# ON THE BEHAVIOR OF ZEROS AND POLES OF BEST UNIFORM POLYNOMIAL AND RATIONAL APPROXIMANTS

R. Grothmann*
Mathem. Geogr. Fakultät
Katholische Universität
Eichstätt
Ostenstrasse 18
D-8078 Eichstätt
WEST GERMANY

E.B. Saff**
Institute for Constructive
Mathematics
Department of Mathematics
University of South Florida
Tampa, Florida 33620
USA

ABSTRACT. We investigate the behavior of zeros of best uniform polynomial approximants to a function $f$, which is continuous in a compact set $E \subset \mathbb{C}$ and analytic on $\overset{\circ}{E}$, but not on $E$. Our results are related to a recent theorem of Blatt, Saff, and Simkani which roughly states that the zeros of a *subsequence* of best polynomial approximants distribute like the equilibrium measure for $E$. In contrast, we show that there might be another subsequence with zeros essentially all tending to $\infty$. Also, we investigate near best approximants. For rational best approximants we prove that its zeros and poles cannot all stay outside a neighborhood of $E$, unless $f$ is analytic on $E$.

*The research of this author was done while visiting the Institute for Constructive Mathematics, University of South Florida, Tampa.

**The research of this author was supported, in part, by the National Science Foundation under grant DMS-8620098.

AMS Classification: 41A20
Key words and phrases: Polynomial approximation, rational approximation, best approximants, zero distributions.



A. Cuyt (ed.), Nonlinear Numerical Methods and Rational Approximation, 57–75.

## 1. STATEMENT OF RESULTS

We investigate the behavior of zeros and poles of best uniform approximants in the complex plane. Throughout this paper we will assume that $E \subset \mathbb{C}$ is a compact set and that $\overline{\mathbb{C}}\setminus E$ is connected. Using the Chebyshev norm on $E$,

$$(1.1)\qquad \|g\|_E := \sup_{z\in E} |g(z)|,$$

we will approximate a function $f$, analytic on $\overset{o}{E}$ (the interior of $E$) and continuous in $E$, with respect to $\Pi_n$, the set of algebraic polynomials of degree at most $n$, or with respect to $\mathcal{R}_{m,n}$, the rational functions with numerator in $\Pi_m$ and denominator in $\Pi_n$.

We denote by $p_n^*(f)$ the best uniform approximant to $f$ on $E$ with respect to $\Pi_n$, i.e.

$$(1.2)\qquad e_n(f) := \|f - p_n^*(f)\|_E \le \|f - p_n\|_E$$

for all $p_n \in \Pi_n$. By a theorem of Mergelyan, we know $e_n(f) \to 0$. With $p_n^*(f)$ we associate a unit measure $\upsilon_n^*$, called the *zero distribution* of $p_n^*(f)$, by

$$(1.3)\qquad \upsilon_n^*(A) := \frac{\text{number of zeros of } p_n^*(f) \text{ in } A}{\deg p_n^*(f)}$$

for Borel sets $A \subset \mathbb{C}$, where we count the zeros according to their multiplicity.

The location of zeros of $p_n^*(f)$ or other converging sequences of polynomials has been investigated by several authors. Jentzsch [J] proved in 1914 that the partial sums $s_n$ of a power series with finite radius of convergence $r > 0$ have the property that every point on the circle $|z| = r$ is a limit point of zeros of the $s_n$. The related question in our setting is whether every point on the boundary of $E$ is a limit point of zeros of the $p_n^*(f)$, if $f$ is not analytic on $E$. The most complete and general answer to this question was obtained by Blatt, Saff, and Simkani [BSS] in 1986. Earlier investigations are due to Borwein [Bo] and Blatt and Saff [BS]. We should also mention similar results concerning sequences of so-called maximally convergent polynomials by Walsh [W2], who studied the case when $f$ is analytic on $E$ but not entire.

The results in [BSS] deal primarily with the limiting *distribution* of the zeros of the $p_n^*(f)$. The classical result concerning the distribution of zeros of partial sums of a power series is due to Szegö [Sz] and improves the above mentioned theorem of Jentzsch.

Before stating the known and new results we recall the definition of the *equilibrium measure* $\mu_E$ of $E$. Throughout the paper we will assume that $\overline{\mathbb{C}}\setminus E$ is *regular*; that is, $\overline{\mathbb{C}}\setminus E$ has a classical Green's function $G$ with pole at $\infty$, which is continuous on $\partial E$ (the boundary of $E$) and has the value zero on $\partial E$ (cf. [T]). The function $G$ is harmonic in $\mathbb{C}\setminus E$ and $G(z) - \log|z|$ is harmonic at $\infty$ and assumes the value $-\log[\mathrm{cap}(E)]$ at $\infty$, where $\mathrm{cap}(E) > 0$ is the *logarithmic capacity* of $E$ (cf.[T]). The measure $\mu_E$ is the unique unit measure that is supported on $\partial E$ and satisfies

$$G(z) = \int \log|z - t| d\mu_E(t) - \log[\mathrm{cap}(E)]. \tag{1.4}$$

It is also the unique unit measure that is supported on E and minimizes the energy integral

$$I(\mu) := \iint \log|z - t|^{-1} d\mu(t) d\mu(z). \tag{1.5}$$

We can now state the main result of Blatt, Saff, and Simkani.

*THEOREM* 1.1 ([BSS]). *Let* $E \subset \mathbb{C}$ *be compact,* $\overline{\mathbb{C}} \setminus E$ *be connected and regular. Let* $f$ *be continuous on* $E$, *analytic on* $\overset{o}{E}$ *but not on* $E$. *Furthermore, assume that* $f$ *does not vanish identically on any component of* $\overset{o}{E}$. *Then the sequence* $\{\nu_n^*\}$ *in* (1.3) *possesses a subsequence* $\{\nu_{\ell(n)}^*\}$ *that converges weakly to the equilibrium measure* $\mu_E$ *of* $E$.

By the weak (vague) convergence of $\nu_{\ell(n)}^*$ we mean

$$\lim_{n\to\infty} \int \phi \, d\nu_{\ell(n)}^* = \int \phi \, d\mu_E$$

for all continuous $\phi$ on $\mathbb{C}$ having compact support.

Our first result shows that there may be another subsequence of $\{\nu_n^*\}$ that has a completely different behavior.

*THEOREM* 1.2. *Let* $E \subset \mathbb{C}$ *be compact with a connected and regular complement. Then there is a function* $f$ *on* $E$ *such that*

(a) $f$ *is continuous on* $E$, *analytic on* $\overset{o}{E}$, *but not analytic on* $E$,

(b) *f has no zeros in* $E$, *and*
(c) *for a subsequence* $\{\nu^*_{k(n)}\}$ *of the measures* (1.3)

$$\lim_{n\to\infty} \nu^*_{k(n)}(S) = 0, \quad \textit{for all bounded } S \subset \mathbb{C}. \tag{1.6}$$

The proof of Theorem 1.2 will be given in Section 2.

We now discuss "near best approximants," i.e. we assume that we have a sequence $q_n \in \Pi_n$ such that

$$\|f - q_n\|_E \le C\|f - p^*_n(f)\|_E, \quad \text{all } n \in \mathbf{N}, \tag{1.7}$$

for some constant $C$. (We caution the reader that our definition of "near best" is different from that in [BS].) As shown in the next example, we can no longer expect that every point on the boundary of $E$ is a limit point of zeros of the $q_n$, let alone that the zero distributions $\nu_n$ associated with the $q_n$ have a subsequence converging weakly to $\mu_E$.

*Example* 1.1. Let $f(z) := \sqrt{z}$ for $\operatorname{Re} z \ge 0$ and set

$$K_\alpha := \{z \in \mathbb{C}: |z - \alpha| \le \alpha\}. \tag{1.8}$$

Then, for $p_n \in \Pi_n$ and $\alpha > 0$,

$$\left\|\frac{1}{\sqrt{\alpha}}\, p_n(\alpha z) - f(z)\right\|_{K_1} = \frac{1}{\sqrt{\alpha}}\, \|p_n(w) - f(w)\|_{K_\alpha}, \tag{1.9}$$

where $\|\cdot\|_M$ denotes the Chebyshev norm on $M$. Thus, if $\{q_n\}_1^\infty$ is the sequence of best uniform approximants to $f$

on $K_2$, it satisfies (1.7) with $E := K_1$. But no point of $E\setminus\{0\}$ is a limit point of zeros of the $q_n$ since $q_n(z) \to \sqrt{z}$ uniformly on $K_2$. An example similar to the above appears in [BIS].

Notice that in Example 1.1, the set $E$ has a non-empty interior. The authors do not know of an analogous example for $E = [-1,1]$; however, the validity of the following conjecture would lead to such a result.

*Conjecture.* Set $E_1 := [-1,1]$,

$$(1.10) \qquad E_2 := \{z \in \mathbb{C} : |\mathrm{Re}\, z| \le 2,\ |\mathrm{Im}\, z| \le |\mathrm{Re}\, z|^2\},$$

and define

$$(1.11) \qquad f(z) := \begin{cases} z & \text{for } \mathrm{Re}\, z \ge 0, \\ -z & \text{for } \mathrm{Re}\, z < 0, \end{cases}$$

so that $f(x) = |x|$ for $x \in \mathbb{R}$. If we denote by $e_{i,n}$ the error of best uniform approximation of $f$ on $E_i$ with respect to $\Pi_n$, then we conjecture that there is a constant $c$ such that

$$(1.12) \qquad e_{2,n} \le c\, e_{1,n}, \quad \text{all} \quad n \in \mathbf{N}.$$

This conjecture would yield another example of a sequence $\{q_n\}$ satisfying (1.7), where $E = [-1,1]$ and $q_n$ is the best approximant to $f$ on $E_2$ in $\Pi_n$. Notice that no point $x \in [-1,1]\setminus\{0\}$ can be a limit point of zeros of such $q_n$.

The above discussion naturally raises the question as to whether near best approximants (in sense of (1.7)) necessarily have *at least one* limit point of zeros on $\partial E$ when f is not analytic on E. Our next theorem lends evidence in the affirmative direction.

*THEOREM* 1.3. *Let* E *and* f *be as in Theorem* 1.1 *and assume that* f *does not vanish identically on any (open) component of* $\overset{o}{E}$ *or on any (closed) component of* E. *Suppose there exist constants* $c < 1$ *and* $K > 1$ *such that the errors* (1.2) *satisfy*

$$(1.13) \qquad \liminf_{n\to\infty} e_{[cn]}(f)/e_n(f) > K.$$

*If* $q_n \in \Pi_n$ *satisfies*

$$(1.14) \qquad \|f - q_n\|_E \le K\, e_n(f), \qquad \text{all} \quad n \in \mathbf{N},$$

*then at least one point of* $\partial E$ *must be a limit point of zeros of the sequence* $\{q_n\}_{n\in\mathbf{N}}$.

In (1.13) the symbol [cn] denotes the integral part of cn. The proof of Theorem 1.3 will be given in Section 3.

*Example* 1.2. Theorem 1.3 in particular applies if $E = [-1,1]$ and

$$(1.15) \qquad \frac{D_1}{n^\alpha} \le e_n(f) \le \frac{D_2}{n^\alpha}, \qquad \text{all} \quad n \in \mathbf{N},$$

for positive constants $D_1$, $D_2$, $\alpha$. Indeed, (1.15) yields

$$(1.16)\qquad \liminf_{n\to\infty} \frac{e_{[cn]}(f)}{e_n(f)} \geq \frac{D_1}{c^{\alpha}D_2} > K,$$

for $c$ small enough. By a famous result of Bernstein [Be], (1.15) is true with $\alpha = 1$ for the case $E = [-1,1]$ and $f(x) = |x|$. Thus for any $K \geq 1$, every sequence $\{q_n\}$ satisfying

$$(1.17)\qquad \Big\| |x| - q_n(x) \Big\|_{[-1,1]} \leq K\, e_n(|x|), \qquad \text{all} \quad n \in \mathbb{N},$$

must have at least one limit point of zeros in $[-1,1]$. In fact, the origin must be a limit point of zeros of such $q_n$, since (1.17) remains true if $[-1,1]$ is replaced by $[-\epsilon,\epsilon]$, $\epsilon > 0$, for the same sequence $\{q_n\}$.

The behavior of the zeros and poles of best *rational* approximants is far more delicate. An elementary result in this direction is the following.

*THEOREM* 1.4. *Let* $E \subset \mathbb{C}$ *be compact,* $\overline{\mathbb{C}}\setminus E$ *be connected and regular. Assume that* $f$ *is continuous on* $E$, *analytic in* $\overset{o}{E}$, *and that* $r_n \in \mathcal{R}_{n,n}$ *satisfies*

$$(1.18)\qquad \|f - r_n\|_E \leq e_n(f), \qquad \textit{all} \quad n \quad \textit{large}.$$

*Suppose further that* $f$ *does not vanish identically on any (closed) component of* $E$. *If, for* $n$ *large, all poles and zeros of* $r_n$ *are outside a neighborhood* $U$ *of* $E$, *then* $f$ *is analytic on* $E$.

The proof of Theorem 1.4 will be given in Section 4. This theorem should be compared with results of A. Levin [L].

*Remark 1.* Theorem 1.4 applies, if $r_n$ is a best approximant in $\mathcal{R}_{n,m(n)}$ for $m(n) \leq n$ to $f$.

*Remark 2.* In the proof of Theorem 1.4 we establish the stronger conclusion that if all zeros and poles of $r_n$ lie outside the level curve $\Gamma_{\gamma_o} : G(z) = \log \gamma_o$, then $f$ can be analytically continued throughout the interior of $\Gamma_{\gamma_o}$.

*Remark 3.* It seems likely (although the authors cannot now prove it) that Theorem 1.4 should remain true under the weaker assumption that only the poles of $r_n$ lie outside $U$.

## 2. PROOF OF THEOREM 1.2

Denote by $T_n = z^n + \cdots$ the generalized Chebyshev polynomial of degree $n$ on $E$, i.e.

$$(2.1) \qquad \|T_n\|_E = \min_{p \in \Pi_{n-1}} \|z^n - p(z)\|_E ,$$

and for $n \geq 1$ set

$$(2.2) \qquad q_n := \frac{1}{n^2 \|T_n\|_E} T_n .$$

We construct an increasing sequence $\{k(n)\}$ of natural numbers, a sequence $\{m(n)\}$ of natural numbers and a sequence $\{\alpha_n\}$ of real numbers by induction. In each step we set

$$(2.3)\qquad S_N := 1 + \sum_{n=0}^{N} \left[q_{k(n)+m(n)} + \alpha_{n+1} q_{k(n+1)}\right].$$

The desired function $f$ will be the limit of $S_N$, that is

$$(2.4)\qquad f := 1 + \sum_{n=0}^{\infty} \left[q_{k(n)+m(n)} + \alpha_{n+1} q_{k(n+1)}\right].$$

We will require that $k(n)+m(n) < k(n+1)$ and $0 < \alpha_{n+1} < 1$. This ensures that (2.4) converges uniformly on $E$. The truncations of the series (2.4) will reflect the behavior of $p_n^*(f)$ of respective degree.

We start with $k(0) = 0$ and set $S_{-1} = 1$. We choose $m(0)$ such that

$$(2.5)\qquad \sum_{j=m(0)}^{\infty} \frac{1}{j^2} < 1.$$

Thus $f$ will have no zeros in $E$.

Assume now, that $k(n)$, $m(n)$, and $\alpha_n$ have been constructed for $0 \leq n \leq N$. Since the zero polynomial is the best approximant to $q_{k(N)+m(N)}$ in $\Pi_{k(N)}$, $S_{N-1}$ is the best approximant to $S_{N-1} + q_{k(N)+m(N)}$ in $\Pi_{k(N)}$. By the continuity of the best approximation operator and Rouché's theorem, there is an $\epsilon_N > 0$ such that for all $f \in C(E)$ with

$$(2.6)\qquad \left\|f - \left[S_{N-1} + q_{k(N)+m(N)}\right]\right\|_E < \epsilon_N ,$$

we have

(2.7) $\quad p^*_{k(N)}(f)$ has not more zeros in $B_N$ than $S_{N-1}$ has in $B_{2N}$,

where $B_R := \{z \in \mathbb{C}: |z| < R\}$. Notice that $S_{N-1}$ is a polynomial of exact degree $k(N)$. Thus, by the continuity of the best approximation operator, we can assume that $\epsilon_N > 0$ is so small that for all $f$ satisfying (2.6) we also have

(2.8) $\quad p^*_{k(N)}(f)$ is a polynomial of exact degree $k(N)$.

Choose $k(N+1)$ so that

$$k(N+1) > (k(N) + m(N))^2, \tag{2.9}$$

and

$$\sum_{j=k(N+1)}^{\infty} \frac{1}{j^2} < \epsilon_N . \tag{2.10}$$

By Rouché's theorem, we can also choose $0 < \alpha_{N+1} < 1$ such that

(2.11) $\quad S_N = \left[S_{N-1} + q_{k(N)+m(N)}\right] + \alpha_{N+1} q_{k(N+1)}$ has at most $k(N) + m(N)$ zeros in $B_{2N+2}$.

Next we define $m(N+1)$. Again, $S_{N-1} + q_{k(N)+m(N)}$ is the best approximation to $S_N$ in $\Pi_{k(N)+m(N)}$. Its leading coefficient is

(2.12) $$\beta_N := \frac{1}{(k(N)+m(N))^2 \|T_{k(N)+m(N)}\|_E}.$$

By the continuity of the best approximation operator we can choose $\delta_N > 0$ small enough such that for all $f \in C(E)$ with

(2.13) $$\|f - S_N\|_E \le \delta_N ,$$

we have

(2.14) the leading coefficient $\gamma_N$ of $p^*_{k(N)+m(N)}(f)$ satisfies $|\gamma_N| > |\beta_N|/2$.

We now choose $m(N+1)$ so that

(2.15) $$\sum_{j=k(N+1)+m(N+1)}^{\infty} \frac{1}{j^2} < \delta_N.$$

For $f$ in (2.4) we have (2.7), (2.8), (2.11) and (2.14), by (2.10) and (2.15) for $N \ge 2$. Thus $p^*_{k(N)}(f)$ is a polynomial of exact degree $k(N)$. By (2.11) and (2.7), it has at most $k(N-1) + m(N-1)$ zeros in $B_N$. Thus for the measure $\nu^*_n$ we have

(2.16) $$\nu^*_{k(N)}(B_N) \le \frac{k(N-1) + m(N-1)}{k(N)} < \frac{1}{k(N-1) + m(N-1)}.$$

For bounded $S \subset \mathbb{C}$ this implies

(2.17) $$\lim_{N\to\infty} \nu^*_{k(N)}(S) = 0.$$

Write $p^*_n(f) = a_n z^n + \cdots$ . Then by (2.14),

$$\limsup_{n\to\infty} |a_n|^{1/n} \geq \limsup_{n\to\infty} \frac{1}{\|T_n\|^{1/n}} = \frac{1}{\mathrm{cap}(E)} \tag{2.18}$$

(cf.[T]). Thus, by a result of Blatt and Saff [BS], $f$ is not analytic on $E$. □

## 3. PROOF OF THEOREM 1.3.

Assume to the contrary that no point of $\partial E$ is a limit point of zeros of $\{q_n\}_1^\infty$. Since (1.14) implies that $q_n \to f$ uniformly on $E$ and $f$ does not vanish identically on any component of $\mathring{E}$, then the set of zeros of $f$ in $\mathring{E}$ is identical to the set of limit points in $\mathring{E}$ of the zeros of $\{q_n\}_1^\infty$ (recall Hurwitz's theorem). Thus $f$ can have at most finitely many zeros in $\mathring{E}$ since, otherwise, either $f$ vanishes identically on a component of $\mathring{E}$ or a point of $\partial E$ is a limit point of zeros of $\{q_n\}_1^\infty$. Let $z_1, \ldots, z_m$ denote the zeros of $f$ in $\mathring{E}$.

With $G$ defined as in (1.4), set

$$E_\gamma := E \cup \{z \in \mathbb{C}\setminus E : G(z) \leq \log \gamma\}, \quad \gamma > 1. \tag{3.1}$$

The assumption on the zeros of $q_n$ implies that there exists $\gamma_o > 1$ such that for all $n$ large, say $n \geq n_o$, the set $E_{\gamma_o}$ contains precisely $m$ zeros $z_{1,n}, \ldots, z_{m,n}$ of $q_n$, where $z_{j,n} \to z_j$ as $n \to \infty$. We claim that for each $\gamma_1$, with $1 < \gamma_1 < \gamma_o$,

$$(3.2)\qquad \limsup_{n\to\infty} \|q_n\|_{E_{\gamma_1}}^{1/n} = 1.$$

To establish (3.2) we first define

$$(3.3)\qquad \hat{q}_n(z) := q_n(z) / \prod_{j=1}^{m} (z - z_{j,n}), \quad n \geq n_o.$$

Then $\hat{q}_n \in \Pi_{n-m}$ and $\hat{q}_n$ is zero-free in $E_{\gamma_o}$. Furthermore, since the $q_n$ are uniformly bounded on $E$, the Bernstein-Walsh lemma (cf.[W1,p.77]) implies that

$$(3.4)\qquad \limsup_{n\to\infty} \|\hat{q}_n\|_{E_{\gamma_o}}^{1/n} \leq \gamma_o.$$

Next we note that $\overset{o}{E}_{\gamma_o}$ (the interior of $E_{\gamma_o}$) consists of finitely many simply connected components which are bounded by Jordan curves (cf.[W1,p.66]). On any such component $\Omega$ of $\overset{o}{E}_{\gamma_o}$ we can define a single-valued analytic branch of $[\hat{q}_n]^{1/n}$, for $n \geq n_o$. Moreover, from (3.4), the $[\hat{q}_n]^{1/n}$, $n \geq n_o$, form a normal family in $\Omega$. Since $\Omega$ must contain a component $C$ of $E$ and $f$ does not vanish identically on $C$, then

$$\lim_{n\to\infty} \hat{q}_n(z) = f(z) / \prod_{j=1}^{m} (z - z_j) \neq 0$$

for infinitely many points of $C$ (recall that $\overline{\mathbb{C}}\backslash E$ is regular, so $E$ can contain no isolated points). Of course, for such points of $C$ we have

$$(3.5) \qquad \lim_{n\to\infty} |\hat{q}_n(z)|^{1/n} = 1,$$

and so, from the normality property, we deduce that (3.5) holds uniformly on each closed subset of $\Omega$. This fact yields the assertion of (3.2).

Next, let $p_{[cn]} \in \Pi_{[cn]}$ be the Lagrange interpolant to $q_n$ in the $[cn] + 1$ Fekete points for the set $E$. Then it follows from (3.2), the properties of the Fekete points (cf.[W1,p.174]), and the Hermite remainder formula, that

$$(3.6) \qquad \limsup_{n\to\infty} \|q_n - p_{[cn]}\|_E^{1/n} \le 1/\gamma_1^c < 1.$$

Furthermore, by (1.14) and the definition of $e_n(f)$ we have

$$e_{[cn]}(f) \le \|f - p_{[cn]}\|_E \le \|f - q_n\|_E + \|q_n - p_{[cn]}\|_E$$

$$\le K\, e_n(f) + \|q_n - p_{[cn]}\|_E ,$$

and so

$$(3.7) \qquad \frac{e_{[cn]}(f)}{e_n(f)} \le K + \frac{1}{e_n(f)} \|q_n - p_{[cn]}\|_E .$$

But as $f$ is not analytic on $E$, a theorem of Walsh [W1,p.78] asserts that

$$\limsup_{n\to\infty} [e_n(f)]^{1/n} = 1.$$

Letting $\Lambda$ denote a subsequence of $\mathbf{N}$ for which $[e_n(f)]^{1/n} \to 1$ as $n \to \infty$, $n \in \Lambda$, we deduce from (3.6) that

$$\lim_{\substack{n\to\infty \\ n\in\Lambda}} \frac{1}{e_n(f)} \|q_n - p_{[cn]}\|_E = 0.$$

Hence, in view of (3.7), we have

$$\liminf_{n\to\infty} e_{[cn]}(f)/e_n(f) \leq K,$$

which contradicts assumption (1.13). □

## 4. PROOF OF THEOREM 1.4

As in Section 3, let $G$ be the classical Green's function on $\overline{\mathbb{C}}\backslash E$ with pole in $\infty$. Define $E_\gamma$, $\gamma > 1$, as in (3.1). For $z_o \in \overline{\mathbb{C}}\backslash E$ let $G(z,z_o)$ be the classical Green's function on $\overline{\mathbb{C}}\backslash E$ with pole at $z_o$ (cf.[T]).

Write $r_n = p_n/q_n$, where $p_n \in \Pi_n$ has the zeros $z_1^{(n)},\ldots,z_{k(n)}^{(n)}$ and $q_n \in \Pi_n$ has the zeros $w_1^{(n)},\ldots,w_{\ell(n)}^{(n)}$, where each zero is listed according to its multiplicity. Then

$$(4.1) \qquad h_n(z) := \log |r_n(z)| + \sum_{\upsilon=1}^{k(n)} G(z,z_\upsilon^{(n)}) - \sum_{\upsilon=1}^{\ell(n)} G(z,w_\upsilon^{(n)})$$

$$- (k(n) - \ell(n))G(z)$$

is harmonic in $\overline{\mathbb{C}}\backslash E$ and, by the maximum principle, satisfies

$$(4.2) \qquad h_n(z) \le \log \|r_n\|_E , \qquad \text{for} \quad z \in \overline{\mathbb{C}}\backslash E.$$

Choose $\gamma_o > 1$ such that $E_{\gamma_o} \subset U$ and set $\Gamma_{\gamma_o} := \partial E_{\gamma_o}$. Then there are constants $1 < d < D$ such that for all $z_o \in \overline{\mathbb{C}}\backslash U$ and $z \in \Gamma_{\gamma_o}$

$$(4.3) \qquad \log d \le G(z,z_o) \le \log D.$$

Using (4.1), (4.2), and (4.3) we get, for $z \in \Gamma_{\gamma_o}$,

$$(4.4) \qquad |r_n(z)| \le \|r_n\|_E \frac{D^{\ell(n)}}{d^{k(n)}} \gamma_o^{k(n)-\ell(n)} \le \|r_n\|_E (D\gamma_o)^n.$$

By the maximum principle, (4.4) holds for all $z \in E_{\gamma_o}$.

From (1.18) we know that $r_n \to f$ uniformly on $E$. Using arguments similar to those of Section 3 we deduce that

$$(4.5) \qquad \limsup_{n\to\infty} \|r_n\|_{E_{\gamma_1}}^{1/n} = 1$$

for every $1 < \gamma_1 < \gamma_o$. Next, let $p_{n-1} \in \Pi_{n-1}$ be the Lagrange interpolant to $r_n$ in the $n$ Fekete points for $E$. As before (cf.(3.6)), equation (4.5) implies that

$$(4.6)\qquad \limsup_{n\to\infty} \|r_n - p_{n-1}\|_E^{1/n} \le 1/\gamma_1 < 1.$$

From (1.18) we deduce that

$$e_{n-1}(f) \le \|f - p_{n-1}\|_E \le \|f - r_n\|_E + \|r_n - p_{n-1}\|_E$$

$$\le e_n(f) + \|r_n - p_{n-1}\|_E ,$$

so that

$$\limsup_{n\to\infty} [e_{n-1}(f)-e_n(f)]^{1/n} \le \limsup_{n\to\infty} \|r_n-p_{n-1}\|_E^{1/n} \le 1/\gamma_1.$$

Hence

$$(4.7)\qquad \limsup_{n\to\infty} [e_n(f)]^{1/n} \le 1/\gamma_1 < 1,$$

which implies $f$ is analytic on $E$. □

Finally we note that (4.7) yields the stronger conclusion that $f$ is analytic in the interior of $E_{\gamma_1}$ and, since $\gamma_1 < \gamma_o$ is arbitrary, $f$ is analytic in the interior of $E_{\gamma_o}$ as claimed in Remark 2.

## REFERENCES

[Be] S.N. Bernstein, 'Sur la valeur asymptotique de la meilleure approximation de $|x|$', *Acta Mathematica* 37 (1913), 1-57.

[Bo] P.B. Borwein, 'The relationship between the zeros of best approximants and differentiability', *Proc. Amer. Math. Soc.* 92 (1984), 528-532.

[BIS] H.-P. Blatt, A. Iserles, E.B. Saff, 'Remarks on the behavior of best approximating polynomials and rational functions', In: *Algorithms for Approximation* (IMA conference series: new ser. 10) (J.C. Mason and M.G. Cox, ed), Oxford University Press (1987), pp. 437-445.

[BS] H.-P. Blatt and E.B. Saff, 'Behavior of zeros of polynomials of near best approximation', *J. Approx. Theory* 46, No. 4 (1986), 323-344.

[BSS] H.-P. Blatt, E.B. Saff, M. Simkani, 'Jentzsch-Szegö type theorems for the zeros of best approximants', (to appear).

[J] R. Jentzsch, 'Untersuchungen zur Theorie Analytischer Funktionen' Inaugural-Dissertation, Berlin (1914).

[L] A. Levin, 'The distribution of the poles of best approximating rational functions and the analytical properties of the approximated function', *Israel Journal of Mathematics*, 24, No. 2 (1976), 139-144.

[Sz] G. Szegö, 'Uber die Nullstellen von Polynomen, die in einem Kreis gleichmäßig konvergieren', *Sitzungsbericht Ber. Math. Ges.* 21 (1922), 59-64.

[T] M. Tsuji, *Potential Theory in Modern Function Theory*, Chelsea Publishing Company, New York (1958).

[W1] J.L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, American Math. Soc. Coll. Publ., Vol. 20 (1935), 5th ed. (1969).

[W2] J.L. Walsh, 'The analogue for maximally convergent polynomials of Jentzsch's theorem', *Duke Math. J.* 26 (1959), 605-616.