ is any compact set, we denote by $\mathscr{F}_n(E)$ any $n$-point subset $S$ of $E$ for which the Vandermonde expression

$$V(S) := \left[ \prod_{z,t \in S, z \neq t} |z-t| \right]^{1/2}$$

is as large as possible. The points in $\mathscr{F}_n(E)$ are called Fekete points of $E$. Their relation to the capacity is given by

$$\lim_{n \to \infty} V(\mathscr{F}_n(K))^{2/n(n-1)} = \text{cap}(K).$$

With any finite point set $S$ we associate $\nu_S$, the counting measure on $S$; that is,

$$\nu_S(E) := \text{number of points of } S \cap E.$$

Finally, the weak-star convergence of measures will be denoted by $\rightharpoonup$. 
Our main result is

**Theorem 1.** Let \( K \) be a compact subset of \( \mathbb{C} \) with connected complement and positive capacity, \( f \in C_{\Lambda}(K) \), and let \( A_n(f) \) be the set of extreme points defined in (1.1). Then, for some subsequence \( \mathcal{N} \) of the natural numbers,

\[
\frac{1}{n+1} \nu_{\mathcal{S}_{n+1}(A_{n-1}(f))} \Rightarrow \mu_K, \quad n \in \mathcal{N}, \quad n \to \infty,
\]

where \( \mu_K \) is the equilibrium measure for \( K \).

We note that the counterexample given in [6] implies that (2.2) need not be true for every \( \mathcal{N} \). Note also that if \( A_{n-1}(f) \) happens to be an \( (n+1) \)-point set for \( n \in \mathcal{N} \), then (2.2) gives the limiting distribution of these sets \( A_{n-1}(f) \). However, \( A_{n-1}(f) \) can actually coincide with \( \partial K \) for every \( n \); as a nontrivial example, consider \( K := \{ z : |z| \leq 1 \} \cup \{ 2 \} \) and

\[
f(z) := \begin{cases} 0 & \text{if } |z| \leq 1, \\ 1 & \text{if } z = 2. \end{cases}
\]

Our second result concerns the points where the Chebyshev polynomials assume their maximum absolute value. For any compact set \( K \subset \mathbb{C} \), we set

\[
t_n(K) := \inf_{p_n \in \Pi_n} \| z^n - p_{n-1} \|_K = \inf_{z^n + \cdots + \Pi_n} \| z^n + \cdots \|_K.
\]

As is well known (see [10]), the constants \( t_n(K) \) are related to the capacity of \( K \) via the formula

\[
\lim_{n \to \infty} t_n(K)^{1/n} = \text{cap}(K).
\]

If \( K \) contains at least \( n \) points, there is a unique monic polynomial \( T_n^* = T_n^*(K; z) = z^n + \cdots \in \Pi_n \) such that

\[
t_n(K) = \| T_n^* \|_K.
\]

We call \( T_n^* \) the **Chebyshev polynomial** of degree \( n \) associated with \( K \).

Our next result concerns the extremal point sets

\[
A_n^*(K) := \{ z \in K : |T_n^*(K; z)| = t_n(K) \}.
\]

We shall prove

**Theorem 2.** Let \( K \) be compact with \( \text{cap}(K) > 0 \). Then

\[
\frac{1}{n+1} \nu_{\mathcal{S}_{n+1}(A_n^*(K))} \Rightarrow \mu_K \quad \text{as} \quad n \to \infty.
\]

We emphasize that, unlike Theorem 1, the weak-star convergence in Theorem 2 holds for the full sequence of nonnegative integers.
For the case when the boundary of $K$ is sufficiently smooth, we shall prove a result concerning the rate of convergence: let $K$ be a closed region bounded by a Jordan curve $\Gamma$, where $\Gamma$ has a continuous tangent. Moreover, assume that the angle of inclination $\theta(s)$ of the tangent to $\Gamma$ as a function of the arclength $s$ along $\Gamma$ satisfies a Hölder condition; that is,

\begin{equation}
|\theta(s) - \theta(s')| \leq L|s - s'|^\alpha,
\end{equation}

where $L$, $\alpha$ are constants and $0 < \alpha \leq 1$. Then we say that $\Gamma$ belongs to the class $C^{1,H}$.

As in [5], we define, for any signed measure $\sigma$ on $\Gamma$, the number

\begin{equation}
[\sigma] := \sup |\sigma(\gamma)|,
\end{equation}

where the supremum is taken over all subarcs $\gamma$ of the boundary $\Gamma$. Then the rate of convergence of

\begin{equation}
\mu_n := \frac{1}{n+1} \nu_{\tau_n(A_n(f))}
\end{equation}

to the equilibrium measure $\mu_K$ can be estimated as follows.

**Theorem 3.** Let $K$ be a closed Jordan region with boundary $\Gamma \in C^{1,H}$, and let $f \in C_K(K)$. Then there is a constant $c$ (depending on $\Gamma$) and a subsequence $N$ of natural numbers such that

\begin{equation}
[\mu_K - \mu_n] \leq \frac{\log n}{\sqrt{n}},
\end{equation}

whenever $n \in N$.

This means that when $K$ is a sufficiently smooth Jordan region, the weak-star convergence established in Theorem 1 can be quantified and the rate of convergence is at least $(\log n)/\sqrt{n}$. It remains open to determine the best possible rate of convergence.

If $K$ is a compact subset of the real line, a quantified version of Theorem 1 can also be given. For example, it is possible to show that when $K$ is the finite union of compact intervals, there exists a constant $c$ and a subsequence $N$ of natural numbers such that, for any interval $[a, b]$,

\begin{equation}
|(\mu_K - \mu_n)[a, b]| \leq c \left(\frac{\log n}{n}\right)^{1/3},
\end{equation}

whenever $n \in N$. Since the proof of (2.9) depends substantially on generalizations of the Erdős-Turán inequalities [3], we leave the verification of (2.9) for another occasion.

Finally, we mention that our results hold for weighted approximation. Let $w \geq 0$ be a continuous weight function on $K$ and set

\[ A_n(f, w) := \{ z \in K : w(z) |f(z) - p_n^*(f ; z)w| = \| w(f - p_n^*(f))w \|_K \}, \]
where \( p^w_n(f)_w \) denotes the best weighted approximation of \( f \) out of \( \Pi_n \) with weight \( w \). If \( E \) is a compact subset of \( K \), then \( \mathcal{F}_n(E)_w \) denotes an \( n \)-point subset \( \{z_0, \ldots, z_{n-1}\} \) of \( E \) maximizing the absolute value of the determinant
\[
|w(z_i)z_j|_{0 \leq i \leq n-1, 0 \leq j \leq n-1}.
\]

With these notations we can prove the following generalization of Theorem 1.

**Theorem 4.** Let \( K \) be compact with \( \text{cap}(K) > 0 \) and connected complement, and assume that \( f \in C_n(K) \). Furthermore, let \( w \) be a nonnegative continuous weight function on \( K \) such that the (inner logarithmic) capacity of the set \( \{z \in K : w(z) > 0\} \) equals the capacity of \( K \). Then, for some subsequence \( \mathcal{N} \) of the natural numbers,
\[
\frac{1}{n+1} \nu_{S_n+1} \xrightarrow{\mathcal{N}} \mu_K \quad \text{as} \quad n \to \infty, \quad n \in \mathcal{N},
\]
where \( S_{n+1} := \mathcal{F}_{n+1}(A_{n-1}(f,w))_w \). In particular, this holds if \( w \) vanishes only on a set of capacity zero.

We also mention that the weighted analogue of Theorem 2 holds as well.

**Corollary.** With the assumptions of Theorem 4, every point in the support of \( \mu_K \) is a limit point of the sets \( A_n(f,w) \).

We remark that the support of \( \mu_K \) lies on the boundary \( \partial K \) of \( K \), and so the corollary means that most of the points of \( \partial K \) (i.e., except for a set of capacity zero) attract extremal points. This is fairly surprising since \( w \) can vanish on a large part of \( \partial K \); consider, for example,
\[
w(z) = 1 - |z|
\]
on the unit disk \( \{z : |z| \leq 1\} \).

## 3. Proofs

Our proofs are based on two simple lemmas.

**Lemma 1.** Suppose \( f \in C_n(K) \) and the complement of \( K \) is connected. Then there exists a subsequence \( \mathcal{N} \) of the natural numbers with the property
\[
t_n(A_{n-1}(f)) \geq \frac{1}{n^2} t_n(K) \quad \text{for} \quad n \in \mathcal{N},
\]
where \( t_n \) is defined in (2.3).

For the proof see Lemma 2.3 and (2.18) of [6].

**Lemma 2.** For any compact set \( E \) containing at least \( n+1 \) points,
\[
t_n(\mathcal{F}_{n+1}(E)) \geq \frac{1}{n+1} t_n(E).
\]

For the proof see Section 7.9, Lemma, p. 177, of [11].
Combining Lemmas 1 and 2 we see that
\begin{equation}
 t_n(\mathscr{F}_{n+1}(A_{n-1}(f))) \geq (n+1)^{-3} t_n(K), \quad n \in \mathcal{N}.
\end{equation}
Let \(\mathscr{F}_{n+1}(A_{n-1}(f)) = \{z_{0,n}, \ldots, z_{n,n}\}\). Since this is an \((n+1)\)-point set, we can easily compute its \(n\)th Chebyshev polynomial \(T_n^*\) and Chebyshev number \(t_n(\mathscr{F}_{n+1}(A_{n-1}(f))) = t_n\). In fact, \(T_n^*\) must have the same absolute value, \(t_n\), at every \(z_{k,n}\), and must have leading coefficient 1, which easily yields
\begin{equation}
 T_n^*(z) = \left(\sum_{k=0}^{n} \frac{\omega_{n+1}(z) \text{ sgn } \omega'_{n+1}(z_k,n)}{\omega'_{n+1}(z_k,n)(z - z_k,n)}\right) \left(\sum_{k=0}^{n} \frac{1}{\omega'_{n+1}(z_k,n)}\right)^{-1},
\end{equation}
where
\begin{equation}
 \omega_{n+1}(z) := \prod_{k=0}^{n} (z - z_k,n)
\end{equation}
and
\[ t_n = \left(\sum_{k=0}^{n} \frac{1}{\omega'_{n+1}(z_k,n)}\right)^{-1}. \]
Indeed, if \(T_n\) is any \(n\)th degree polynomial with
\[ T_n(z_{k,n}) = \varepsilon_{k,n} t, \quad 0 \leq k \leq n, \]
with \(t > 0\) and \(|\varepsilon_{k,n}| = 1\) that has leading coefficient 1, then, by the Lagrange interpolation formula,
\begin{equation}
 T_n(z) = \sum_{k=0}^{n} \frac{\omega_{n+1}(z)}{\omega'_{n+1}(z_k,n)(z - z_k,n)} \varepsilon_{k,n} t
\end{equation}
and
\[ t \sum_{k=0}^{n} \frac{\varepsilon_{k,n}}{\omega'_{n+1}(z_k,n)} = 1, \]
which implies
\[ t \geq \left(\sum_{k=0}^{n} \frac{1}{\omega'_{n+1}(z_k,n)}\right)^{-1}. \]
The expression for \(t_n\) and (3.1) show that
\begin{equation}
 |\omega_{n+1}(z_{k,n})| = \prod_{j=0}^{n} |z_{k,n} - z_{j,n}| \geq (n+1)^{-3} t_n(K)
\end{equation}
for every \(0 \leq k \leq n\) and \(n \in \mathcal{N}\).

**Proof of Theorem 1.** We keep the above notations and give two proofs, a complex function theoretical one and a real variable argument.

Let \(\mathscr{F}_{n+1}(K) = \{y_{0,n+1}, \ldots, y_{n,n+1}\}\) be a set of \((n+1)\)th Fekete points of \(K\). The polynomials
\[ \tau_{n+1}(z) = \tau_{n+1}(K, z) := \prod_{k=0}^{n} (z - y_{k,n+1}) \]
are called *Fekete polynomials* of $K$. By the Lagrange interpolation formula, we have for the polynomials $w_{n+1}$ of (3.2)

$$
\tau_{n+1}(z) - w_{n+1}(z) = \sum_{k=0}^{n} \left( \tau_{n+1}(z_{k,n}) - w_{n+1}(z_{k,n}) \right) \frac{w_{n+1}(z)}{\omega_{n+1}(z_{k,n})(z-z_{k,n})}
$$

(note that $\tau_{n+1} - w_{n+1}$ is of degree at most $n$). Since $w_{n+1}(z_{k,n}) = 0$ for every $k$ and

$$
|\tau_{n+1}(z_{k,n})| \leq \|\tau_{n+1}\|_{K} = d_{n+1}(K),
$$

we see that for the $n$'s satisfying (3.3), i.e., for $n \in \mathcal{N}$,

$$
|\tau_{n+1}(z) - w_{n+1}(z)| \leq (n+1)^{3} \frac{d_{n+1}(K)}{t_{n}(K)} |w_{n+1}(z)| \sum_{k=0}^{n} \frac{1}{|z-z_{k,n}|},
$$

$$
\left| \frac{\tau_{n+1}(z)}{w_{n+1}(z)} \right| \leq (n+1)^{3} \frac{d_{n+1}(K)}{t_{n}(K)} \sum_{k=0}^{n} \frac{1}{|z-z_{k,n}|} + 1,
$$

which implies that, for $n \in \mathcal{N}$,

$$
\frac{1}{n+1} \log \left| \frac{\tau_{n+1}(z)}{w_{n+1}(z)} \right| \leq C_{H} \varepsilon_{n+1}, \quad z \in H,
$$

where

$$
\varepsilon_{n+1} := \frac{\log(n+1)}{n+1} + \frac{1}{n+1} \log \frac{d_{n+1}(K)}{t_{n}(K)},
$$

uniformly on every closed subset $H$ of $\overline{\mathbb{C}} \setminus K$, the complement of $K$ with respect to the closed Riemann sphere. By Lemma 11.2 of [9]

$$
\lim_{n \to \infty} d_{n}(K)^{1/n} = \text{cap}(K)
$$

which, together with (2.4), imply that $\varepsilon_{n} \to 0$ as $n \to \infty$. It is known (see Theorem 11.1 of [9] and Chapters I and III of [10]) that, for $z \in \mathbb{C} \setminus K$,

$$
\lim_{n \to \infty} \frac{1}{n+1} \log |\tau_{n+1}(z)| = g(z) + \log[\text{cap}(K)],
$$

where $g$ is Green's function for $\mathbb{C} \setminus K$ with the pole at infinity.

Since all the zeros of $\omega_{n+1}$ and $\tau_{n+1}$ lie in $K$, the functions

$$
\frac{1}{n+1} \log \left| \frac{\tau_{n+1}(z)}{\omega_{n+1}(z)} \right|, \quad n \in \mathcal{N},
$$

are harmonic in $\overline{\mathbb{C}} \setminus K$, vanish at $\infty$, and have the upper bound $C_{H} \varepsilon_{n+1}$ on every closed subset $H$ of $\mathbb{C} \setminus K$ (cf. (3.4)). Thus we can apply Harnack's inequality and get that, uniformly on closed subsets $H'$ of the interior of $H$,

$$
\left| \frac{1}{n+1} \log \left| \frac{\tau_{n+1}(z)}{\omega_{n+1}(z)} \right| \right| \leq C_{H} \varepsilon_{n+1}, \quad n \in \mathcal{N}.
$$
Making use of (3.5) we finally obtain that, uniformly on compact subsets of \( C \setminus K \),
\[
(3.6) \quad \lim_{n \to \infty} \frac{1}{n+1} \log |\omega_{n+1}(z)| = g(z) + \log(\text{cap}(K)).
\]
Denoting the potential corresponding to a measure \( \mu \) by \( U(\mu; z) \); that is,
\[
U(\mu; z) := \int \log |z - t|^{-1} d\mu(t),
\]
then (3.6) can be written in the equivalent form
\[
(3.7) \quad \lim_{n \to \infty} \frac{1}{n+1} \nu_{\mathcal{F}_n(A_{n-1}(f))} = U(\mu; z), \quad z \in C \setminus K,
\]
where
\[
\mu_n := \frac{1}{n+1} \nu_{\mathcal{F}_n(A_{n-1}(f))}.
\]
Proceeding as in the proof of Theorem 2.1 of [2], we let \( \mu \) be a weak-star limit of \( \{\mu_n\}_{n \in \mathcal{N}} \); that is, there exists \( \mathcal{N}_1 \subseteq \mathcal{N} \) such that
\[
\mu_n \rightharpoonup \mu \quad \text{as} \quad n \to \infty, \quad n \in \mathcal{N}_1.
\]
Note that since the Fekete points of \( A_{n-1}(f) \) lie on the boundary \( \partial K \) of \( K \), then \( \text{supp}(\mu) \subseteq \partial K \) and \( \mu(\partial K) = 1 \). From (3.7) we get
\[
(3.8) \quad U(\mu; z) = U(\mu_K; z), \quad z \in C \setminus K.
\]
Further, since \( U(\mu_K; z) \leq I(\mu_K) \) for all \( z \in C \) (see Theorem III.12 of [10]) and since \( U(\mu; z) \) is lower semicontinuous, we obtain from (3.8) that
\[
U(\mu; z) \leq I(\mu_K), \quad z \in \partial K.
\]
Integrating this inequality with respect to \( d\mu \) over \( \partial K \) we see that the energy \( I(\mu) \) satisfies \( I(\mu) \leq I(\mu_K) \). Thus, by the unicity of \( \mu_K \), we get \( \mu = \mu_K \) and so the whole sequence \( \{\mu_n\}_{n \in \mathcal{N}} \) converges to \( \mu_K \) in the weak-star topology.

Our second proof also starts with (3.3). The inequality implies that
\[
V(\mathcal{F}_{n+1}(A_{n-1}(f))) = \left( \prod_{k=0}^{n} \prod_{j=0}^{n} |z_{k,n} - z_{j,n}| \right)^{1/2} \geq (n+1)^{-3(n+1)/2} (\text{cap}(K))^{(n+1)/2},
\]
and so taking into account (2.4), all we have to prove is the following proposition which is interesting in itself.

**Proposition.** Assume \( K \) is compact with \( \text{cap}(K) > 0 \) and, for every \( n \in \mathcal{N} \), where \( \mathcal{N} \) is a subsequence of the natural numbers, let
\[
Z_n := \{z_{1,n}, \ldots, z_{n,n}\}
\]
be an \( n \)-point subset of \( K \). If
\[
(3.10) \quad \liminf_{n \to \infty} \frac{1}{n} V(Z_n)^{2/n(n-1)} \geq \text{cap}(K),
\]
then

\[ \frac{1}{n} \nu_{Z_n} \Rightarrow \mu_\mathcal{K} \quad \text{as} \quad n \to \infty, \quad n \in \mathcal{N}. \]

Recall that \( V(Z_n) \) was defined in (2.1).

**Proof.** Let \( \lambda_n := (1/n) \nu_{Z_n} \) and let \( \lambda \) be a weak-star limit point of \( \{\lambda_n\}_{n \in \mathcal{N}} \), say

\[ \lambda_n \xrightarrow{\text{weak-star}} \lambda \quad \text{as} \quad n \to \infty, \quad n \in \mathcal{N}_1 \subseteq \mathcal{N}. \]

We have to show that \( \lambda = \mu_\mathcal{K} \). First we mention that \( \lambda \) does not have point masses. In fact, if we have \( \lambda([z_0]) = c > 0 \), then for every \( \delta > 0 \) and large \( n \in \mathcal{N}_1 \), there would be at least \((c/2)n\) points from \( Z_n \) in a \( \delta \) neighborhood of \( z_0 \) and so we would have

\[ V(Z_n) \leq \delta^{(c/2)n-1/2} (1 + \text{diam} \ K)^{n^2}. \]

But, for small \( \delta \), this contradicts (3.10).

Thus, by Tonelli's theorem, we get that

\[ I(\lambda) = \int \int \log \frac{1}{|z - t|} \, d\lambda(z) \, d\lambda(t) = \int \int \log \frac{1}{|z - t|} \, d\lambda(z) \, d\lambda(t). \]

Set

\[ g_\varepsilon(t) := \begin{cases} \log \frac{1}{t} & \text{if} \quad t > 2\varepsilon, \\ 0 & \text{if} \quad t < \varepsilon, \\ \frac{1}{\varepsilon} (t - \varepsilon) \log \frac{1}{2\varepsilon} & \text{if} \quad \varepsilon \leq t \leq 2\varepsilon. \end{cases} \]

Then \( g_\varepsilon \) is continuous and, as \( \varepsilon \to 0^+ \), tends monotonically to \( \log(1/t) \) for \( t > 0 \).

Thus, by the monotone convergence theorem, (3.10), and (3.11), we have for the energy integral

\[ I(\lambda) = \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \int \int g_\varepsilon(|z - t|) \, d\lambda_n(z) \, d\lambda_n(t), \quad n \in \mathcal{N}_1, \]

\[ \leq \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \int \int \log \frac{1}{|z - t|} \, d\lambda_n(z) \, d\lambda_n(t) \]

\[ = \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{n^2} \log \frac{1}{V(Z_n)^2} \leq \log \frac{1}{\text{cap}(K)} = I(\mu_\mathcal{K}). \]
Each \( \lambda_n, n \in \mathbb{N}_1 \), is supported on \( K \) and \( \lambda_n(K) = 1 \). Therefore, the weak-star limit \( \lambda \) also has these properties, and so \( \lambda = \mu_K \) follows from the unicity of \( \mu_K \).

**Proof of Theorem 2.** It is well known and easy to see that \( \tau_n(A^*_{n}(K)) = \tau_n(K) \). Thus, by Lemma 2,

\[
\tau_n(F_{n+1}(A^*_{n}(K))) \geq (n+1)^{-1} \tau_n(K), \quad n = 0, 1, \ldots
\]

Since the preceding proofs used only the estimate (3.1), they can be applied in the present situation as well and we are done.

**Proof of Theorem 3.** Let \( z_{0, n}, z_{1, n}, \ldots, z_{n, n} \) be an \( (n+1) \)-point Fekete set of \( A_{n-1}(f) \) and denote by \( \mu_n \) the discrete unit measure with mass \( 1/(n+1) \) at each point \( z_{i,n} \) (cf. (2.7)).

We follow a technique of Kleiner [5] and consider the circles \( K_{i,n} = K(z_{i,n}, 1/(n+1)^3) \) with center at \( z_{i,n} \) and radius \( 1/(n+1)^3 \). We denote by \( \mu_n^* \) the unit measure that has constant density \((n+1)^3/\pi\) on \( \bigcup_{i=0}^n K_{i,n} \).

Furthermore, we measure the angle between two lines in the absolute sense; that is, such an angle is between 0 and \( \pi/2 \). We say that a subarc \( \gamma \) of \( \Gamma \) is inclined to the line \( l \) with angle \( \geq \varepsilon \) if all tangents to \( \gamma \) have this property. Now, if \( z_0 \in \Gamma \), there exists a real number \( r = r(z_0) > 0 \) such that the subarc

\[
\Gamma \cap \{ z : |z - z_0| \leq r \}
\]

is inclined to the normal line \( l_0 \) of \( \Gamma \) in \( z_0 \) with an angle \( \geq \pi/3 \).

Let \( \gamma \) be a subarc of \( \Gamma \) with endpoints \( a, b \). Define \( A \) (resp. \( B \)) as the set of points \( \xi \) on the normal line in \( a \) (resp. \( b \)) such that \( |\xi - a| \leq r(a) \) (resp. \( |\xi - b| \leq r(b) \)). Then, for \( n \) large, the point set

\[
\Gamma^* := \Gamma \cup K_{0, n} \cup K_{1, n} \cup \cdots \cup K_{n, n}
\]

is separated by \( A \cup B \) into two connected components, one of which, say \( \gamma^* \), contains the subarc \( \gamma \).

Now, for any (bounded) signed measure \( \sigma \) on \( \Gamma^* \) we define

\[
[\sigma] := \sup_{\gamma \in \Gamma} |\sigma(\gamma^*)|,
\]

which coincides with the definition in (2.6) if \( \text{supp}(\sigma) \subseteq \Gamma \).

To avoid minor difficulties, we assume hereafter that the length of \( \Gamma \) is less than one. Then, again following Kleiner [5, p. 133], we obtain

\[
[\mu_K - \mu_n] - [\mu_K - \mu_n^*] \leq c_1/n^3,
\]

for some constant \( c_1 \), and

\[
\|\mu_K\|^2 \geq \|\mu_n^*\|^2 \geq \|\mu_K\|^2 - \frac{1}{(n+1)^3 \text{cap}(K)}.
\]
where \( \mu^* \) is the equilibrium measure for \( \Gamma^* \) (cf. (3.12)), and where, for a signed measure on \( \Gamma^* \), we use the energy norm
\[
\| \sigma \| := \left( \iint \log |z-t|^{-1} \, d\sigma(z) \, d\sigma(t) \right)^{1/2}
\]
(see p. 82 of [7]). Now since
\[
\langle \mu^*, \mu^* - \mu_n^* \rangle := \iint \log |z-t|^{-1} \, d\mu^*(z) \, d(\mu^* - \mu_n^*)(t) = \int U(\mu^*; t) \, d(\mu^* - \mu_n^*)(t),
\]
where \( U(\mu^*; t) \) is the logarithmic potential of \( \mu^* \) (so that \( U(\mu^*; t) = \| \mu^* \|^2 \) for all \( t \in \Gamma^* \)), we obtain
\[
\langle \mu^*, \mu^* - \mu_n^* \rangle = 0
\]
and consequently
\[
(3.16) \quad \| \mu^* - \mu_n^* \|^2 = \| \mu_n^* \|^2 - \| \mu^* \|^2.
\]
Similarly, we obtain
\[
\| \mu_K - \mu^* \|^2 = \| \mu_K \|^2 - \| \mu^* \|^2,
\]
and by (3.15) we get
\[
(3.17) \quad \| \mu_K - \mu^* \|^2 \leq \frac{1}{(n+1)^2 \text{cap}(K)}.
\]
Now,
\[
\| \mu_n^* \|^2 = \iint \log |z-t|^{-1} \, d\mu_n^*(z) \, d\mu_n^*(t)
= \sum_{j=0}^{n} \int_{K_{i,n}} \left( \sum_{i=0}^{n} \int_{K_{j,n}} \log |z-t|^{-1} \, d\mu_n^*(z) \right) \, d\mu_n^*(t).
\]
Since, for fixed \( t \), the function \( \log |z-t|^{-1} \) is superharmonic in \( z \), we have
\[
\int_{K_{i,n}} \log |z-t|^{-1} \, d\mu_n^*(z) = \frac{(n+1)^2}{2\pi} \int_{K_{i,n}} \log |z-t|^{-1} \, |dz|
\leq \frac{1}{n+1} \log |t-z_{i,n}|^{-1}.
\]
Hence
\[
\| \mu_n^* \|^2 \leq \frac{1}{n+1} \sum_{i,j=0}^{n} \int_{K_{i,n}} \log |t-z_{i,n}|^{-1} \, d\mu_n^*(t),
\]
and since \( \log |t-z_{i,n}|^{-1} \) is superharmonic in \( t \) it follows that
\[
\int_{K_{i,n}} \log |t-z_{i,n}|^{-1} \, d\mu_n^*(t) \leq \frac{1}{n+1} \log |z_{j,n} - z_{i,n}|^{-1}, \quad \text{for} \ i \neq j,
\]


