WEIGHTED POLYNOMIAL APPROXIMATION OF ANALYTIC FUNCTIONS

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ABSTRACT

A necessary and sufficient condition is given for a weight such that the geometric order of weighted polynomial approximation of a function f is equivalent to the analyticity of f.

1. *Background and statements of results for L_p, $0 < p < \infty$*

Let $d\mu$ be a Borel measure (not necessarily finite) on $[-1, 1]$ with infinite support and set

$$E_n(f)_{L_p(d\mu)} := \inf_{p_n \in \Pi_n} \|f - p_n\|_{L_p(d\mu)}, \qquad n = 0, 1, \ldots, \tag{1.1}$$

which is the error in best approximation of f in $L_p(d\mu)$ by algebraic polynomials of degree at most n. We consider the sets

$$A_\mu := \{f \mid f \text{ coincides } d\mu\text{–a.e. with some } g \text{ analytic on } [-1, 1]\} \tag{1.2}$$

and, for $0 < p \leqslant \infty$,

$$B_{\mu, p} := \{f \mid \beta(f)_{\mu, p} < 1\}, \tag{1.3}$$

where

$$\beta(f)_{\mu, p} := \limsup_{n \to \infty} [E_n(f)_{L_p(d\mu)}]^{1/n}. \tag{1.4}$$

The purpose of this paper is to address the following.

PROBLEM. Determine necessary and sufficient conditions on $d\mu$ such that $A_\mu = B_{\mu, p}$.

For $d\mu(x) = dx$, that is, for Lebesgue measure on $[-1, 1]$, S. N. Bernstein's classical theorem (cf. [**2**; **7**, §5.4.1, 6.5.1]) asserts that, for $p = \infty$, the classes A_μ and $B_{\mu, p}$ coincide and, moreover, the formula

$$\beta(f)_{\mu, p} = \alpha(f) \tag{1.5}$$

holds, where

$$\alpha(f) := \inf\{1/r \mid g \text{ can be analytically extended to the interior of the ellipse } [2x/(r + r^{-1})]^2 + [2y/(r - r^{-1})]^2 = 1, \text{ where } g \text{ is the function in (1.2)}\}. \tag{1.6}$$

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In this section, we consider the above problem for $0 < p < \infty$. In Section 2, we provide the proofs of the results stated in Section 1 and, in Section 3, we investigate the case when $p = \infty$.

Our main result for $0 < p < \infty$ is the following.

THEOREM 1.1. *Let $0 < p < \infty$ be fixed. Then $A_\mu = B_{\mu, p}$ if and only if there exists a polynomial $Q(\not\equiv 0)$ in $L_p(d\mu)$ such that the nth root of the leading coefficient of the nth degree orthonormal polynomial for the weight $|Q|^p \, d\mu$ tends to 2 as $n \to \infty$. Moreover, if the latter condition holds for some $Q(\not\equiv 0)$ in $L_p(d\mu)$, then it holds for every $Q(\not\equiv 0)$ in $L_p(d\mu)$.*

We shall also show that the equality of the sets A_μ and $B_{\mu, p}$ is 'p-independent' in the following sense.

COROLLARY 1.2. *If $0 < p, q < \infty$, then $A_\mu = B_{\mu, p}$ if and only if $A_\mu = B_{\mu, q}$.*

Moreover, Bernstein's formula persists in this general setting. Namely, we have the following.

COROLLARY 1.3. *If $A_\mu = B_{\mu, p}$, then (1.5) holds for every $f \in A_\mu$.*

Theorem 1.1 can also be formulated in terms of the limiting distribution of the zeros of orthonormal polynomials. For this purpose, let us assume that $Q \not\equiv 0$ is a polynomial that belongs to $L_p(d\mu)$ and let

$$R_n(|Q|^p \, d\mu, z) = \gamma_n(|Q|^p \, d\mu) \, z^n + \dots \; \# \in \Pi_n \tag{1.7}$$

be the nth orthonormal polynomial corresponding to the measure $|Q|^p \, d\mu$. We say that $|Q|^p \, d\mu$ is an *arcsine measure* if the zeros $\{x_{i,n}\}_{i=1}^n$ of R_n have the arcsine distribution as $n \to \infty$; that is,

$$\lim_{n \to \infty} \frac{\# \{x_{i,n} \mid x_{i,n} \in (a, b)\}}{n} = \frac{1}{\pi} \int_a^b \frac{dx}{(1-x^2)^{\frac{1}{2}}}$$

for every interval $(a, b) \subset [-1, 1]$. Since arcsine measures play an important role in the theory of orthogonal polynomials, we formulate the following.

COROLLARY 1.4. *Suppose that $d\mu$ is not a singular measure with respect to Lebesgue measure on $[-1, 1]$. Assume that $0 < p < \infty$ and $Q \not\equiv 0$ is a polynomial in $L_p(d\mu)$. Then $A_\mu = B_{\mu, p}$ if and only if $|Q|^p \, d\mu$ is an arcsine measure.*

As proved in [**3**, **4**], if $d\mu$ is finite and $\mu' > 0$ almost everywhere on $[-1, 1]$, then $d\mu$ is an arcsine measure and, therefore, $A_\mu = B_{\mu, p}$. Note, however, that a construction given in [**9**] shows that there are sets $E \subset [-1, 1]$ with arbitrarily small Lebesgue measure such that if $\mu' > 0$ almost everywhere on E, then

$$\lim_{n \to \infty} [\gamma_n(\mu' |_E)]^{1/n} = 2.$$

Since $\operatorname{supp}(d\mu) \subset [-1, 1]$ implies that

$$\liminf_{n \to \infty} [\gamma_n(d\mu)]^{1/n} \geqslant 2$$

(cf. [3]) and $\gamma_n(d\mu)$ is a decreasing function of the measure $d\mu$ (cf. [5, p. 50]), it follows that for such a measure

$$\lim_{n\to\infty} [\gamma_n(d\mu)]^{1/n} = 2.$$

Thus, there are sets E of arbitrarily small measures such that if $\mu' > 0$ almost everywhere on E, then $A_\mu = B_{\mu,p}$.

We also remark that there are many equivalent formulations of the condition '$[\gamma_n(|Q|^p\,d\mu)]^{1/n} \to 2$ as $n\to\infty$' of Theorem 1.1. One of them is

$$\lim_{n\to\infty} |R_n(|Q|^p\,d\mu, z)|^{1/n} = |z+(z^2-1)^{\frac{1}{2}}| \tag{1.8}$$

uniformly on compact subsets of $\mathbb{C}\backslash[-1, 1]$ (see, for example, [5, §III, Theorems 7.1 and 9.1] and Lemma 2.2 below).

Theorem 1.1 immediately follows from the following more complete result.

THEOREM 1.5. *Let $0 < p < \infty$ be fixed. Then*

(i) *$A_\mu \subset B_{\mu,p}$ if and only if there is a polynomial $Q(\not\equiv 0)$ in $L_p(d\mu)$.*

(ii) *$B_{\mu,p} \subset A_\mu$ if and only if either*

(a) *the only polynomial in $L_p(d\mu)$ is the zero polynomial, or*

(b) *there exists a polynomial $Q(\not\equiv 0)$ in $L_p(d\mu)$ such that*

$$\lim_{n\to\infty} [\gamma_n(|Q|^p\,d\mu)]^{1/n} = 2, \tag{1.9}$$

which, in turn, is equivalent to (1.9) *holding for every $Q(\not\equiv 0)$ in $L_p(d\mu)$.*

2. *Proofs of results for $0 < p < \infty$*

We begin with the following.

Proof of Theorem 1.5. To establish (i), suppose first that $A_\mu \subset B_{\mu,p}$. Then by considering polynomial approximants to a non-polynomial analytic function (say, $f(x) = e^x$), it is easy to construct a polynomial $Q(\not\equiv 0) \in L_p(d\mu)$.

In the converse direction, we shall show that for every polynomial $Q \not\equiv 0$ and function $f \in A_\mu$ there are polynomials $P_n \in \Pi_n$, $n = 0, 1, \ldots$, such that

$$\limsup_{n\to\infty} \|(f-P_n)Q^{-1}\|_\infty^{1/n} \leqslant \alpha(f) < 1, \tag{2.1}$$

where $\|\cdot\|_\infty$ denotes the supremum norm on $[-1, 1]$ and $\alpha(f)$ is defined in (1.6). That $f \in B_{\mu,p}$ then follows from

$$\limsup_{n\to\infty} \|f-P_n\|_{L_p(d\mu)}^{1/n} \leqslant \alpha(f) \limsup_{n\to\infty} \left(\int_{-1}^{1} |Q|^p\,d\mu\right)^{1/pn} = \alpha(f). \tag{2.2}$$

Consequently our proof also shows that $\beta(f)_{\mu,p} \leqslant \alpha(f)$ whenever $A_\mu = B_{\mu,p}$.

To establish (2.1) we note that by Bernstein's formula (1.5), which holds for $p = \infty$ and $d\mu(x) = dx$, there exist polynomials $q_n(x) \in \Pi_n$, $n = 0, 1, \ldots$, such that

$$\limsup_{n\to\infty} \|f-q_n\|_\infty^{1/n} \leqslant \alpha(f). \tag{2.3}$$

It follows easily from (2.3) that the sequences of derivatives also satisfy

$$\limsup_{n\to\infty} \|f^{(j)}-q_n^{(j)}\|_\infty^{1/n} \leqslant \alpha(f), \qquad j = 1, 2, \ldots. \tag{2.4}$$

Consequently, by subtracting a suitable Hermite interpolant of $f-q_n$ we can construct polynomials $P_n \in \Pi_n$, $n \geqslant n_0$, such that

$$\limsup_{n\to\infty} \|f-P_n\|_\infty^{1/n} \leqslant \alpha(f) \tag{2.5}$$

and $f-P_n$ vanishes at those zeros of Q (counting multiplicity) that lie on $[-1, 1]$. A standard application of Bernstein's inequality (cf. [**1**; **7**, 2.13.27]) then implies that for the supremum norm on the ellipse

$$\Gamma_\rho: [2x/(\rho+\rho^{-1})]^2 + [2y/(\rho-\rho^{-1})]^2 = 1 \tag{2.6}$$

(cf. (1.6)) we have, for every $1 < \rho < 1/\alpha(f)$,

$$\limsup_{n\to\infty} \|f-P_n\|_{\Gamma_\rho}^{1/n} \leqslant \alpha(f)\,\rho.$$

Hence, for ρ close to 1,

$$\limsup_{n\to\infty} \|(f-P_n)\,Q^{-1}\|_{\Gamma_\rho}^{1/n} \leqslant \alpha(f)\,\rho. \tag{2.7}$$

But, as $(f-P_n)\,Q^{-1}$ is analytic inside Γ_ρ for ρ close to 1, the maximum principle implies that (2.7) holds on $[-1, 1]$. Thus, on letting $\rho \to 1$, we get (2.1). This completes the proof of statement (i).

The basis of the proof of statement (ii) of Theorem 1.5 is the following.

LEMMA 2.1. *$B_{\mu,p} \subset A_\mu$ if and only if the following condition holds:*

$D(d\mu, p)$: $$\limsup_{n\to\infty} \Big[\sup_{\substack{P_n \in \Pi_n \\ P_n \not\equiv 0}} \|P_n\|_\infty / \|P_n\|_{L_p(d\mu)} \Big]^{1/n} \leqslant 1.$$

Proof. Suppose that $D(d\mu, p)$ holds and $f \in B_{\mu,p}$. Let $p_n \in \Pi_n$, $n = 1, 2, \ldots$, satisfy

$$\limsup_{n\to\infty} \|f-p_n\|_{L_p(d\mu)}^{1/n} = \beta(f)_{\mu,p} < 1. \tag{2.8}$$

Then

$$\limsup_{n\to\infty} \|p_n - p_{n-1}\|_{L_p(d\mu)}^{1/n} \leqslant \beta(f)_{\mu,p},$$

and so, by $D(d\mu, p)$,

$$\limsup_{n\to\infty} \|p_n - p_{n-1}\|_\infty^{1/n} \leqslant \beta(f)_{\mu,p}.$$

Hence the p_n converge geometrically in the uniform norm to some function g which, by Bernstein's theorem, must be analytic on $[-1, 1]$. Since (2.8) implies that $f-p_{n_k} \to 0$ $d\mu$-a.e. for some subsequence $\{n_k\}$, then $f = g$ $d\mu$-a.e., and so $f \in A_\mu$.

Now suppose that $B_{\mu,p} \subset A_\mu$ and assume to the contrary that for some $Q_{n_k} \in \Pi_{n_k}$ and $\beta > 1$,

$$\|Q_{n_k}\|_\infty / \|Q_{n_k}\|_{L_p(d\mu)} > \beta^{n_k}, \qquad n_k \to \infty.$$

On squaring the Q_{n_k} and dividing by $\beta^{-n_k/2}\|Q_{n_k}\|_\infty^2$ we get a sequence of non-negative polynomials $P_{n_k} \in \Pi_{2n_k}$ satisfying

$$\|P_{n_k}\|_\infty = \gamma^{n_k} \quad \text{but} \quad \|P_{n_k}\|_{L_p(d\mu)} < \gamma^{-n_k}, \tag{2.9}$$

where $\gamma := \sqrt{\beta} > 1$. Setting $f := \sum P_{n_k}$, we have

$$E_n(f)_{L_p(d\mu)} = O(\gamma^{-n/2}), \qquad n = 1, 2, \ldots,$$

that is, $f \in B_{\mu,p}$. But observe that for every constant K the set

$$\{x \in [-1, 1] \,|\, f(x) > K\}$$

contains an interval I_K. If we had $f \in A_\mu$, then f would be bounded by some K^* $d\mu$-a.e.; hence we would have $\mu(I_{K^*}) = 0$. But this is impossible since if c is the centre of I_{K^*} and $Q(\not\equiv 0) \in L_p(d\mu)$, then one would get $Q(x)|x-c|^{-1} \in B_{\mu,p}$ (cf. [**11**, p. 75]) while $Q(x)|x-c|^{-1} \notin A_\mu$ (recall that $d\mu$ has infinite support) contradicting $B_{\mu,p} \subset A_\mu$. Lemma 2.1 is now proved.

Notice that the first part of the proof of Lemma 2.1 yields via Bernstein's theorem that if $A_\mu = B_{\mu,p}$, then $\alpha(f) \leqslant \beta(f)_{\mu,p}$. This and the remark following (2.2) justify Corollary 1.3.

We now return to the proof of statement (ii) of Theorem 1.5. First suppose that $d\mu$ is finite. It is easy to see that condition $D(d\mu, p)$ for $p = 2$ is the same as

$$\lim_{n\to\infty} \|R_n(d\mu)\|_\infty^{1/n} = 1, \tag{2.10}$$

where $R_n(d\mu)$ are the orthonormal polynomials with respect to $d\mu$ (simply expand any P_n in terms of the R_m). Moreover, as we shall show in Lemma 2.2 below, relation (2.10) is equivalent to $[\gamma_n(d\mu)]^{1/n} \to 2$ as $n \to \infty$. Hence for finite $d\mu$ all we have to show (cf. also the completion of the proof in the general case below) is that $D(d\mu, p)$ is the same condition for every p. Let $0 < p < q < \infty$. By Hölder's inequality, $D(d\mu, p)$ implies $D(d\mu, q)$. Now assume $D(d\mu, q)$. Then for every $P_n \in \Pi_n$,

$$\|P_n\|_{L_q(d\mu)} \leqslant \|P_n\|_\infty^{1-p/q} \|P_n\|_{L_p(d\mu)}^{p/q},$$

which implies that

$$\|P_n\|_\infty / \|P_n\|_{L_p(d\mu)} \leqslant [\|P_n\|_\infty / \|P_n\|_{L_q(d\mu)}]^{q/p}.$$

Together with $D(d\mu, q)$ this last inequality yields $D(d\mu, p)$.

Finally, let $d\mu$ be arbitrary. By Lemma 2.1, we need to show that $D(d\mu, p)$ is equivalent to the statement (a) or (b) of part (ii) of Theorem 1.5. Clearly (a) implies $D(d\mu, p)$ and so, from the discussion for finite measures, it suffices to prove that the conditions $D(d\mu, p)$ and $D(|Q|^p\, d\mu, p)$ are equivalent for every polynomial $Q(\not\equiv 0) \in L_p(d\mu)$. That $D(|Q|^p\, d\mu, p)$ implies $D(d\mu, p)$ is obvious. Now suppose that $Q \in \Pi_s$. One can easily see that, for $P_n \in \Pi_n$,

$$\|P_n\|_\infty \leqslant Cn^{2s} \|P_n Q\|_\infty$$

for some constant C independent of n. Hence, if we assume $D(d\mu, p)$, we get for $\varepsilon > 0$

$$\begin{aligned} \|P_n\|_\infty &\leqslant Cn^{2s} \|P_n Q\|_\infty \leqslant CC_\varepsilon n^{2s}(1+\varepsilon)^{n+s} \|P_n Q\|_{L_p(d\mu)} \\ &= CC_\varepsilon n^{2s}(1+\varepsilon)^{n+s} \|P_n\|_{L_p(|Q|^p d\mu)}, \end{aligned}$$

and this implies $D(|Q|^p\, d\mu, p)$. The proof of Theorem 1.5 is now complete.

In the above proof we used the equivalence given in the following.

LEMMA 2.2. *For a finite Borel measure $d\nu$ on $[-1, 1]$, the two statements*

$$\lim_{n\to\infty} \|R_n(d\nu)\|_\infty^{1/n} = 1 \tag{2.11}$$

and

$$\lim_{n\to\infty} [\gamma_n(d\nu)]^{1/n} = 2 \tag{2.12}$$

are equivalent.

This is more or less a known result, but for completeness we present a proof.

Proof. First suppose that for every $\varepsilon > 0$,

$$\|R_n(d\nu)\|_\infty \leqslant (1+\varepsilon)^n$$

for all large n. Then, by Bernstein's inequality [**1**], we have

$$|R_n(d\nu, x)| \leqslant |T_n(x)|(1+\varepsilon)^n \quad \text{for } |x| \geqslant 1, n \geqslant n_\varepsilon, \tag{2.13}$$

where $T_n(x) := \cos(n \arccos x)$ is the nth Chebyshev polynomial. Hence, by letting $x \to \infty$ and making use of the fact that $T_n(x)$ has 2^{n-1} as its leading coefficient, we get

$$\gamma_n(d\nu) \leqslant 2^{n-1}(1+\varepsilon)^n, \qquad n \geqslant n_\varepsilon$$

and so

$$\limsup_{n\to\infty} [\gamma_n(d\nu)]^{1/n} \leqslant 2.$$

On the other hand,

$$\liminf_{n\to\infty} [\gamma_n(d\nu)]^{1/n} \geqslant 2$$

is always true if $\operatorname{supp}(d\nu) \subset [-1, 1]$ (see [**3**, Lemma 2.1]). These facts prove that (2.11) implies (2.12).

Conversely, (2.12) implies that $d\nu$ is arcsine [**3**, Theorem 1.1] and hence (cf. [**4**, Lemma 3.1])

$$\lim_{n\to\infty} \|R_n(d\nu)/\gamma_n(d\nu)\|_\infty^{1/n} = \tfrac{1}{2},$$

which, together with (2.12), gives (2.11).

Proof of Corollary 1.2. Obviously, for $0 < p, q < \infty$ there is a non-trivial polynomial Q^* in $L_p(d\mu)$ if and only if there is one, say Q^{**}, in $L_q(d\mu)$. We then have $Q^*Q^{**} \in L_p(d\mu) \cap L_q(d\mu)$ and since the $Q(\not\equiv 0)$ in Theorem 1.1 is arbitrary in $L_p(d\mu)$, we get that the condition of Theorem 1.1 is simultaneously satisfied or not for p and q, which proves the corollary.

Proof of Corollary 1.4. If $d\mu$ has no carrier (that is, a set $E \subset [-1, 1]$ with $\mu(E) = \mu([-1, 1])$ of capacity zero, then the conditions in Corollary 1.4 and Theorem 1.1 are equivalent by [**10**, Theorem 1]. Now if $d\mu$ is not singular with respect to Lebesgue measure, then every carrier has positive Lebesgue measure and hence positive capacity (cf. [**8**, Theorem III.10]). Thus the corollary follows.

3. *Supremum norm results*

Having settled the cases when $p < \infty$, we now turn to $p = \infty$. Since for $L_\infty(d\mu)$ the norm involves only the support of $d\mu$ and not $d\mu$ itself, it is appropriate to reformulate the Problem of Section 1 in the following way.

Let $w \geqslant 0$ be a Lebesgue measurable function on $[-1, 1]$ that is positive on a set of positive measure, and let

$$E_n(f)_w^* := \inf_{p_n \in \Pi_n} \|w(f-p_n)\|_{L_\infty}, \tag{3.1}$$

$$\beta(f)_w^* := \limsup_{n\to\infty} [E_n(f)_w^*]^{1/n}, \tag{3.2}$$

$$B_w^* := \{f \mid \beta(f)_w^* < 1\}. \tag{3.3}$$

The reformulated problem is the characterization of those w for which $A_w^* = B_w^*$, where A_w^* stands for the set of functions that coincide $w(x)\,dx$–a.e. with a function analytic on $[-1, 1]$.

Just as in Section 2, one can show that $A_w^* \subset B_w^*$ if and only if there is a non-trivial polynomial Q such that $Qw \in L_\infty$; and $B_w^* \subset A_w^*$ if and only if

$$\limsup_{n\to\infty} \Big[\sup_{\substack{P_n\in\Pi_n \\ P_n \not\equiv 0}} \|P_n\|_\infty / \|wP_n\|_{L_\infty} \Big]^{1/n} \leqslant 1. \tag{3.4}$$

However, as we show in the following example, the case when $p = \infty$ differs from $p < \infty$ in that (3.4) is no longer equivalent to the condition appearing in Theorem 1.1.

EXAMPLE. There exists a set $E \subset [-1, 1]$ such that for the characteristic function χ_E for E we have $A_{\chi_E}^* = B_{\chi_E}^*$ but

$$[\gamma_n(\chi_E)]^{1/n} \nrightarrow 2 \text{ as } n \to \infty, \tag{3.5}$$

where $\gamma_n(\chi_E)$ is the leading coefficient of the nth orthonormal polynomial belonging to the measure $\chi_E(x)\,dx$.

CONSTRUCTION. For a natural number n and $\delta < 1/n^3$ set

$$E_{n,\delta} := \bigcup_{s=-n^3+1}^{n^3-1} \left[\frac{s}{n^3} - \delta, \frac{s}{n^3} + \delta\right]. \tag{3.6}$$

It is obvious from Markoff's inequality

$$\|P_n'\|_\infty \leqslant n^2 \|P_n\|_\infty, \qquad P_n \in \Pi_n,$$

that for any choice of numbers $0 < \delta_n < 1/n^3$, the set

$$E := \bigcup_{n=1}^{\infty} E_{n,\delta_n} \tag{3.7}$$

has the property

$$\limsup_{n\to\infty} \Big[\sup_{\substack{P_n\in\Pi_n \\ P_n \not\equiv 0}} \|P_n\|_\infty / \|P_n\|_E \Big] \leqslant 1, \tag{3.8}$$

where $\|\cdot\|_E$ denotes the L_∞-norm on E. Hence by the remarks made above, $A_{\chi_E}^* = B_{\chi_E}^*$.

Next we show that for an appropriate sequence $\{\delta_n\}$, the assertion of (3.5) is also true. We require that

$$\delta_{n+1} < \tfrac{1}{2}\delta_n, \qquad n = 1, 2, \ldots, \tag{3.9}$$

and inductively define the δ_n and two sequences $\{n_k\}$ and $P_{n_k} \in \Pi_{n_k}$ as follows.

Let $C(F)$ denote the (inner) logarithmic capacity of the Borel set F (see, for example [**8**, p. 55]). We need the following elementary properties (cf. [**8**, Chapter III, Sections 2, 3]).

(a) If $T: x' = ax + b$ is a linear transformation and $E_1 = T(E)$, then $C(E_1) = |a|\,C(E)$.
(b) $C(F_1) \leqslant C(F_2)$ whenever $F_1 \subset F_2$.
(c) $C([a, b]) = \frac{1}{4}(b - a)$.
(d) If $F_n \subset [-\frac{1}{2}, \frac{1}{2}]$, $n = 1, 2, \ldots$, and $F = \bigcup_{n=1}^{\infty} F_n$, then

$$\log C(F) \leqslant \left[\sum_{n=1}^{\infty} 1/\log C(F_n) \right]^{-1}.$$

One can easily justify from these properties that the following construction can be carried out.

Let δ_1 be so small that

$$C(E_{1,\delta_1}) < \tfrac{1}{4}. \tag{3.10}$$

By a well-known result of Szegö [6] (cf. [8, Theorem III.26]), if $F \subset [-1, 1]$ is closed, then

$$\lim_{n\to\infty} [\inf_{p_{n-1}\in\Pi_{n-1}} \|x^n - p_{n-1}\|_F]^{1/n} = C(F).$$

Hence there is $n_1 > 1$ and a monic polynomial $P_{n_1} \in \Pi_{n_1}$ such that

$$\|P_{n_1}\|_{E_{1,\delta_1}} < 4^{-n_1}$$

which implies that

$$\|P_{n_1}\|_{L_2(E_{1,\delta_1})} < 4^{-n_1}.$$

If δ_2 is sufficiently small, then (3.9) implies that for the set E in (3.7) we have

$$\|P_{n_1}\|_{L_2(E)} < 4^{-n_1}. \tag{3.11}$$

In fact we can choose $\delta_2, ..., \delta_{n_1}$ so small that with $E_m := \bigcup_{n=1}^m E_{n,\delta_n}$ both (3.11) and

$$C(E_{n_1}) < \tfrac{1}{4}$$

hold (cf. (3.10)). Then there is $n_2 > n_1$ and a monic $P_{n_2} \in \Pi_{n_2}$ such that

$$\|P_{n_2}\|_{E_{n_1}} < 4^{-n_2}.$$

If we choose $\delta_{n_1+1}, ..., \delta_{n_2}$ sufficiently small, then

$$\|P_{n_2}\|_{L_2(E)} < 4^{-n_2}$$

and

$$C(E_{n_2}) < \tfrac{1}{4}$$

hold. By continuing in this manner we obtain a sequence of monic polynomials $P_{n_k} \in \Pi_{n_k}$ such that

$$\|P_{n_k}\|_{L_2(E)} < 4^{-n_k}, \qquad n_k \to \infty.$$

But this means (cf. [5, p. 50]) that

$$[\gamma_{n_k}(\chi_E)]^{1/n_k} > 4;$$

that is, (3.5) holds.

In order to formulate a positive theorem, we call a Borel set $E \subset [-1, 1]$ of *stable capacity* $\frac{1}{2}$ (cf. [9, 3]) if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$C(E \backslash F) > \tfrac{1}{2} - \varepsilon$$

whenever $m(F) < \delta$ (m denotes the Lebesgue measure on $\mathbb{R}$). Noting that $C([-1, 1]) = \frac{1}{2}$, we now prove the following (compare [3, Theorem 1.3a]).

THEOREM 3.1. $A_w^* = B_w^*$ *for every bounded* w *that is positive almost everywhere on* $E \subset [-1, 1]$ *if and only if* E *has stable capacity* $\frac{1}{2}$.

Proof. Since w is bounded, $A_w^* \subset B_w^*$ is immediate from Bernstein's theorem (cf. (1.5) for $p = \infty$), so we have only to consider $B_w^* \subset A_w^*$.

If E has stable capacity $\frac{1}{2}$ and $w(x) > 0$ almost everywhere on E, then $d\mu(x) := w(x)\,dx$ is an arcsine measure (see [3, Theorem 1.3a]) and so, by Corollary 1.4 we have $B_{\mu,1} \subset A_w^*$ and thus $B_w^* \subset A_w^*$.

Now suppose that E does not have stable capacity $\frac{1}{2}$. Then there is $\varepsilon > 0$ such that for every $\delta > 0$ there is $F_\delta \subset E$ with $m(F_\delta) < \delta$ and $C(E \backslash F_\delta) < \frac{1}{2} - \varepsilon$. Since $C(E \backslash F_\delta)$ denotes inner capacity, there is a set F'_δ with $m(F'_\delta) < \delta$ such that $E \backslash F'_\delta$ is compact and

$$C(E \backslash F'_\delta) < \tfrac{1}{2} - \varepsilon.$$

By applying the previously mentioned theorem of Szegö to the sets $E \backslash F'_\delta$ and $[-1, 1]$, we get that there is a monic polynomial $P_\delta \in \Pi_{n(\delta)}$ of degree $n(\delta) > 1/\delta$ such that

$$[\|P_\delta\|_\infty / \|P_\delta\|_{E \backslash F'_\delta}]^{1/n(\delta)} > 1/(1-2\varepsilon). \tag{3.12}$$

Therefore, if

$$0 \leqslant w(x) \leqslant 1, \qquad x \in [-1, 1],$$

and

$$w(x) < (1-2\varepsilon)^{n(\delta)} \quad \text{for } x \in F'_\delta, \tag{3.13}$$

then we have from (3.12) that

$$(\|P_\delta\|_\infty / \|wP_\delta\|_E)^{1/n(\delta)} > 1/(1-2\varepsilon). \tag{3.14}$$

Since $m(F'_\delta) < \delta$, almost every x belongs to at most finitely many of the sets $F'_{2^{-n}}$, $n = 1, 2, \ldots$. Hence we can define a weight w $(0 \leqslant w \leqslant 1)$ that is positive almost everywhere on E, zero outside E and for which (3.13) holds for every $\delta = 2^{-n}$, $n = 1, 2, \ldots$. But then (3.14) shows that (3.4) is not true and hence $B_w^* \not\subset A_w^*$.

Let us finally mention that Theorem 3.1 extends to any $\infty > p > 0$. Let $E \subseteq [-1, 1]$ be a Borel set. Then $A_\mu = B_{\mu, p}$ for every finite measure μ for which $\mu'(x) > 0$ almost everywhere on E if and only if E has stable capacity $\frac{1}{2}$. This follows from [**3**, Theorem 1.3a] and Corollary 1.4.

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