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THE DENSITY OF ALTERNATION POINTS IN RATIONAL APPROXIMATION

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ABSTRACT. We investigate the behavior of the equioscillation (alternation) points for the error in best uniform rational approximation on $[-1, 1]$. In the context of the Walsh table (in which the best rational approximant with numerator degree $\leq m$, denominator degree $\leq n$, is displayed in the nth row and the mth column), we show that these points are dense in $[-1, 1]$, if one goes down the table along a ray above the main diagonal ($n = [cm], c < 1$). A counterexample is provided showing that this may not be true for a subdiagonal of the table. In addition, a Kadec-type result on the distribution of the equioscillation points is obtained for asymptotically horizontal paths in the Walsh table.

1. STATEMENT OF RESULTS

Denote by $\mathscr{R}_{m,n}$ the rational functions with numerator in Π_m, the set of algebraic real polynomials of degree at most m, and denominator in Π_n. Then the best approximation $r^*_{m,n} = p^*_{m,n}/q^*_{m,n}$ in $\mathscr{R}_{m,n}$ to $f \in C[-1, 1]$ with respect to the uniform norm

$$(1.1) \qquad \|g\|_{[-1,1]} := \sup\{|g(x)| : x \in [-1, 1]\}$$

is unique and is characterized by an equioscillation property [M], i.e., there are $m + n + 2 - d$ points

$$(1.2) \qquad -1 \leq x_1^{(m,n)} < \cdots < x_{m+n+2-d}^{(m,n)} \leq 1,$$

where

$$(1.3) \qquad d := d(m, n) := \min\{m - \deg p^*_{m,n}, n - \deg q^*_{m,n}\},$$

such that for a $\sigma = \pm 1$ and all $k = 1, \ldots, m + n + 2 - d$

$$(1.4) \qquad f(x_k^{(m,n)}) - r^*_{m,n}(x_k^{(m,n)}) = \sigma(-1)^k \|f - r^*_{m,n}\|_{[-1,1]}.$$

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(Here and below, we assume that $p^*_{m,n}$ and $q^*_{m,n}$ do not have a common factor.)

Much is known about the behavior of alternation points for best polynomial approximation $(n = 0)$. For this case, Lorentz [L] and Kroó and Saff [KS] give examples showing that for a subsequence $\{m_k\}$ the alternation points (1.2), with $m = m_k$, $n = 0$, may avoid a subinterval of $[-1, 1]$. However, Kadec [K] proved that there is always a subsequence such that the alternation points behave like the extremal points of the Chebyshev polynomial of degree $m+1$, that is, like $\{\cos[k\pi/(m+1)]\}_{k=0}^{m+1}$. For polynomial approximation, this implies the denseness of the alternation points in $[-1, 1]$.

For rational approximation, given m and n, we pick any alternation set (1.2) and write

$$\rho_{m,n}(f) := \sup_{x\in[-1,1]} \min_k |x - x_k^{(m,n)}| \tag{1.5}$$

as a measure for the density of the alternation set in $[-1, 1]$. We shall prove

Theorem 1.1. *Let* $n = n(m)$ *satisfy*

$$n(m) \le n(m+1) \le n(m) + 1, \qquad n(m) \le m, \tag{1.6}$$

for $m = 0, 1, \ldots$. *If* $f \in C[-1, 1]$, $f \notin \mathscr{R}_{m,n(m)}$, $m = 0, 1, \ldots$, *then*

$$\liminf_{m\to\infty} \left(\frac{m - n(m)}{\log m}\right) \rho_{m,n(m)}(f) < \infty. \tag{1.7}$$

The proof of Theorem 1.1 will be given in §2.

Remark. Theorem 1.1 applies in the case $n(m) = [cm]$ for any constant $c \le 1$, where $[\cdot]$ denotes the greatest integer function. If $c < 1$, we deduce from (1.7) that

$$\liminf_{m\to\infty} \rho_{m,[cm]}(f) = 0, \tag{1.8}$$

which implies that the alternation points are dense in $[-1, 1]$ for such a "ray sequence" of best approximants. On the other hand, we show in Theorem 1.3 below that this density may not hold when $m/n(m) \to 1$.

Our second result is similar to Kadec's result [K] on polynomial approximation. We write for $-1 \le \alpha < \beta \le 1$ (with $x_k^{(m,n)}$ as in (1.2))

$$N_{m,n}(\alpha, \beta) := \#\{x_k^{(m,n)} : \alpha \le x_k^{(m,n)} \le \beta,\ k = 1, \ldots, m+n+2-d\}. \tag{1.9}$$

Theorem 1.2. *Assume, in addition to the hypotheses of Theorem* 1.1, *that*

$$\lim_{m\to\infty} \frac{n(m)}{m} = 0. \tag{1.10}$$

Then there exists a subsequence Ω *of* $\mathbb{N}$ *such that for all* $[\alpha, \beta] \subseteq [-1, 1]$,

$$\lim_{\substack{m\to\infty \\ m\in\Omega}} \frac{N_{m,n(m)}(\alpha, \beta)}{N_{m,n(m)}(-1, 1)} = \frac{\arccos\alpha - \arccos\beta}{\pi}. \tag{1.11}$$

Finally, we give a counterexample, which shows that Theorems 1.1 and 1.2 cannot be proved for a subdiagonal of the Walsh table. Indeed, for approximation in $\mathscr{R}_{n-1,n}$ it is possible that, for all n, the extremal points all reside in an arbitrarily small interval.

Theorem 1.3. *For every* $2 > \varepsilon > 0$, *there is a function* $f \in C[-1,1]$ *such that for each* $n = 1, 2, \ldots$ *the error* $f - r^*_{n-1,n}(f)$ *has no alternation points in* $(-1+\varepsilon, 1]$.

The proof of Theorem 1.3 will be given in §3. The results of this paper should be compared with those of Kroó and Peherstorfer [KP] for L_1-approximation.

2. Proofs of Theorems 1.1 and 1.2

We need the following lemma, which follows easily from classical results. We include the proof for the sake of completeness.

Lemma 2.1. *Given* $-1 \le \alpha < \beta \le 1$ *and* $n \in \mathbb{N}$ *there exists a* $p_n \in \Pi_n$ *with*

$$\|p_n\|_{[-1,\alpha]\cup[\beta,1]} < 1, \tag{2.1}$$

and

$$\|p_n\|_{[\alpha,\beta]} > c_1 e^{c_2 n(\beta-\alpha)}, \tag{2.2}$$

where $c_1, c_2 > 0$ *are constants independent of* α, β *and* n.

In (2.1) and (2.2) the norms are again the sup norms over the indicated set.

Proof. Let

$$T_m(x) := \cos(m \arccos x) \tag{2.3}$$

denote the Chebyshev polynomial of degree m. For $m := [n/2]$, $\tau := (\beta-\alpha)/2$, set

$$q_n(x) := \frac{1}{2} T_m\left(1 + \frac{\tau^2}{2} - \left(2 + \frac{\tau^2}{2}\right)\frac{x^2}{4}\right). \tag{2.4}$$

Since $\tau \le 1$ and

$$T_m(1+\eta) \ge \tfrac{1}{2}(1+\sqrt{2\eta})^m, \qquad \eta > 0, \tag{2.5}$$

we have for some constants $c_1, c_2 > 0$:

$$q_n(0) = \frac{1}{2} T_m\left(1 + \frac{\tau^2}{2}\right) \ge \frac{1}{4}(1+\tau)^m > c_1 e^{nc_2(\beta-\alpha)}. \tag{2.6}$$

For $x \in [-2, -\tau] \cup [\tau, 2]$,

$$-1 \le 1 + \frac{\tau^2}{2} - \left(2 + \frac{\tau^2}{2}\right)\frac{x^4}{4} < 1. \tag{2.7}$$

The lemma now follows with $p_n(x) := q_n(x - (\alpha+\beta)/2)$. □

Proof of Theorem **1.1**. Set $E_m(f) := \|f - r^*_{m,n(m)}\|_{[-1,1]}$ for $m \in \mathbb{N}$. Since $f \notin \mathcal{R}_{m,n(m)}$, we have $E_m(f) > 0$ for all $m \in \mathbb{N}$. Also, from (1.6), it follows that $E_m(f) \downarrow 0$, and so from elementary theorems about series (cf. [K])

$$\sum_{m=0}^{\infty} \frac{E_m(f) - E_{m+1}(f)}{E_m(f) + E_{m+1}(f)} = \infty. \tag{2.8}$$

Thus there is a subsequence Ω of $\mathbb{N}$ with $E_m(f) - E_{m+1}(f) \neq 0$ and

$$\frac{E_m(f) + E_{m+1}(f)}{E_m(f) - E_{m+1}(f)} < m^2 \tag{2.9}$$

for all $m \in \Omega$.

For $m \in \Omega$, set

$$R_m := \frac{1}{E_m(f) - E_{m+1}(f)}(r^*_{m,n(m)} - r^*_{m+1,n(m+1)}). \tag{2.10}$$

At the alternation points

$$-1 \leq x_1^{(m)} < \cdots < x^{(m)}_{m+n(m)+2-d(m)} \leq 1 \tag{2.11}$$

of $f - r^*_{m,n(m)}$ we have with $\sigma = \pm 1$

$$\sigma(-1)^k R_m(x_k^{(m)}) \geq 1, \qquad k = 1, \ldots, m + n(m) + 2 - d(m). \tag{2.12}$$

Moreover, from (1.6) and (2.12) it follows that $R_m = P_m/Q_m$ with

$$\deg P_m = m + n(m) + 1 - d(m), \tag{2.13}$$

$$\deg Q_m \leq 2n(m) + 1 - d(m). \tag{2.14}$$

Thus $R_m - q$ can have at most $m + n(m) + 1 - d(m)$ zeros, if $q \in \Pi_{m-n(m)}$. Let c_1, c_2 be as in Lemma 2.1. For $m \in \Omega$, let $x^*_m \in [-1, 1]$ satisfy

$$\min_k |x^*_m - x^{(m)}_k| = \rho_{m,n(m)}(f) =: t_m. \tag{2.15}$$

If $x^*_m \in [-1, x_1^{(m)}]$, we let $p_{m-n(m)}$ be the polynomial that satisfies Lemma 2.1 with $\alpha = -1$ and $\beta = x_1^{(m)}$. From (2.12) and (2.1) it follows that $R_m \pm p_{m-n(m)}$ has $m+n(m)+1-d(m)$ zeros in $(x_1^{(m)}, 1]$ and hence is zero-free in $[-1, x_1^{(m)}]$. Thus

$$c_1 e^{c_2 t_m (m-n(m))} \leq \|R_m\|_{[-1,1]} < m^2, \tag{2.16}$$

where the last inequality follows from (2.9). If $x^*_m \in [x^{(m)}_{m+n(m)+2-d(m)}, 1]$, we use Lemma 2.1 with $\alpha = x^{(m)}_{m+n(m)+2-d(m)}$ and $\beta = 1$ and again we get (2.16). Otherwise denote the zeros of R_m by $y_k^{(m)}$, where

$$x_1^{(m)} < y_1^{(m)} < x_2^{(m)} < \cdots < y^{(m)}_{m+n(m)+1-d(m)} < x^{(m)}_{m+n(m)+2-d(m)}, \tag{2.17}$$

and set $y_0^{(m)} := x_1^{(m)}, y^{(m)}_{m+n(m)+2-d(m)} := x^{(m)}_{m+n(m)+2-d(m)}$. Then $|y_k^{(m)} - y_{k+1}^{(m)}| \geq t_m$ for some $k = k^*$. As above, counting the zeros of $R_m \pm (p_{m-n(m)} - 1)/2$, where $p_{m-n(m)}$ satisfies Lemma 2.1 with $\alpha = y^{(m)}_{k^*}$ and $\beta = y^{(m)}_{k^*+1}$, yields

$$\tfrac{1}{2}\left(c_1 e^{c_2 t_m (m-n(m))} - 1\right) < m^2. \tag{2.18}$$

By (2.16) or (2.18) we get for a constant $c_3 > 0$

$$t_m(m - n(m)) \leq c_3 \log m , \tag{2.19}$$

which yields (1.7). □

Proof of Theorem 1.2. It suffices to prove (1.11) for the case $\alpha = -1$. In fact, it is enough to show that

$$\limsup_{\substack{m\to\infty \\ m\in\Omega}} \frac{N_{m,n(m)}(-1,\beta)}{N_{m,n(m)}(-1,1)} \leq \frac{\pi - \arccos\beta}{\pi}, \tag{2.20}$$

since replacing x by $-x$ and β by $-\beta$, we get the corresponding lower estimate for lim inf. Let $m-n(m) = s(m)+l(m)$, where $s(m)$ is to be determined later. With the notations of the previous proof, set for $m \in \Omega$,

$$q_m(x) := \tfrac{1}{2} T_{s(m)}(x + 1 - \beta) T_{l(m)}(x), \tag{2.21}$$

$$N(m) := \#\{x \in (\beta, 1]: |T_{l(m)}(x)| = 1,\ |q_m(x)| > m^2\}, \tag{2.22}$$

where T_k denotes the kth the Chebyshev polynomial, $\|T_k\|_{[-1,1]} = 1$. Then $R_m - q_m$ has at least $N(m) - 1$ zeros in $(\beta, 1]$. Thus it can have at most $m + n(m) + 2 - d(m) - N(m)$ zeros in $[-1, \beta]$. Hence

$$\begin{aligned}\limsup_{m\to\infty} \frac{N_{m,n(m)}(-1,\beta)}{N_{m,n(m)}(-1,1)} &\leq \limsup_{m\to\infty} \frac{m + n(m) + 2 - d(m) - N(m)}{m + n(m) + 2 - d(m)} \\ &= 1 - \liminf_{m\to\infty} \frac{N(m)}{m},\end{aligned} \tag{2.23}$$

since between two alternation points of R_m in $[-1, \beta]$ is one zero of $R_m - q_m$ and since $n(m)/m \to 0$ $(d(m) \leq n(m))$. In (2.23) and the rest of the proof, all limits are for $m \in \Omega$. Now choose $s(m)$ such that

$$\lim_{m\to\infty} \frac{s(m)}{\log m} = \infty, \quad \lim_{m\to\infty} \frac{l(m)}{m} = 1. \tag{2.24}$$

Then the first equation in (2.24) together with (2.5) yields

$$\lim_{m\to\infty} (\inf\{x \in (\beta, 1]: \tfrac{1}{2} T_{s(m)}(x + 1 - \beta) > m^2\}) = \beta. \tag{2.25}$$

Also, for $\beta < \widetilde{\beta} < 1$, it follows from the second equation in (2.24) that

$$\lim_{m\to\infty} \frac{\#\{x \in (\widetilde{\beta}, 1]: |T_{l(m)}(x)| = 1\}}{m} = \frac{\arccos\widetilde{\beta}}{\pi}. \tag{2.26}$$

Finally, (2.25) and (2.26) yield (with (2.21) and (2.22))

$$\liminf_{m\to\infty} \frac{N(m)}{m} \geq \frac{\arccos\beta}{\pi}, \tag{2.27}$$

which together with (2.23) gives (2.20). □

where p^*_{n-2} is of degree $n-2$ and

$$a_2 < a_3^* < a_4 < \cdots < a^*_{2n-1} < a_{2n} < 0. \tag{3.16}$$

Thus, by the equioscillation property (1.4), there is a constant c_n such that

$$S_n - r^*_{n-2,n-1}(S_n) = c_n r_{2n-2}(a_2, a_3^*, a_4, \dots, a_{2n}). \tag{3.17}$$

Since $r_{2n-2}(a_1, \dots, a_{2n-1})$ has no alternation point in $(\varepsilon/2, 1]$, Lemma 3.1 shows that $S_n - r^*_{n-2,n-1}(S_n)$ has no alternation point in $(\varepsilon/2, 1]$. We choose the b_k's such that

(3.18) the series $f(x) = \sum_{k=1}^{\infty} \frac{b_k}{x - a_{2k}}$ converges uniformly on $[0, 1]$,

and

(3.19) $r^*_{n-2,n-1}(f)$ is close enough to $r^*_{n-2,n-1}(S_n)$ to guarantee that $f - r^*_{n-2,n-1}(f)$ has no alternation point in $(\varepsilon, 1]$.

For (3.19) we used the fact (cf. [W]) that the best approximation operator is continuous in S_n, since $r^*_{n-2,n-1}(S_n)$ is nondegenerate (i.e. $d = 0$ in (1.3)). □

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For every $\varepsilon > 0$, there is a polynomial $\tilde{q} \in \Pi_n$ with highest coefficient $C_\alpha - C_\beta$ and real zeros $\tilde{\xi}_1 \leq \cdots \leq \tilde{\xi}_n$ such that

$$\tilde{q}(x_n) > 0, \tilde{q}(x_{n-1}) < 0, \ldots, \tilde{q}(0) < 0 \quad \text{for } n \text{ even}, \tag{3.8}$$
$$\tilde{q}(0) > 0 \quad \text{for } n \text{ odd},$$

$$|\tilde{\xi}_k - \xi_k| < \varepsilon, \qquad k = 1, \ldots, n. \tag{3.9}$$

It follows that $\tilde{\xi}_1 < 0$ and thus $x_k > \tilde{\xi}_k$ for $k = 1, \ldots, n$. Since $\varepsilon > 0$ is arbitrary, this implies $x_k \geq \xi_k \geq y_k$ for $k = 1, \ldots, n$. □

Lemma 3.2. *Given $0 < \varepsilon < 1$ there is an increasing sequence $a_1 < a_2 < \cdots < 0$, such that $r_{n-1}(a_1, \ldots, a_n)$ has no extremal points in $(\varepsilon, 1]$ for $n \geq 1$.*

Proof. Set $a_1 := -\varepsilon/4$. We will construct a_n by induction such that the function

$$f_n(x) := \frac{\prod_{k=2}^n (x + 2a_k)}{\prod_{k=1}^n (x - a_k)} \tag{3.10}$$

alternates in sign in the points

$$0 = \delta_{1,n} < \cdots < \delta_{n,n} < \varepsilon \tag{3.11}$$

and satisfies

$$|f_n(\delta_{k,n})| > |f_n(x)| \qquad \text{for } k = 1, \ldots, n, \ x \in [\varepsilon, 1]. \tag{3.12}$$

If we have this sequence, $r_{n-1}(a_1, \ldots, a_n)$ cannot have an alternation point in $(\varepsilon, 1]$ for $n \geq 2$, since otherwise for a suitably chosen $\gamma \in \mathbb{R}$ the function $f_n - \gamma r_{n-1}(a_1, \ldots, a_n)$ has a zero in each interval $(\delta_{k,n}, \delta_{k+1,n})$ and an additional zero in $(\delta_{n,n}, 1)$.

We observe now that $f_1(x) = 1/(x - a_1)$ is decreasing in $[0, 1]$ and satisfies (3.12) with $\delta_{1,1} = 0$. Having constructed $a_1, \ldots, a_{n-1}$, we observe that

$$\lim_{a_n \to 0^-} \frac{x + 2a_n}{x - a_n} = 1 \text{ uniformly on } [\lambda, 1] \tag{3.13}$$

for all $\lambda > 0$. Thus, for $|a_n|$ sufficiently small, (3.12) will be satisfied for $\delta_{k,n+1} := \delta_{k-1,n}$, $k = 3, \ldots, n+1$ and for $\delta_{1,n+1} := \delta_{1,n} = 0$. Thus it remains to show the existence of $\delta_{2,n+1}$. Since $f_{n+1}(0) = -2f_n(0)$, this follows from (3.13) by choosing $|a_n|$ small enough. □

Proof of Theorem 1.3. We will prove the theorem on the interval $[0, 1]$. Choose $a_1, a_2, \ldots$ as in Lemma 3.2 with $\varepsilon/2$ replacing ε. Let, for $b_k > 0$,

$$S_n(x) := \sum_{k=1}^n \frac{b_k}{x - a_{2k}}. \tag{3.14}$$

We now use a result in [B] stating that the best approximation to S_n out of $\mathscr{R}_{n-2,n-1}$ has the form

$$r^*_{n-2,n-1}(S_n)(x) = \frac{p^*_{n-2}(x)}{\prod_{k=1}^{n-1}(x - a^*_{2k+1})}, \tag{3.15}$$

where p^*_{n-2} is of degree $n-2$ and

$$a_2 < a_3^* < a_4 < \cdots < a^*_{2n-1} < a_{2n} < 0. \tag{3.16}$$

Thus, by the equioscillation property (1.4), there is a constant c_n such that

$$S_n - r^*_{n-2,n-1}(S_n) = c_n r_{2n-2}(a_2, a_3^*, a_4, \dots, a_{2n}). \tag{3.17}$$

Since $r_{2n-2}(a_1, \dots, a_{2n-1})$ has no alternation point in $(\varepsilon/2, 1]$, Lemma 3.1 shows that $S_n - r^*_{n-2,n-1}(S_n)$ has no alternation point in $(\varepsilon/2, 1]$. We choose the b_k 's such that

(3.18) the series $f(x) = \sum_{k=1}^{\infty} \frac{b_k}{x - a_{2k}}$ converges uniformly on $[0, 1]$,

and

(3.19) $r^*_{n-2,n-1}(f)$ is close enough to $r^*_{n-2,n-1}(S_n)$ to guarantee that $f - r^*_{n-2,n-1}(f)$ has no alternation point in $(\varepsilon, 1]$.

For (3.19) we used the fact (cf. [W]) that the best approximation operator is continuous in S_n, since $r^*_{n-2,n-1}(S_n)$ is nondegenerate (i.e. $d = 0$ in (1.3)). □

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