STRONG ASYMPTOTICS FOR L_p EXTREMAL POLYNOMIALS ($1 < p \leq \infty$) ASSOCIATED WITH WEIGHTS ON [-1,1]

by

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Abstract. While Szegö type asymptotics of orthonormal polynomials are classical, there has been a longstanding lack of corresponding results for L_p extremal polynomials, $p \neq 2$. In particular, in a 1969 paper, Widom raised the question of $p = \infty$. Here we fill some of the gaps for $1 < p \leq \infty$.

1. Introduction

Let $0 < p \leq \infty$, and $w \in L_p[-1,1]$ be non-negative in $[-1,1]$ and positive on a set of positive Lebesgue measure. We can then define for $n = 1,2,3,\ldots$,

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$$(1.1) \qquad E_{np}(w) := \inf_{P \in \mathcal{P}_{n-1}} \|\{x^n - P(x)\}w(x)\|_{L_p[-1,1]} \,,$$

where $\mathcal{P}_{n-1}$ denotes the class of real polynomials of degree at most n-1. It is easily seen that there is at least one monic polynomial $T_{np}(w,x) = x^n + \cdots \in \mathcal{P}_n$ such that

$$(1.2) \qquad \|T_{np}(w,x)w(x)\|_{L_p[-1,1]} = E_{np}(w).$$

We call $T_{np}(w,x)$ an L_p extremal polynomial for w. We define also the normalized extremal polynomials

$$(1.3) \qquad p_{np}(w,x) := T_{np}(w,x)/E_{np}(w),$$

$n = 1,2,3,\ldots$, satisfying

$$(1.4) \qquad \|p_{np}(w,x)w(x)\|_{L_p[-1,1]} = 1.$$

When $p = 2$, $p_{np}(w,x)$ is just the orthonormal polynomial of degree n for the weight w^2.

This paper addresses the asymptotics of $T_{np}(w,x)$ in $\mathbb{C}\backslash[-1,1]$ as $n \to \infty$. Under general conditions on w, Fekete and Walsh [3] and Widom [11] established nth root asymptotics. For example, if $w(x) > 0$ a.e. in $[-1,1]$, their results imply that

$$(1.5) \qquad \lim_{n\to\infty} [T_{np}(w,z)]^{1/n} = \varphi(z)/2,$$

locally uniformly in $\mathbb{C}\backslash[-1,1]$, where

$$(1.6) \qquad \varphi(z) := z + \sqrt{z^2-1} \,, \qquad z \in \mathbb{C}\backslash[-1,1],$$

is the usual conformal map of $\mathbb{C}\backslash[-1,1]$ onto $\{\zeta : |\zeta| > 1\}$. Here the branch of the nth root is chosen so that $[T_{np}(w,z)]^{1/n}$ behaves like z at ∞.

The asymptotics for $T_{np}(w,z)$ itself have proved more elusive. In his 1969 paper, Widom [12, p.205] remarked that even in the case of weights on $[-1,1]$, Szegö type asymptotics for $T_{n\infty}(w,z)$ had not yet been established. While Widom obtained asymptotics for $E_{n\infty}(w)$ and its analogue in more general situations than that treated here, he could not turn these into asymptotics for the polynomials. In this paper, we shall fill this gap, at least for $1 < p \le \infty$, $p \ne 2$.

Of course for $p = 2$, everything is classical: Assuming the Szegö condition,

$$\int_{-1}^{1} \log w(x)\, dx/\sqrt{1-x^2} > -\infty, \tag{1.7}$$

Szegö (see [10]) proved that locally uniformly in $\mathbb{C}\backslash[-1,1]$,

$$\lim_{n\to\infty} p_{n2}(w,z)/\varphi(z)^n = (2\pi)^{-1/2} D^{-2}(F(\phi);\ \varphi(z)^{-1}), \tag{1.8}$$

where

$$F(\phi) := w(\cos\phi)|\sin\phi|^{1/2}, \quad \phi \in \mathbb{R}, \tag{1.9}$$

and $D(\cdot;\cdot)$ is the Szegö function

$$D(F(\phi);\ u) := \exp\left[\frac{1}{4\pi}\int_{-\pi}^{\pi} \log F(\phi)\, \frac{1+ue^{-i\phi}}{1-ue^{-i\phi}}\, d\phi\right], \tag{1.10}$$

$|u| < 1$. Taking $z = \infty$ formally in (1.8), we see also that

$$\begin{aligned} \lim_{n\to\infty} E_{n2}(w)2^n &= (2\pi)^{1/2} D^2(F(\phi);\ 0) \\ &= (2\pi)^{1/2}\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log F(\phi)d\phi\right] \\ &= \pi^{1/2} G[w], \end{aligned} \tag{1.11}$$

after some elementary manipulations, where

$$G[w] := \exp\left[\pi^{-1}\int_{-1}^{1} \log w(x)dx/\sqrt{1-x^2}\right] \tag{1.12}$$

is a geometric mean of w. See [5,8] for recent reviews.

We have used the term "strong asymptotics" in our title to describe (1.8). Other commonly used names are power asymptotic, Szegö asymptotic, or full exterior asymptotic. We can now state our main result:

Theorem 1.1. Let $1 < p < \infty$ and $w \in L_p[-1,1]$ be a non-negative function such that for each $r < \infty$, $w^{-1} \in L_r[-1,1]$. Let

$$\sigma_p := \{\Gamma(1/2)\Gamma((p+1)/2)/\Gamma(p/2+1)\}^{1/p}, \tag{1.13}$$

and

$$F_p(\phi) := w(\cos\phi)|\sin\phi|^{1/p}, \quad \phi \in \mathbb{R}. \tag{1.14}$$

Then

$$\lim_{n\to\infty} E_{np}(w)2^{n-1+1/p} = \sigma_p G[w]. \tag{1.15}$$

Furthermore, uniformly in closed subsets of $\mathbb{C}\backslash[-1,1]$, we have

$$\lim_{n\to\infty} T_{np}(w,z)/\{\varphi(z)/2\}^n = D^{-2}(F_p(\phi);\ \varphi(z)^{-1})D^2(F_p(\phi);\ 0), \tag{1.16}$$

and

$$\lim_{n\to\infty} p_{np}(w,z)/\varphi(z)^n = (2\sigma_p)^{-1}D^{-2}(F_p(\phi);\ \varphi(z)^{-1}). \tag{1.17}$$

For $p = \infty$, we shall prove:

Theorem 1.2. Let $w(x)$ be positive and continuous in $[-1,1]$, and let

$$\sigma_\infty := 1, \tag{1.18}$$

and

$$F_\infty(\phi) := w(\cos\phi), \quad \phi \in \mathbb{R}. \tag{1.19}$$

Then (1.15), (1.16) and (1.17) remain valid for $p = \infty$.

We note that our condition on w^{-1} in Theorem 1.1 implies Szegö's condition and severely restricts the zeros of w: It allows zeros of logarithmic, but not algebraic, strength. We shall prove a lim sup result corresponding to (1.15) under only Szegö's condition - see Theorem 2.2. If we could prove a matching lim inf result, then at least for $2 < p < \infty$, the Szegö-type asymptotics (1.16) and (1.17) would follow under only Szegö's condition (1.7).

This paper is organized as follows: In Section 2, we obtain asymptotics for $E_{np}(w)$ and in Section 3, we obtain the asymptotics for $T_{np}(w,z)$.

2. Asymptotics for $E_{np}(w)$.

First we list Bernstein's explicit formula for $E_{np}(w)$ and $T_{np}(w,x)$ for special w:

Lemma 2.1. *Let* q *be a positive integer and* $S(x)$ *be a polynomial of degree* $2q$, *positive in* $(-1,1)$, *possibly with simple zeros at* ± 1, *and let*

(2.1) $$V(x) := \{(1-x^2)/S(x)\}^{1/2}, \quad x \in (-1,1),$$

and for $0 < p \leq \infty$,

(2.2) $$V_p(x) := (1-x^2)^{-1/(2p)} V(x), \quad x \in (-1,1).$$

Further, let σ_p *be defined by (1.13) for* $0 < p < \infty$ *and by (1.18) for* $p = \infty$, *and let the Szegö function and geometric mean be defined by (1.10) and (1.12) respectively.* *Let* $n \geq q$.

(a) *Then for* $1 \leq p \leq \infty$,

(2.3) $$E_{np}(V_p) = \sigma_p 2^{-n+1-1/p} G[V_p],$$

and for $0 < p < 1$,

(2.4) $$E_{np}(V_p) \leq \sigma_p 2^{-n+1-1/p} G[V_p].$$

(b) *Let*

(2.5) $\tau_n(x) := 2^{-n-1/p}G[V_p]\{z^{-n}D^{-2}(V(\cos\phi);z) + z^nD^{-2}(V(\cos\phi);z^{-1})\}$,

$x := \cos\theta$; $z := e^{i\theta}$; $\theta \in [0,\pi]$. _Then_ $\tau_n(x)$ _is a monic polynomial of degree_ n, _and for_ $0 < p \le \infty$,

(2.6) $\|\tau_n V_p\|_{L_p[-1,1]} = \sigma_p 2^{-n+1-1/p}G[V_p]$,

while for $1 \le p \le \infty$,

(2.7) $T_{np}(V_p,x) = \tau_n(x)$.

Finally,

(2.8) $|\tau_n(x)V(x)| \le 2^{-n+1-1/p}G[V_p]$, $x \in [-1,1]$,

and for $u \in \mathbb{C}\backslash[-1,1]$,

(2.9) $|\tau_n(u)/\{2^{-n-1/p}G[V_p]\varphi(u)^nD^{-2}(V(\cos\phi);\ \varphi(u)^{-1})\} - 1|$

$$\le |\varphi(u)|^{2q-2n-2}.$$

Proof. See Theorem 13.1 in [7], which is just a reformulation of statements in Achieser [1,pp.250-4]. □

We can now prove:

Theorem 2.2. _Let_ $0 < p < \infty$ _and_ $w \in L_p[-1,1]$ _be a non-negative function_. _Then_

(2.10) $\limsup_{n\to\infty} E_{np}(w)2^{n-1+1/p} \le \sigma_p G[w]$,

where, if the integral in the definition (1.12) of G[w] _diverges to_ $-\infty$, _we interpret_ G[w] _as_ 0.

Proof. We remark first that $G[w] < \infty$ is an easy consequence of the arithmetic-geometric mean inequality (cf.[10]) and the fact that $w \in L_p[-1,1]$ implies $w \in L_s[-1,1]$ for $s < p$. For a given n, and

$0 < p < \infty$, let $V_p(x)$ be a function given by (2.2), fulfilling the hypotheses of Lemma 2.1. Let $\epsilon \in (0,1)$ and $w_\epsilon(x) := \max\{w(x), \epsilon\}$, $x \in [-1,1]$. Further let $\mathcal{F}$ be a measurable subset of $[-1,1]$ with $\|w\|_{L_\infty(\mathcal{F})} < \infty$, and let $\mathcal{E} := [-1,1]\backslash\mathcal{F}$. With the notation of Lemma 2.1, we have, for $n \geq q$,

$$(2.11) \quad E_{np}^p(w) \leq E_{np}^p(w_\epsilon) \leq \|\tau_n w_\epsilon\|_{L_p[-1,1]}^p = \|\tau_n w_\epsilon\|_{L_p(\mathcal{F})}^p + \|\tau_n w_\epsilon\|_{L_p(\mathcal{E})}^p$$

$$\leq \|\tau_n V_p\|_{L_p(\mathcal{F})}^p \|V_p^{-1} w_\epsilon\|_{L_\infty(\mathcal{F})}^p$$

$$+ \{2^{-n+1-1/p} G[V_p]\}^p \|V^{-1} w_\epsilon\|_{L_p(\mathcal{E})}^p$$

$$\leq \{\sigma_p 2^{-n+1-1/p} G[V_p]\}^p$$

$$\times \left\{\|V_p^{-1} w_\epsilon\|_{L_\infty(\mathcal{F})}^p + \sigma_p^{-p} \|V^{-1} w_\epsilon\|_{L_p(\mathcal{E})}^p\right\}.$$

Now taking $S(x) := (1-x^2)R^2(x)$ in Lemma 2.1, where $R(x)$ is a polynomial positive in $[-1,1]$, we see that

$$(V_p^{-1} w_\epsilon)(x) = (1-x^2)^{1/(2p)} R(x) w_\epsilon(x)$$

and

$$(V^{-1} w_\epsilon)(x) = R(x) w_\epsilon(x).$$

Let $g(x)$ be a continuous positive function in $[-1,1]$. We can choose a sequence $R = R_{n-1} \in \mathcal{P}_{n-1}$ of polynomials converging uniformly in $[-1,1]$ to g as $n \to \infty$. Then (2.11) yields

$$(2.12) \quad \limsup_{n\to\infty} \left\{E_{np}(w) 2^{n-1+1/p}\right\}^p$$

$$\leq \sigma_p^p G[(1-x^2)^{-1/(2p)} g(x)^{-1}]^p$$

$$\times \left\{\|(1-x^2)^{1/(2p)} g(x) w_\epsilon(x)\|_{L_\infty(\mathcal{F})}^p + \sigma_p^{-p} \|g w_\epsilon\|_{L_p(\mathcal{E})}^p\right\}.$$

Next, we claim that (2.12) holds more generally for any measurable function $g(x)$ that is bounded above and below by positive constants. To see this, note first that for such a $g(x)$, we can choose continuous functions $g_m(x)$, $m = 1, 2, \ldots$, bounded above and below by positive constants independent of n, such that

$$\lim_{m\to\infty} g_m(x) = g(x) \qquad \text{a.e. in } [-1,1].$$

For example, we can choose

$$g_m(x) := \int_{[x-1/m,x+1/m]\cap[-1,1]} g(t)dt \Big/ \int_{[x-1/m,x+1/m]\cap[-1,1]} dt.$$

Furthermore, we can choose a measurable set $\hat{\mathcal{F}} \subset \mathcal{F}$ such that $\text{meas}(\mathcal{F}\setminus\hat{\mathcal{F}})$ is as small as we please and

$$\lim_{m\to\infty} g_m(x) = g(x) \qquad \text{uniformly on } \hat{\mathcal{F}}.$$

Let $\hat{\mathcal{E}} := [-1,1]\setminus\hat{\mathcal{F}}$. As g_m and g are bounded above and below by positive constants independent of n, inequality (2.12) yields for each m,

$$\text{(2.13)} \qquad \limsup_{n\to\infty} \left\{E_{np}(w)2^{n-1+1/p}\right\}^p$$

$$\le \sigma_p^p \, G[(1-x^2)^{-1/(2p)}g_m(x)^{-1}]^p$$

$$\times \left\{\|(1-x^2)^{1/(2p)}g_m(x)w_\epsilon(x)\|^p_{L_\infty(\hat{\mathcal{F}})} + \sigma_p^{-p}\|g_m w_\epsilon\|^p_{L_p(\hat{\mathcal{E}})}\right\} .$$

and so, on letting $m \to \infty$, we get

$$\limsup_{n\to\infty} \left\{E_{np}(w)2^{n-1+1/p}\right\}^p$$

$$\le \sigma_p^p \, G[(1-x^2)^{-1/(2p)}g(x)^{-1}]^p$$

$$\times \left\{\|(1-x^2)^{1/(2p)}g(x)w_\epsilon(x)\|^p_{L_\infty(\hat{\mathcal{F}})} + \sigma_p^{-p}\|gw_\epsilon\|^p_{L_p(\hat{\mathcal{E}})}\right\} .$$

Since we can choose $\hat{\mathcal{F}} \subset \mathcal{F}$ such that $\mathrm{meas}(\hat{\mathcal{E}}\setminus\mathcal{E})$ is as small as desired, we obtain via Lebesgue's Dominated Convergence Theorem, that (2.12) holds (as claimed) for any measurable g that is bounded above and below by positive constants.

Now, we can choose $\mathcal{F}$ to omit small intervals containing -1 and 1. Then $(1-x^2)^{1/(2p)}w_\epsilon(x)$ is bounded above and below by positive constants on $\mathcal{F}$, and the same is true for

$$g(x) := \begin{cases} (1-x^2)^{-1/(2p)}w_\epsilon(x)^{-1}, & x \in \mathcal{F} \\ 1, & x \in \mathcal{E}. \end{cases}$$

Thus, if $\chi_{\mathcal{F}}$ and $\chi_{\mathcal{E}}$ denote the characteristic functions of $\mathcal{F}$ and $\mathcal{E}$ respectively, we obtain from (2.12)

$$(2.14) \qquad \limsup_{n\to\infty}\left\{E_{np}(w)2^{n-1+1/p}\right\}^p$$

$$\le \sigma_p^p\, G[w_\epsilon(x)\chi_{\mathcal{F}}(x) + (1-x^2)^{-1/(2p)}\chi_{\mathcal{E}}(x)]^p ,$$

$$\times \left\{1 + \sigma_p^{-p}\|w_\epsilon\|^p_{L_p(\mathcal{E})}\right\} .$$

Choosing $\mathcal{F}$ such that $\mathrm{meas}(\mathcal{E})$ is as small as desired, (2.14) yields

$$\limsup_{n\to\infty}\left\{E_{np}(w)2^{n-1+1/p}\right\}^p \le \sigma_p^p\, G[w_\epsilon]^p .$$

Finally, (2.10) follows by writing

$$\log w_\epsilon^{-1} = \log^+ w_\epsilon^{-1} - \log^+ w_\epsilon ,$$

and observing that since $\log^+ w_\epsilon(\cos\phi)$ is bounded above by the integrable function $\log^+(w(\cos\phi) + 1)$ for $\epsilon < 1$, we have

$$\lim_{\epsilon\to0+}\int_0^\pi \log^+ w_\epsilon(\cos\phi)d\phi = \int_0^\pi \log^+ w(\cos\phi)d\phi$$

and, by the Monotone Convergence Theorem,

$$\lim_{\epsilon\to 0+} \int_0^{\pi} \log^+ w_\epsilon(\cos\phi)^{-1} d\phi = \int_0^{\pi} \log^+ w(\cos\phi)^{-1} d\phi. \qquad \square$$

<u>Theorem 2.3</u>. <u>Let</u> $w(x)$ <u>be a bounded non-negative Riemann integrable function on</u> $[-1,1]$. <u>Then</u>

$$\limsup_{n\to\infty} E_{n\infty}(w)2^{n-1} \le G[w]. \tag{2.15}$$

<u>Proof</u>. Let V_∞ be a function given by (2.2), fulfilling the hypotheses of Lemma 2.1. We obtain

$$E_{n\infty}(w) \le \|\tau_n V_\infty\|_{L_\infty[-1,1]} \|V_\infty^{-1} w\|_{L_\infty[-1,1]} = 2^{-n+1} G[V_\infty] \|V_\infty^{-1} w\|_{L_\infty[-1,1]} \cdot$$

by (2.6) and (1.18). Choosing $S(x) := (1-x^2)R^2(x)$ as before, and then choosing $R(x)$ to approximate the reciprocal of a continuous positive function g, we obtain

$$\limsup_{n\to\infty} E_{n\infty}(w)2^{n-1} \le G[g] \|g^{-1} w\|_{L_\infty[-1,1]} \cdot$$

Since w is Riemann integrable, a theorem of M. Riesz on one-sided approximation [4,p.73] ensures that we can choose a continuous function $g = g_\delta$ (even a polynomial) such that

$$w(x) \le g(x) \quad \text{in} \quad [-1,1]$$

and

$$\int_{-1}^{1} (g(x)-w(x))dx < \delta,$$

for any given $\delta > 0$. This g yields the desired result if w has a positive lower bound in $[-1,1]$. When the latter condition fails, replace w by w_ϵ as in the previous proof, and then let $\epsilon \to 0+$. $\square$

We now turn to the corresponding asymptotic lower bounds. Together with Theorems 2.2 and 2.3, they immediately yield (1.15).

Lemma 2.4. *Let* $1 < p < \infty$ *and* $w \in L_p[-1,1]$ *be a non-negative function such that* $w^{-1} \in L_r[-1,1]$ *for every* $r < \infty$. *Then*

$$(2.16) \qquad \liminf_{n\to\infty} E_{np}(w)2^{n-1+1/p} \ge \sigma_p G[w].$$

If $p = 1$ *or* $p = \infty$, *(2.16) remains valid provided* w *is positive and continuous in* $[-1,1]$.

Proof. Let $1 < p < \infty$ and $r > s > 1$, with $r^{-1} + s^{-1} = 1$, and $s \le p$. An easy consequence of Hölder's inequality is that if $H^{-1} \in L_{pr/s}[-1,1]$ and $JH \in L_p[-1,1]$, then

$$(2.17) \qquad \|JH\|_{L_p[-1,1]} \ge \|J\|_{L_{p/s}[-1,1]}\|H^{-1}\|^{-1}_{L_{pr/s}[-1,1]}$$

(see [6,Lemma 3.1]). Let $V_{p/s}(x)$ be a function given by (2.2), fulfilling the hypotheses of Lemma 2.1. Applying (2.17) with $J(x) := \{x^n-P(x)\}V_{p/s}(x)$ and $H(x) := V_{p/s}(x)^{-1}w(x)$, we obtain

$$E_{np}(w) \ge E_{n,p/s}(V_{p/s})\left\|V_{p/s}w^{-1}\right\|^{-1}_{L_{pr/s}[-1,1]}$$

$$= 2^{-n+1-s/p}\sigma_{p/s}G[V_{p/s}]\left\|V_{p/s}w^{-1}\right\|^{-1}_{L_{pr/s}[-1,1]} ,$$

by (2.3) and as $p/s \ge 1$. Choosing $S(x) := R^2(x)$, where $R(x)$ does not vanish in $[-1,1]$, and choosing $R(x)$ to uniformly approximate a function $g(x)$ positive and continuous in $[-1,1]$, we obtain

$$\liminf_{n\to\infty} E_{np}(w)2^{n-1+s/p} \ge$$

$$\sigma_{p/s}G[(1-x^2)^{(1-s/p)/2}g(x)^{-1}]\|(1-x^2)^{(1-s/p)/2}g(x)^{-1}w(x)^{-1}\|^{-1}_{L_{pr/s}[-1,1]} .$$

Choosing $g(x)$ to approximate $(1-x^2)^{(1-s/p)/2}w(x)^{-1}$ in a suitable sense, we obtain

$$\liminf_{n\to\infty} E_{np}(w)2^{n-1+s/p} \ge \sigma_{p/s}G[w]\|1\|^{-1}_{L_{pr/s}[-1,1]} .$$

Here $\|1\|_{L_{pr/s}[-1,1]} = 2^{s/(pr)} \to 1$ as $r \to \infty$, and also then $s \to 1$, so $\sigma_{p/s} \to \sigma_p$. Then (2.16) follows.

Finally in the cases $p = 1, \infty$, the proof is much easier: Since w is positive and continuous, one can choose a sequence $\{R_n\}_1^\infty$ of polynomials such that

$$R_n w \to 1 \quad \text{as} \quad n \to \infty,$$

uniformly in $[-1,1]$. By suitable choice of V_1 and V_∞, we easily obtain (2.16). □

We remarked in Section 1 that our conditions on w^{-1} imply Szegö's condition (1.7), and we now briefly justify this. Note that then also $G[w] > 0$. Suppose that for some $s > 0$, $w^{-1} \in L_s[-1,1]$. By the arithmetic-geometric mean inequality [10,p.2],

$$\exp\left[\pi^{-1} \int_{-1}^{1} \log\, w^{-s/3}(x)dx/\sqrt{1-x^2}\right] \leq \pi^{-1} \int_{-1}^{1} w^{-s/3}(x)dx/\sqrt{1-x^2}$$

$$\leq \pi^{-1} \|w^{-1}\|^{s/3}_{L_s[-1,1]} \|(1-x^2)^{-1/2}\|_{L_{3/2}[-1,1]} < \infty.$$

Hence (1.7).

3. Asymptotics for Extremal Polynomials.

The main new ideas of this paper are contained in Lemmas 3.1 and 3.2, where standard L_2 techniques [8] are turned into L_p ones:

Lemma 3.1. *Let* $2 \leq p \leq \infty$ *and* $w \in L_p[-1,1]$ *be a non-negative function that is positive on a set of positive measure. Let* $n \geq 1$, $P(z)$ *be a polynomial of degree* n *with leading coefficient* Δ, *and let*

$$\text{(3.1)} \qquad A := E_{np}(w)\Delta.$$

Then for $z \in \mathbb{C}\backslash[-1,1]$,

$$(3.2)\qquad |P(z)/p_{np}(w,z)-1| \le d(z)^{-1}\left\{\|Pw\|^2_{L_p[-1,1]} - A^2\right\}^{1/2} + |A-1|,$$

<u>where</u> $d(z)$ <u>denotes the distance from</u> z <u>to</u> $[-1,1]$.

<u>Proof</u>. Suppose first $p < \infty$. Note that as $p_{np}(w,x)$ is an extremal polynomial for w^p, we have

$$\int_{-1}^{1} |p_{np}(w,x)|^{p-2} p_{np}(w,x)\pi(x)w^p(x)dx = 0,$$

for each $\pi \in \mathcal{P}_{n-1}$ (cf.[2,p.9]). Let

$$\hat{w}(x) := |p_{np}(w,x)|^{p-2}w^p(x), \qquad x \in [-1,1].$$

By its normalization, we see that $p_{np}(w,x)$ is the orthonormal polynomial of degree n for $\hat{w}(x)$, that is

$$\int_{-1}^{1} p_{np}(w,x)^2\hat{w}(x)dx = 1,$$

and

$$\int_{-1}^{1} p_{np}(w,x)\pi(x)\hat{w}(x)dx = 0, \qquad \pi \in \mathcal{P}_{n-1}.$$

Let

$$q(x) := P(x) - Ap_{np}(w,x),$$

a polynomial of degree $\le n-1$. Let $\ell_{jn}(x)$, $j = 1,2,\dots,n$, denote the fundamental polynomials of Lagrange interpolation at the zeros $x_{1n}, x_{2n}, \dots, x_{nn}$ of $p_{np}(w,x)$. We have

$$q(x) = \sum_{j=1}^{n} q(x_{jn})\ell_{jn}(x).$$

By a well-known formula [8,p.6], [4,p.114, eqn.(6.3)], we have

$$\ell_{jn}(x) = \lambda_{jn} p_{n-1}(x_{jn})(\gamma_{n-1}/\gamma_n) p_{np}(w,x)/(x-x_{jn}),$$

where λ_{jn}, $j = 1,2,\ldots,n$, are the Gauss-Christoffel numbers of order n for $\hat{w}$, while $p_{n-1}(x)$ is the orthonormal polynomial of degree $n-1$ for $\hat{w}$ and γ_{n-1} and γ_n are the leading coefficients of $p_{n-1}(x)$ and $p_{np}(w,x)$, respectively. Here, as $\hat{w}$ is a weight on $[-1,1]$,

$$\gamma_{n-1}/\gamma_n \leq 1$$

[4,p.41] and so we obtain for $z \in \mathbb{C}\backslash[-1,1]$,

$$(3.3)\qquad \begin{aligned} |P(z)/p_{np}(w,z) - A| &= |q(z)/p_{np}(w,z)| \\ &\leq \sum_{j=1}^{n} \lambda_{jn} |p_{n-1}(x_{jn})|\,|q(x_{jn})|/|z-x_{jn}| \\ &\leq d(z)^{-1} \left\{\sum_{j=1}^{n} \lambda_{jn} p_{n-1}^2(x_{jn})\right\}^{1/2} \left\{\sum_{j=1}^{n} \lambda_{jn} q^2(x_{jn})\right\}^{1/2} \\ &= d(z)^{-1} \left\{\int_{-1}^{1} q^2(x)\hat{w}(x)dx\right\}^{1/2}, \end{aligned}$$

by orthonormality, and the exactness of the Gauss quadrature formula. Here

$$I := \int_{-1}^{1} q^2(x)\hat{w}(x)dx = \int_{-1}^{1} P^2(x)\hat{w}(x)dx - 2A\int_{-1}^{1} P(x)p_{np}(w,x)\hat{w}(x)dx + A^2$$

$$= \int_{-1}^{1} P^2(x)\hat{w}(x)dx - A^2,$$

since A is the coefficient of $p_{np}(w,x)$ in the expansion of $P(x)$ in the orthonormal polynomials for $\hat{w}$. Taking account of the definition of $\hat{w}$ and using Hölder's inequality with parameters $2/p$ and $1-2/p$, we obtain

$$I \leq \|Pw\|^2_{L_p[-1,1]} \|p_{np}(w,x)w(x)\|^{p-2}_{L_p[-1,1]} - A^2 = \|Pw\|^2_{L_p[-1,1]} - A^2.$$

Then (3.2) follows from (3.3).

Finally, if $p = \infty$, we can simply let $p \to \infty$ in (3.2), noting that (as is well-known) $E_{np}(w)$ and $p_{np}(w,x)$ respectively converge to $E_{n\infty}(w)$ and $p_{n\infty}(w,x)$ as $p \to \infty$. □

<u>Proof of (1.16) and (1.17) when</u> $2 \le p \le \infty$. Suppose first $p < \infty$. Let V_p be a function given by (2.2), fulfilling the hypotheses of Lemma 2.1. Note that by (2.3), (2.7) and (2.8),

$$(3.4) \qquad |p_{np}(V_p,x)V(x)| \le \sigma_p^{-1}, \quad x \in [-1,1],$$

while by (2.3), (2.7) and (2.9), for $u \in \mathbb{C}\backslash[-1,1]$,

$$(3.5) \qquad |p_{np}(V_p,u)/\{(2\sigma_p)^{-1}\varphi(u)^n D^{-2}(V(\cos\phi);\ \varphi(u)^{-1})\} - 1|$$
$$\le |\varphi(u)|^{2q-2n-2},$$

where $2q$ is the degree of S. Let $\mathcal{F} \subset [-1,1]$ be a measurable set for which

$$\|w\|_{L_\infty(\mathcal{F})} < \infty,$$

and let $\mathcal{E} := [-1,1]\backslash\mathcal{F}$. Further, let

$$A := E_{np}(w)/E_{np}(V_p).$$

Substituting $P := p_{np}(V_p)$ in (3.2) yields for $z \in \mathbb{C}\backslash[-1,1]$,

$$(3.6) \qquad |p_{np}(V_p,z)/p_{np}(w,z) - 1|$$
$$\le d(z)^{-1}\left\{\|p_{np}(V_p,x)w(x)\|^2_{L_p[-1,1]} - A^2\right\}^{1/2} + |A-1|$$
$$\le d(z)^{-1}\left\{\left[\|p_{np}(V_p,x)V_p(x)\|_{L_p(\mathcal{F})}\|V_p^{-1}w\|_{L_\infty(\mathcal{F})}\right.\right.$$
$$\left.\left. + \sigma_p^{-1}\|V^{-1}w\|_{L_p(\mathcal{E})}\right]^2 - A^2\right\}^{1/2} + |A-1|$$

$$\le d(z)^{-1}\left\{\left[\|V_p^{-1}w\|_{L_\infty(\mathcal{F})} + \sigma_p^{-1}\|V^{-1}w\|_{L_p(\mathcal{E})}\right]^2 - A^2\right\}^{1/2} + |A-1|.$$

A glance at the right-hand side of (2.11) indicates its similarity to this last right-hand side and we proceed in a like fashion. Take $S(x) := (1-x^2)R^2(x)$ and $R = R_n(x)$ where $R_n(x)$ has degree $q-1 = q_n-1 \le (n-2)/2$ and $R_n(x) \to g(x)$ uniformly in $[-1,1]$ as $n \to \infty$, where $g(x)$ is positive and continuous in $[-1,1]$. Set

$$h_n(f,z) := (2\sigma_p)^{-1}\varphi(z)^n D^{-2}(f(\cos\phi);\ \varphi(z)^{-1}),$$

and with $f_p(x) := w(x)(1-x^2)^{1/(2p)}$ (so that $f_p(\cos\phi) = F_p(\phi)$), write

$$\frac{h_n(f_p,z)}{p_{np}(w,z)} = \alpha_n\beta_n\lambda_n, \tag{3.7}$$

where

$$\alpha_n = \alpha_n(V_p,z) := \frac{p_{np}(V_p,z)}{p_{np}(w,z)}, \quad \beta_n = \beta_n(V_p,z) := \frac{h_n(V,z)}{p_{np}(V_p,z)},$$

$$\lambda_n = \lambda_n(V_p,z) := \frac{h_n(f_p,z)}{h_n(V,z)}.$$

Then, for $V(x) = R_n(x)^{-1}$, $V_p(x) = R_n(x)^{-1}(1-x^2)^{-1/(2p)}$, inequality (3.5) together with the fact that $q \le n/2$ imply

$$\lim_{n\to\infty} \beta_n = 1, \tag{3.8}$$

uniformly in closed subsets of $\mathbb{C}\setminus[-1,1]$. Also, since $V \to g^{-1}$ uniformly on $[-1,1]$, we see from (1.10) that

$$\lim_{n\to\infty} \lambda_n = D^{-2}((f_p g)(\phi);\ \varphi(z)^{-1}), \tag{3.9}$$

uniformly in closed subsets of $\mathbb{C}\setminus[-1,1]$. Furthermore, from (1.15) and (3.6) we get for any closed set $K \subset \mathbb{C}\setminus[-1,1]$,

$$(3.10) \qquad \limsup_{n\to\infty} \|\alpha_n - 1\|_{L_\infty(K)} \le \delta_K\, B(g,\mathcal{F},K),$$

where $\delta_K := \|d(z)^{-1}\|_{L_\infty(K)}$ and

$$B(g,\mathcal{F},K) := \left\{\left[\|gf_p\|_{L_\infty(\mathcal{F})} + \sigma_p^{-1}\|gw\|_{L_p(\mathcal{E})}\right]^2 - G[gf_p]^2\right\}^{1/2}$$

$$+ |G[gf_p] - 1|.$$

Since

$$|\alpha_n\beta_n\lambda_n - 1| \le |\alpha_n - 1||\beta_n - 1||\lambda_n - 1| + |\alpha_n||\beta_n - 1| + |\beta_n||\lambda_n - 1| + |\lambda_n||\alpha_n - 1|,$$

it follows from (3.7) - (3.10) that

$$(3.11) \qquad \limsup_{n\to\infty} \left\|\frac{h_n(f_p,z)}{p_{np}(w,z)} - 1\right\|_{L_\infty(K)}$$

$$\le \|D^{-2}((f_pg)(\phi);\ \varphi(z)^{-1}) - 1\|_{L_\infty(K)}$$

$$+ \|D^{-2}((f_pg)(\phi);\ \varphi(z)^{-1})\|_{L_\infty(K)}\delta_K B(g,\mathcal{F},K).$$

Much as in the proof of Theorem 2.2, we note that (3.11) holds, more generally, when g is any measurable function bounded above and below by positive constants. Choosing $\mathcal{F}$ such that f_p is bounded above and below by positive constants in $\mathcal{F}$, we take

$$g(x) := \begin{cases} f_p(x)^{-1}, & x \in \mathcal{F} \\ 1, & x \in \mathcal{E}. \end{cases}$$

Since we can choose $meas(\mathcal{E})$ as small as desired (note that $w(x)$ is positive a.e. in $[-1,1]$), we get $B(g,\mathcal{F},K) \to 0$ and $D^{-2}((f_pg)(\phi);$ $\varphi(z)^{-1}) \to 1$ uniformly on K, as $meas(\mathcal{E}) \to 0$. Then (3.11) yields (1.17).

Finally, if $p = \infty$, the proof is substantially easier, since then our hypotheses on w ensure that we can choose $V_\infty = V_{\infty,n}$ in

Lemma 2.1 such that $V_\infty \to w$ uniformly in $[-1,1]$ as $n \to \infty$. □

Next, we turn to the cases $1 < p < 2$. The method below works also for $2 \le p \le \infty$ under the hypotheses of Theorems 1.1 and 1.2, but we preferred to include Lemma 3.1 because of its greater potential: it requires little more than asymptotics for $E_{np}(w)$, whereas Lemma 3.2 places awkward restrictions on w^{-1}:

<u>Lemma 3.2</u>. <u>Let</u> $1 < p < 2 < q$ <u>satisfy</u> $p^{-1} + q^{-1} = 1$ <u>and let</u> u, w <u>be non-negative functions on</u> $[-1,1]$ <u>that are positive on a set of positive measure and satisfy</u> $u \in L_1[-1,1]$ <u>and</u> $uw^{-1} \in L_q[-1,1]$. <u>Let</u> $\{p_n\}_0^\infty$ <u>denote the orthonormal polynomials for</u> u <u>satisfying</u>

$$\int_{-1}^{1} p_n(x)p_m(x)u(x)dx = \delta_{mn}, \quad m,n = 0,1,2,\ldots .$$

<u>Let</u> $P(z)$ <u>be a polynomial of degree</u> n, <u>with leading coefficient</u> Λ, <u>and let</u> A <u>and</u> $d(z)$ <u>be as in Lemma 3.1</u>. <u>Let</u>

$$(3.12) \qquad \Psi(x) := \left[((x^p+1)/2)^{1/(p-1)} - 1\right]^{(p-1)/p}, \qquad x \in [1,\infty).$$

<u>Then for</u> $z \in \mathbb{C}\backslash[-1,1]$,

$$(3.13) \qquad |P(z)/p_{np}(w,z) - 1|$$

$$\le |A-1| + 2d(z)^{-1}|p_n(z)/p_{np}(w,z)|\,|A|\Psi\left[\|Pw\|_{L_p[-1,1]}\Lambda^{-1}\right]$$

$$\times \{\|p_{n-1}uw^{-1}\|_{L_q[-1,1]} + d(z)^{-1}\|p_nuw^{-1}\|_{L_q[-1,1]}\}.$$

<u>Proof</u>. We shall use Clarkson's inequalities [2,p.3] for $1 < p < 2$: For $f,g \in L_p[-1,1]$,

$$\left\|f+g\right\|_{L_p[-1,1]}^{p/(p-1)} + \left\|f-g\right\|_{L_p[-1,1]}^{p/(p-1)}$$

$$\le 2\left\{\left\|f\right\|_{L_p[-1,1]}^{p} + \left\|g\right\|_{L_p[-1,1]}^{p}\right\}^{1/(p-1)}.$$

Letting $\hat{P} := P/\Lambda$, $f := \hat{P}w$ and $g := T_{np}(w,\cdot)w$, we note that

$$\|f+g\|_{L_p[-1,1]} = 2\|(f+g)/2\|_{L_p[-1,1]} \geq 2E_{np}(w),$$

and so Clarkson's inequalities yield

$$\left\|(\hat{P} - T_{np}(w))w\right\|^{p/(p-1)}_{L_p[-1,1]}$$

$$\leq 2\left\{\|\hat{P}w\|^p_{L_p[-1,1]} + E^p_{np}(w)\right\}^{1/(p-1)} - (2E_{np}(w))^{p/(p-1)},$$

and so using (3.1) and (3.12), we obtain

$$\|(\hat{P} - T_{np}(w))w\|_{L_p[-1,1]} \leq 2E_{np}(w)\Psi\left[\|Pw\|_{L_p[-1,1]}A^{-1}\right]. \tag{3.14}$$

Next, let γ_j denote the leading coefficient of $p_j(x)$, and

$$K_n(x,t) := \{\gamma_{n-1}/\gamma_n\}\{p_n(x)p_{n-1}(t) - p_n(t)p_{n-1}(x)\}/(x-t).$$

As in Lemma 3.1, $\gamma_{n-1}/\gamma_n \leq 1$, and so for $t \in [-1,1]$ and $x \in \mathbb{C}\backslash[-1,1]$,

$$|K_n(x,t)| \leq d(x)^{-1}\{|p_n(x)p_{n-1}(t)| + |p_n(t)p_{n-1}(x)|\}. \tag{3.15}$$

Further as $\hat{P}(x) - T_{np}(w,x)$ has degree $\leq n-1$, we have [4,Ch.1],

$$\hat{P}(x) - T_{np}(w,x) = \int_{-1}^{1}(\hat{P}(t) - T_{np}(w,t))K_n(x,t)u(t)dt.$$

Then (3.15) and Hölder's inequality yield for $x \in \mathbb{C}\backslash[-1,1]$,

$$|\hat{P}(x) - T_{np}(w,x)| \leq \|(\hat{P} - T_{np}(w))w\|_{L_p[-1,1]} \tag{3.16}$$

$$\times\, d(x)^{-1}\left\{|p_n(x)|\|p_{n-1}uw^{-1}\|_{L_q[-1,1]} + |p_{n-1}(x)|\|p_nuw^{-1}\|_{L_q[-1,1]}\right\}.$$

Here, much as in the proof of Lemma 3.1,

$$p_{n-1}(x) = \frac{\gamma_{n-1}}{\gamma_n} \sum_{j=1}^{n} \lambda_{jn} p_{n-1}^2(x_{jn}) p_n(x)/(x-x_{jn}),$$

where $\{x_{jn}\}_1^n$ are the zeros of p_n and $\{\lambda_{jn}\}_1^n$ are the Gauss quadrature weights of order n for u. Then for $x \in \mathbb{C}\backslash[-1,1]$,

$$|p_{n-1}(x)/p_n(x)| \le d(x)^{-1} \sum_{j=1}^{n} \lambda_{jn} p_{n-1}^2(x_{jn}) = d(x)^{-1}.$$

Together with (3.14) and (3.16), this yields for $x \in \mathbb{C}\backslash[-1,1]$,

$$(3.17) \qquad |\hat{P}(x) - T_{np}(w,x)| \le 2d(x)^{-1} E_{np}(w) \Psi\left\{ \|Pw\|_{L_p[-1,1]} A^{-1} \right\}$$
$$\times |p_n(x)| \left\{ \|p_{n-1} u w^{-1}\|_{L_q[-1,1]} + d(x)^{-1} \|p_n u w^{-1}\|_{L_q[-1,1]} \right\}.$$

Finally,

$$|P(x)/p_{np}(w,x) - 1| = |A\hat{P}(x) - T_{np}(w,x)|/|T_{np}(w,x)|$$
$$\le |A| |\hat{P}(x) - T_{np}(w,x)|/|T_{np}(w,x)| + |A - 1|.$$

Substituting the estimate (3.17) into this last inequality yields (3.13). □

The difficulty above is choosing u so that $\|p_k u w^{-1}\|_{L_q[-1,1]}$ is bounded independent of k. Note that if

$$(3.18) \qquad u(x) := (1-x^2)^{1/2}, \quad x \in [-1,1],$$

then [4, p.35],

$$(3.19) \qquad \|p_n u\|_{L_\infty[-1,1]} \le (2/\pi)^{1/2}, \quad n = 1,2,3,\ldots .$$

<u>Proof of (1.16) and (1.17) when</u> $1 < p < 2$. Choose u by (3.18), and let V_p be a function given by (2.2), fulfilling the hypotheses of Lemma 2.1. Substituting $P(x) := p_{np}(V_p, x)$ in (3.13) and using

(3.19), we obtain

$$|p_{np}(V_p,z)/p_{np}(w,z) - 1|$$

$$\leq |A-1| + 2d(z)^{-1}|p_n(z)/p_{np}(w,z)||A|\Psi\left[\|p_{np}(V_p)w\|_{L_p[-1,1]}A^{-1}\right]$$

$$\times (2/\pi)^{1/2}\|w^{-1}\|_{L_q[-1,1]}\{1 + d(z)^{-1}\},$$

where $A = E_{np}(w)/E_{np}(V_p)$. Much as in the previous proof, we can choose a sequence of functions V_p such that $A \to 1$, $n \to \infty$, and

$$\|p_{np}(V_p)w\|_{L_p[-1,1]} \to 1, \quad n \to \infty.$$

Since $\Psi(1) = 0$, the Szegö type asymptotics for $p_n(z)$ and the above estimates easily yield (1.17) and hence (1.16). □

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