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POLYNOMIAL AND RATIONAL APPROXIMATION IN THE COMPLEX DOMAIN

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ABSTRACT. Approximation theory in the complex variable setting has its roots in classical function theory, but is rich in modern applications. Moreover, it is a subject that lends much insight into real approximation problems. Starting with the example of Taylor series, we describe methods (such as Faber series and interpolation) for generating good polynomial approximants to a function analytic on a compact set in the plane. We also discuss characterizations for polynomials of best uniform approximation and the "near circularity property." An introduction is given to the theory of Padé approximants, which are rational function analogues of the Taylor sections. We conclude by discussing some contrasts between the theories of polynomial and rational approximation.

1. TAYLOR SECTIONS.

The properties of the Taylor sections for an analytic function are a convenient starting point for approximation and interpolation in the complex z-plane (denoted by $\mathbb{C}$). This is because Taylor sections are least squares polynomial approximants as well as interpolating polynomials. Indeed, if f is analytic at $z = 0$, then the Taylor sections

$$s_n(z) = s_n(f;z) := \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k \tag{1.1}$$

satisfy the interpolation conditions

$$s_n^{(j)}(0) = f^{(j)}(0), \qquad j = 0,1,\ldots,n. \tag{1.2}$$

Moreover, the polynomials $1, z, z^2,\ldots$ are orthogonal with respect to the inner product

$$(g,h) := \frac{1}{2\pi r} \int_{C_r} g(z)\overline{h(z)}|dz|, \qquad C_r : |z| = r, \tag{1.3}$$

and, if f is analytic on $|z| \le r$,

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What can be said about the rate of convergence of the Taylor sections? The answer is intimately related to the familiar Cauchy-Hadamard formula for the radius of convergence ρ of a power series $\sum_{k=0}^{\infty} c_k z^k$. That is,

$$\frac{1}{\rho} = \limsup_{k\to\infty} |c_k|^{1/k}. \tag{1.10}$$

The basic convergence result is the following.

Theorem 1.3. *Let f be analytic in an open set that contains the closed unit disk Δ. Then for the sup norm (1.6), the Taylor sections s_n satisfy*

$$\limsup_{n\to\infty} \|f - s_n\|^{1/n} = 1/\rho < 1, \tag{1.11}$$

where ρ is the radius of the largest open disk centered at the origin throughout which f has a single-valued analytic continuation. Moreover, the sequence s_n converges to f for $|z| < \rho$.

The above theorem, which provides a model for more general results to be mentioned later, nicely illustrates the relationship between the degree of convergence and the maximal circular region of analyticity for f; that is, the larger this circular region, the faster the convergence. In particular, for *entire* functions f

$$\lim_{n\to\infty} \|f - s_n\|^{1/n} = 0. \tag{1.12}$$

While the proof of Theorem 1.3 can be deduced via (1.10), it is more instructive to give an argument based on the interpolation property (1.2) of Taylor sections. For this purpose we appeal to the *Hermite representation* (cf. Walsh [62, §3.1]) for interpolating polynomials.

Lemma 1.4. *Suppose f is analytic inside and on the simple closed contour Γ that surrounds the $n + 1$ points $z_0, z_1, \ldots, z_n$. If p is the unique polynomial in Π_n that interpolates f in these points, then*

$$f(z) - p(z) = \frac{1}{2\pi i} \int_\Gamma \frac{w(z)f(t)}{w(t)(t-z)}\,dt, \qquad z \text{ inside } \Gamma, \tag{1.13}$$

where $w(z) := \prod_{k=0}^{n} (z - z_k)$.

Proof. Replacing $f(z)$ by its Cauchy integral representation

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t - z}\,dt, \qquad z \text{ inside } \Gamma,$$

2. POLYNOMIAL APPROXIMATIONS FOR FUNCTIONS ANALYTIC ON E.

Given a compact set E in the z-plane and a function f analytic on E (i.e., f is analytic on an open set $G \supset E$), how do we generate good polynomial approximations to f on E? When E is a closed disk, we can use Taylor sections which are "good" in the sense of Theorem 1.3. For general sets E we need a procedure that likewise reflects the geometry of E.

First, we insist that E does not separate the plane; that is, $\mathbb{C}\setminus E$ is connected. This assumption is necessary if we expect to get uniform convergence (of polynomials) to an arbitrary function analytic on E. For example, the function $f(z) = 1/z$ is analytic on the circle $E : |z| = 1$, but is not the uniform limit on E of any sequence of polynomials because (by the maximum principle) uniform convergence on $|z| = 1$ implies convergence to an analytic function throughout $|z| \leq 1$.

The connectedness of $\mathbb{C}\setminus E$ is also a sufficient condition for polynomial approximation to functions analytic on E as is stated in the following version of the classical Runge's theorem (cf. [62, §1.10]).

Theorem 2.1. *If f is analytic on a compact set E that does not separate the plane, then there exists a sequence of polynomials that converges uniformly to f on E.*

(The question of polynomial approximation to functions not analytic on E is much more delicate and will be addressed in the next section.)

To prove Theorem 2.1, Runge's approach was to first form Riemann sum approximations to the Cauchy integral representation for f. These Riemann sums are rational functions whose poles lie outside E. Through a process of "pole moving," the rational approximants are converted to polynomial approximants.

For reasonable sets E, we can generate polynomial approximants more directly by constructing an analogue of Taylor series. This was the fruitful approach taken by Faber [15]. To simplify the description of Faber's method we assume that E is a compact set (not a single point) whose complement $C^*\setminus E$ with respect to the extended plane is simply connected. The Riemann mapping theorem asserts that there exists a conformal mapping $w = \phi(z)$ of $C^*\setminus E$ onto the exterior of the unit circle in the w-plane (see Figure 2.1). We can insist that $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$ so that, in a neighborhood of infinity,

$$\phi(z) = \frac{z}{c} + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots, \qquad c > 0. \tag{2.1}$$

$$(2.6)\qquad f(z) = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{f(t)}{t-z}\,dt = \frac{1}{2\pi i}\int_{C_r} \frac{f(\psi(s))}{\psi(s)-z}\,\psi'(s)ds.$$

Since $f(\psi(s))$ is analytic in an annulus of the form $1 < |s| < R$, we can expand this function in a Laurent series:

$$f(\psi(s)) = \sum_{n=-\infty}^{\infty} a_n s^n.$$

Substituting this series into (2.6) and recalling (2.5) we get

$$(2.7)\qquad f(z) = \sum_{n=-\infty}^{\infty} \frac{a_n}{2\pi i}\int_{C_r} \frac{s^n\psi'(s)ds}{\psi(s)-z} = \sum_{n=0}^{\infty} \frac{a_n}{2\pi i}\int_{C_r} \frac{s^n\psi'(s)ds}{\psi(s)-z} = \sum_{n=0}^{\infty} a_n F_n(z)$$

(the integrals with negative n vanish because the integrand is $O(1/s^2)$ near ∞).

To summarize, we obtain the Faber expansion for f by forming the Taylor series for the Cauchy integral of the composition $f\circ\psi$ and substituting F_n for w^n. The process is diagrammed below.

$$f(z) \longrightarrow (f\circ\psi)(w) \longrightarrow \frac{1}{2\pi i}\int_{C_r} \frac{(f\circ\psi)(s)}{s-w}\,ds = \sum_0^{\infty} a_n w^n \longrightarrow \sum_0^{\infty} a_n F_n(z)$$

Exploiting the relationship between Taylor and Faber series leads to the following analogue of Theorem 1.3.

Theorem 2.2. Let f be analytic on E and let $\rho(>1)$ be the largest index such that f has a single-valued analytic continuation throughout the interior of the level curve Γ_ρ. Then the partial sums of the Faber series for f satisfy

$$(2.8)\qquad \limsup_{n\to\infty} \Big\| f - \sum_{k=0}^{n} a_k F_k \Big\|_E^{1/n} = 1/\rho < 1,$$

where $\|\cdot\|_E$ denotes the sup norm on E. Moreover, the Faber series converges to f throughout the interior of Γ_ρ.

What does Theorem 2.2 say to realists who do approximation on an interval? If $E = [-1,1]$, then $\phi(z) = z + \sqrt{z^2-1}$ is just the Joukowski transformation with inverse

$$(2.9)\qquad \psi(w) = \tfrac{1}{2}(w + w^{-1}).$$

For $n \geq 1$, the polynomial part of $\phi(z)^n$ is the same as the polynomial part of

$$\phi(z)^n + \phi(z)^{-n} = w^n + w^{-n}$$

When E is bounded by a smooth Jordan arc or curve, we obtain good points by taking the images under $z = \psi(w)$ of such equally spaced points on $|w| = 1$. For example, if $E = [-1,1]$, the images of the roots of $w^n + i = 0$ under the transformation (2.9) yield the zeros of the Chebyshev polynomial T_n.

There are good points of interpolation that can be determined without knowledge of the mapping function. These are the Fekete points (cf. [62,§7.8]).

Definition 2.4. Let $V_n(z_o,z_1,\ldots,z_n) := \prod_{i<j} (z_i - z_j)$ denote the Vandermonde determinant of order $n + 1$. The points $\beta_k^{(n)} = z_k \in E$ for which the maximum

$$\max \{|V_n(z_o,z_1,\ldots,z_n)|;\ z_k \in E,\ k=0,\ldots,n\}$$

is attained are called Fekete points for E.

The positive constant c that appears in the expansion (2.1) for the mapping function has great importance; it is called the transfinite diameter or logarithmic capacity of E and is denoted by $\operatorname{cap}(E)$. Such terminology arises from an electrostatics problem that we now describe.

For a compact set E (with $\mathbb{C}^*\backslash E$ simply connected) we distribute a unit charge over its boundary ∂E so that equilibrium is reached in the sense that the energy with respect to the logarithmic potential is minimized. This corresponds to the problem of finding the minimum of the energy integral

$$(2.12) \qquad I[\mu] := \int_{\partial E} \int_{\partial E} \log|\xi_1 - \xi_2|^{-1} d\mu(\xi_1)d\mu(\xi_2)$$

over all positive unit measures μ supported on ∂E. The unique measure μ_E that minimizes $I[\mu]$ gives the equilibrium charge distribution with potential

$$(2.13) \qquad U_E(z) := \int_{\partial E} \log|z - t|^{-1} d\mu_E(t).$$

Apart from a small exceptional set, this potential has the constant value $I[\mu_E]$ on the boundary of E. The capacity of E is defined as

$$(2.14) \qquad \operatorname{cap}(E) := \exp(-I[\mu_E]).$$

In this context, the essential criterion for (2.11) to be "good points" of interpolation is that the discrete measures

$$(2.15) \qquad \mu_n := \frac{1}{n+1} \sum_{k=0}^{n} \delta(\beta_k^{(n)}),$$

where $\delta(\beta_k^{(n)})$ denotes the unit measure supported at $\beta_k^{(n)}$, converge to the equilibrium measure μ_E (in the weak-star topology). Such convergence implies set. This is the essential fact that is used to prove the Chebyshev Equioscillation theorem. For complex functions, an analogue of the alternating-sign patterns was developed by Rivlin and Shapiro (cf. [48, §2.6]) and is called the extremal signature.

Let's turn to the geometric aspects of best approximation. We let $A(E)$ denote the collection of functions f that are analytic in the interior of E and continuous on E. If $f \in A(E)$ and E is bounded by a Jordan curve Γ, then best polynomial approximation to f on E reduces to best approximation on Γ; that is, by the maximum principle,

$$\|f - p\|_\Gamma = \|f - p\|_E, \quad \text{all} \quad p \in \Pi_n.$$

The image of Γ under $f - p$ is a curve in the w-plane which we denote by $(f - p)(\Gamma)$ and call an error curve. In this context, the problem of best uniform approximation to f is equivalent to finding an error curve that is contained in a disk of minimal radius about $w = 0$.

It had been observed by some authors and crystallized by Trefethen [53] that the minimal error curve $(f - p_n^*)(\Gamma)$ often has a near circularity property in the sense that it winds around the origin $n + 1$ times and is close to being a perfect circle. Before proceeding with a discussion of this phenomenon we give a consequence of perfect circularity.

Lemma 3.2. *Suppose* E *is bounded by a Jordan curve* Γ, $f \in A(E)$, *and* $p \in \Pi_n$. *If the error curve* $(f - p)(\Gamma)$ *is a perfect circle with center at the origin and winding number* $\geq n + 1$, *then* p *is the polynomial of best uniform approximation to* f *on* E *out of* Π_n.

Proof. If, to the contrary, there exists $q \in \Pi_n$ such that $\|f - q\|_E < \|f - p\|_E$, then

$$|(f - p)(z) - (q - p)(z)| = |(f - q)(z)| < \|f - p\|_E = |(f - p)(z)|$$

for all z on Γ. By Rouché's theorem, this means that $q - p$ and $f - p$ have the same number of zeros interior to Γ. But since this number is at least $n + 1$ and $q - p \in \Pi_n$, we arrive at a contradiction.□

As a simple application of Lemma 3.2, consider the problem of finding the polynomial in Π_n that is of best uniform approximation to $f(z) = z^{n+1}$ on $\Delta : |z| \leq 1$. Since f itself has the perfect circularity property, then $p_n^* \equiv 0$. In other words, the Chebyshev polynomials for the disk Δ are just the powers of z.

Using finite Blaschke products we can produce other examples of per-

Next we solve the CF problem for the inverse polynomial

$$(3.6) \qquad p(z) := z^{N-n-1}q(1/z) \in \Pi_{N-n-1},$$

to obtain the minimal extension Blaschke product

$$B(z) = p(z) + \sum_{k=N-n}^{\infty} c_k^* z^k.$$

Since

$$(3.7) \qquad \|B\|_C = \|z^N B(1/z)\|_C = \|z^{n+1}q(z) + \sum_{k=0}^{n} c_{N-k}^* z^k + \sum_{k=N+1}^{\infty} c_k^* z^{N-k}\|_C,$$

then discarding the terms involving negative powers of z (which have small coefficients), we see that the choice

$$c_k = c_{N-k}^*, \qquad k = 0,1, \ldots, n,$$

in (3.5) gives an error curve with a near circularity property.

The polynomial approximants obtained via this CF method are often much better in the sup norm sense than the Taylor sections. Moreover the technique can be extended to find near best rational approximants (cf. [54], [56]). The theoretical underpinnings of the CF method are contained in a paper of Adamjan, Arov, and Krein [1] who generalized the results of Carathéodory, Fejér, Schur, and Takagi.

Let's now turn to the question of convergence of approximating polynomials. We naturally ask, what is the extension of the Weierstrass theorem to the complex setting? Runge's theorem (Theorem 2.1) is not a true generalization because it assumes far more than continuity - it requires f to be analytic in an open set containing E. Only in 1951 did the Russian mathematician Mergelyan confirm the suspicions of many who had worked on the problem by proving that the assumption on f in Runge's theorem could be weakened.

Theorem 3.4 (Mergelyan [35]). *Let E be a compact set that does not separate the plane. If $f \in A(E)$ (that is, f is analytic in the interior of E and continuous on E), then there exists a sequence of polynomials that converges uniformly to f on E.*

The proof of Mergelyan's theorem (cf. [17], [41]) is a *tour de force* that utilizes the Tietze Extension theorem as well as Koebe's 1/4-theorem. Observe that the Weierstrass theorem is a special case of Theorem 3.4 because an interval has an empty interior and so $A(E)$ reduces to the collection of functions continuous on E.

As an application of Theorem 3.4 we mention the following where p_n^* is the polynomial in Π_n of best uniform approximation to f on E. Then f is analytic on E if and only if

$$(3.11) \qquad \limsup_{n\to\infty} E_n(f)^{1/n} < 1.$$

Proof. In one direction the proof is trivial. Namely, if f is analytic on E, then Theorem 2.2 asserts that the Faber sections and, *a fortiori*, the polynomials of best approximation converge geometrically.

On the other hand, if (3.11) holds, then

$$(3.12) \qquad \limsup_{n\to\infty} \| p_{n+1}^* - p_n^* \|_E^{1/n} < 1.$$

Appealing to Lemma 3.5, we deduce that, for some $r > 1$,

$$\limsup_{n\to\infty} \| p_{n+1}^* - p_n^* \|_{\Gamma_r}^{1/n} < 1.$$

But this means that the sequence $\{p_n^*\}_0^\infty$ converges in the interior of Γ_r, necessarily to an analytic extension of f. □

As with several of the theorems presented, Theorem 3.6 is not stated in its full generality - the assumption on E can be considerably weakened.

4. PADÉ APPROXIMANTS.

Polynomials have the advantage of being easy to evaluate. But the same is true of rational functions. Moreover, rational functions have poles which can imitate the singularities of a function to be approximated. In this section we introduce a class of interpolating rational functions called Padé approximants. These rationals provide a natural extension of the Taylor sections. (Standard references are [39], [4], [5,6]; for a historical treatment, see Brezinski [11].)

Given a formal power series

$$(4.1) \qquad f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

we wish to construct a rational function of a certain type whose Taylor coefficients match those of f as far as possible. To be precise, let

$$(4.2) \qquad \Pi_{m,n} := \{R(z) = P(z)/Q(z) : P \in \Pi_m,\ Q \in \Pi_n,\ Q \not\equiv 0\}.$$

Then the matching condition can be stated as follows: For a fixed pair (m,n), find an $R \in \Pi_{m,n}$ such that

$$(4.3) \qquad (f - R)(z) = O(z^{\ell}),$$

The Padé numerators and denominators are rich in algebraic properties such as the 3-term recurrence relations found by Frobenius (see [4, 24, 39] for a detailed discussion of these properties). Here we pause only to mention a representation for Q_{mn} that illustrates the important role played by the Toeplitz determinants

$$(4.9)\qquad D(m/n) := \begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+n-1} \\ a_{m-1} & a_m & \cdots & a_{m+n-2} \\ \vdots & \vdots & & \vdots \\ a_{m-n+1} & a_{m-n+2} & \cdots & a_m \end{vmatrix} \quad (a_k := 0 \text{ if } k < 0)$$

formed from the coefficients of f.

Theorem 4.2. (Jacobi). *If* $D(m/n) \neq 0$, *then*

$$(4.10)\qquad f(z) - [m/n](z) = O(z^{m+n+1})$$

and the Padé denominator Q_{mn} *normalized by* $Q_{mn}(0) = 1$ *is*

$$(4.11)\qquad Q_{mn}(z) = \frac{1}{D(m/n)} \begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+n} \\ a_{m-1} & a_m & \cdots & a_{m+n-1} \\ \vdots & \vdots & & \vdots \\ a_{m-n+1} & a_{m-n+2} & \cdots & a_{m+1} \\ z^n & z^{n-1} & \cdots & 1 \end{vmatrix} .$$

A fast numerical method (based on the Euclidean algorithm) for solving Toeplitz systems and computing PAs is described in [10].

The PAs for (4.1) are typically displayed in a doubly infinite array known as the Padé table:

$$\begin{array}{|cccc} \hline [0/0] & [1/0] & [2/0] & \cdots \\ [0/1] & [1/1] & [2/1] & \cdots \\ [0/2] & [1/2] & [2/2] & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

$D\backslash\{$ poles of $f\}$. Furthermore, as $m\to\infty$, the poles of $[m/\nu](z)$ tend, respectively, to the ν poles of f in D.

For example, suppose that f is a meromorphic function in the plane whose poles are simple and occur at the points ξ_k, where

$$0 < |\xi_1| < |\xi_2| < \cdots .$$

Then Theorem 4.3 asserts that the poles of $[m/1](z)$ tend to ξ_1; the two poles of $[m/2](z)$ tend to ξ_1, ξ_2; etc.

The proof of Theorem 4.3 is based on the following simple observation (cf. [46]). Since

$$(Q_{m\nu}f - P_{m\nu})(z) = O(z^{m+\nu+1}),$$

then for any $Q \in \Pi_\nu$, the product $QP_{m\nu} \in \Pi_{m+\nu}$ satisfies

$$(Q_{m\nu}Qf - QP_{m\nu})(z) = O(z^{m+\nu+1}),$$

and so $QP_{m\nu}$ is the $(m+\nu)$-th Taylor section of $Q_{m\nu}Qf$. Consequently, we can use the Hermite formula (1.15) to write

$$(4.13) \qquad (Q_{m\nu}Qf - QP_{m\nu})(z) = \frac{1}{2\pi i}\int_{|t|=r} \frac{z^{m+\nu+1}(Q_{m\nu}Qf)(t)}{t^{m+\nu+1}(t-z)}\,dt, \qquad |z| < r,$$

provided $Q_{m\nu}Qf$ is analytic on $|t| \le r$. If Q is chosen to be the monic polynomial whose zeros are the poles of f, then r can be taken arbitrarily close to R. On suitably normalizing the Padé denominators $Q_{m\nu}$ we find that the right-hand side of (4.13) tends to zero in D. In particular, at a zero ξ of Q, we have $(Q_{m\nu}Qf)(\xi) \to 0$ and so $Q_{m\nu}(\xi) \to 0$ because $(Qf)(\xi) \neq 0$. This means that every limit polynomial of the $Q_{m\nu}$'s has zeros at the poles of f (the zeros of Q), which establishes the last assertion of Theorem 4.3. (This same argument can be applied to rational functions that interpolate in the "good points" discussed in §2; see [43].)

In proving convergence theorems for PAs, the essential question is: Where (asymptotically) are the poles of the PAs? In Theorem 4.3, the ν poles of f serve as "attractors" for all the available poles of the $[m/\nu]$ approximants. However, if f has fewer than ν poles, then only a subset of the poles of $[m/\nu](z)$ "know where to go," and the remaining poles may wander aimlessly, destroying convergence. The following simple example illustrates this point.

Consider a sequence of nonzero coefficients a_m for which there is a large discrepancy between the root test and the ratio test:

expansion $f(z) = \sum_{0}^{\infty} (-1)^k c_k z^k$, where the c_k's are the moments

$$(4.16) \qquad c_k := \int_0^b t^k d\mu(t), \qquad k=0,1,\ldots .$$

As we now show, the Padé denominators $Q_{n-1,n}$ for f are related to the polynomials that are orthogonal with respect to $d\mu$. Starting with the defining property

$$(Q_{n-1,n}f - P_{n-1,n})(z) = O(z^{2n}),$$

we replace z by $-1/z$ and multiply by z^n to obtain

$$(4.17) \qquad q_n(z) \int_0^b \frac{z d\mu(t)}{z - t} - z p_{n-1}(z) = O(1/z^n),$$

where $q_n(z) := z^n Q_{n-1,n}(-1/z) \in \Pi_n$ and $p_{n-1}(z) := z^{n-1} P_{n-1,n}(-1/z) \in \Pi_{n-1}$. Then for $j=0,1,\ldots,$ we have

$$(4.18) \qquad q_n(z) \int_0^b \frac{z^j}{z - t} d\mu(t) - z^j p_{n-1}(z) = O(z^j/z^{n+1}).$$

Next, we integrate with respect to z around a simple closed contour containing $[0,b]$ in its interior. Using the Cauchy formula, we find that

$$\int_0^b q_n(t) t^j d\mu(t) = 0, \quad \text{for } j=0,1,\ldots, n-1;$$

that is,

$$(4.19) \qquad q_n(z) = z^n Q_{n-1,n}(-1/z)$$

is the n-th degree orthogonal polynomial for $d\mu$. One consequence of this relation is that the zeros of $Q_{n-1,n}(z)$ must be simple and lie on the cut $(-\infty, -1/b)$. On writing the approximant $[(n-1)/n]$ in the form

$$(4.20) \qquad [(n-1)/n](z) = \frac{P_{n-1,n}(z)}{Q_{n-1,n}(z)} = \sum_{j=1}^{n} \frac{A_{nj}}{1 + z t_{nj}},$$

where the t_{nj}'s are zeros of $q_n(t)$, we deduce in a similar manner from (4.17) that

$$(4.21) \qquad \int_0^b P(t) d\mu(t) = \sum_{j=1}^{n} A_{nj} P(t_{nj})$$

for any polynomial $P \in \Pi_{2n-1}$. Hence, the constants A_{nj} are the Christoffel

$E : r_1 \leq |z| \leq r_2$ is the uniform limit of rational functions that have poles at $z = 0$ and $z = \infty$ (think of its Laurent series!).

To describe the more delicate problem of approximating functions in $A(E)$, we let $\overline{\Pi(E)}$ denote the uniform limits on E of polynomials, and $\overline{R(E)}$ denote the uniform limits on E of rational functions whose poles lie outside E. Then the theorem of Mergelyan (Theorem 3.4) states that $A(E) = \overline{\Pi(E)}$ if and only if $\mathbb{C}\setminus E$ is connected. In contrast, the compact sets E for which $A(E) = \overline{R(E)}$ cannot be characterized topologically; that is, this property is not invariant under a homeomorphism of the plane (cf. [20]). The most popular (and most tasteful) example of a compact set E for which $A(E) \neq \overline{R(E)}$ is the Swiss cheese of A. Roth (cf. [17]), which she manufactured by removing a countable number of disjoint open disks from the closed unit disk. For further discussion of the possibility of rational approximation see Gamelin [18].

Existence of Best Approximants. For an arbitrary compact set E, the existence of best polynomial approximants from Π_m is a simple compactness argument. However, for best rational approximants from $\Pi_{m,n}$ $(n > 0)$, this argument must be modified to handle the possibility of poles tending to the boundary of E. Using normal families, Walsh [62, §12.2] proved that best rational approximants exist provided E contains no isolated points.

Uniqueness of Best Approximants. If $f \in C[a,b]$ is real-valued, then Chebyshev showed that the best uniform approximation to f on $[a,b]$ out of

$$\Pi^r_{m,n} := \{R \in \Pi_{m,n} : R \text{ has real coefficients}\} \tag{5.1}$$

is unique (cf. [34, §9.2]). Surprisingly, this is no longer true if approximation to a real-valued f is done from $\Pi_{m,n}$; that is, if we allow rational approximants with complex coefficients. Indeed, as was shown by Saff and Varga [44], the function $f(x) = x^2$ has no unique best uniform approximation on $[-1,1]$ out of $\Pi_{1,1}$ (any such best rational r_{11} has complex coefficients, so that $\overline{r}_{11} (\neq r_{11})$ is also best). Further examples of this type, as well as non-uniqueness results for approximation on a disk can be found in [25], [42].

Given $f \in A(E)$ we can nonetheless construct a table of best uniform rational approximants to f on E by making a specific choice for each pair (m,n). This analogue of the Padé table is called the Walsh array. The convergence theory for this array closely parallels the theory for the Padé table (e.g. Walsh [64] proved an analogue of Theorem 4.3). Moreover, the Padé table can be viewed as a limiting version of Walsh arrays where best approximation is done on disks $E_\varepsilon : |z| \leq \varepsilon$ with $\varepsilon \to 0$ (cf. [55], [63]).

Degree of Convergence of Best Approximants. For $f \in A(E)$, we set

R_n^* to $|x|$ out of $\Pi_{n,n}$ have all their zeros and poles on the imaginary axis and satisfy (cf. [8])

$$\lim_{n\to\infty} R_n^*(z) = \begin{cases} z & \text{for} \quad \operatorname{Re} z > 0 \\ -z & \text{for} \quad \operatorname{Re} z < 0. \end{cases}$$

REFERENCES

1. V. M. Adamjan, D. Z. Arov, and M. G. Krein, "Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem", Math. USSR Sbornik, 15 (1971), 31-73.

2. J. M. Anderson, "The Faber operator", In: Rational Approximation and Interpolation (P.R. Graves-Morris, E. B. Saff, and R. S. Varga, eds.), Lecture Notes in Math., Vol. 1105, Springer-Verlag, Berlin (1984), 1-10.

3. R. J. Arms and A. Edrei, "The Padé tables and continued fractions generated by totally positive sequences", In: Mathematical Essays Dedicated to A. J. Macintyre, Ohio University Press, Athens, Ohio (1970), 1-21.

4. G. A. Baker, Jr., Essentials of Padé Approximants, Academic Press, New York (1975).

5. G. A. Baker, Jr. and P. R. Graves-Morris, Padé Approximants Part I: Basic Theory, Encyl. of Math., Vol. 13, Cambridge Univ. Press, Cambridge (1981).

6. G. A. Baker, Jr. and P. R. Graves-Morris, Padé Approximants Part II: Extensions and Applications, Encyl. of Math., Vol. 14, Cambridge Univ. Press, Cambridge (1981).

7. H.-P. Blatt and E. B. Saff, "Behavior of zeros of polynomials of near best approximation", J. Approx. Theory, 46 (1986).

8. H.-P. Blatt, A. Iserles, and E. B. Saff, "Remarks on the behavior of zeros of best approximating polynomials and rational functions", (to appear).

9. D. Braess, Nonlinear Approximation, Springer-Verlag, Berlin, (to appear).

10. R. P. Brent, F. G. Gustavson, and D. Y. Yun, "Fast solution of Toeplitz systems of equations and computation of Padé approximants", Journal of Algorithms, 1 (1980), 259-295.

11. C. Brezinski, "The long history of continued fractions and Padé approximants", In: Padé Approximations and Applications (M. G. de Bruin, H. van Rossum, eds.), Lecture Notes in Math., Vol. 888, Springer-Verlag, Berlin (1981), 1-27.

12. A. P. Bulanov, "The asymptotics of the maximum deviation of $|x|$ from rational functions", Mat. Sb. 76 (1968), 288-303.

13. J. H. Curtiss, "Faber polynomials and the Faber series", Amer. Math. Monthly, 78 (1971), 577-596.

31. D. S. Lubinsky and E. B. Saff, "A dictionary of generalized Padé approximants", Institute for Constr. Math. Technical Report (1986), Univ. of South Fla.

32. A. P. Magnus, "Another theorem of convergence of complex weight Padé approximants", Padé Meeting, Luminy, France, 14-18 Oct. 1985.

33. A. Markoff, "Deux demonstrations de la convergence de certains fractions continues", Act. Math., **19** (1895), 93-104.

34. G. Meinardus, Approximation of Functions: Theory and Numerical Methods, Springer-Verlag, Berlin (1967).

35. S. N. Mergelyan, "On the representation of functions by series of polynomials on closed sets", (Russian) Dokl. Akad. Nauk SSSR, 78, 405-408; Translations Amer. Math. Soc. No. 85 (1953).

36. D. J. Newman, "Rational approximation to $|x|$ ", Michigan Math. J., **11** (1964), 11-14.

37. D. J. Newman, Approximation with Rational Functions, Regional Conference Series in Math, Vol. 41, Amer. Math. Soc., Providence, R.I. (1979).

38. J. Nuttall and C. J. Wherry, "Gaussian integration for complex weight Padé functions", J. Inst. Math. Appl., 21 (1978), 165-170.

39. O. Perron, Die Lehre von den Kettenbrüchen, Chelsea Pub. Co., New York (1929).

40. V. A. Popov, "Uniform rational approximation of the class V and its applications", Acta Math. Acad. Sci. Hungar, 29 (1977), 119-129.

41. W. Rudin, Real and Complex Analysis, McGraw-Hill, New York (1974).

42. A. Ruttan, "On the cardinality of a set of best complex rational approximations to a real function", In: Padé and Rational Approximations (E. B. Saff and R. S. Varga, eds.), Academic Press, New York (1977), 303-319.

43. E. B. Saff, "An extension of Montessus de Ballore's theorem on the convergence of interpolating rational functions", Journ. Approx. Theory, **6** (1972), 63-67.

44. E. B. Saff and R. S. Varga, "Nonuniqueness of best complex rational approximation to real functions on real intervals", J. Approx. Theory, 23 (1978), 78-85.

45. E. B. Saff and R. S. Varga, "On the zeros and poles of Padé approximants to e^z , III, Numer. Math., 30 (1978), 241-266.

46. E. B. Saff, "An introduction to the convergence theory of Padé approximants", In: Aspects of Contemporary Complex Analysis (D. A. Brannan, J. G. Clunie, eds.), Academic Press, New York (1980), 493-502.

64. J. L. Walsh, "The convergence of sequences of rational functions of best approximation with some free poles", In: Proc. Sympos. Approximation of Functions (General Motors Res. Lab., 1964), Elsevier, Amsterdam (1965), 1-16.

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