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BOUNDED APPROXIMATION BY POLYNOMIALS WHOSE ZEROS LIE ON A CIRCLE

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ABSTRACT. In a recent paper the first author gave an explicit construction of a sequence of polynomials having their zeros on the unit circumference which converge boundedly to a given bounded zero-free analytic function in the unit disk. In this paper we find the best possible uniform bound for such approximating polynomials and construct a sequence for which this bound is attained. The method is also applied to approximation of an analytic function in the unit disk by rational functions whose poles lie on the unit circumference. Some open problems are discussed.

Let C denote the unit circumference in the z-plane and D its interior. The term *C-polynomial* shall mean a polynomial all of whose zeros lie on C.

In [6] the first author unified the results of MacLane [5] and Chui [2] by proving

THEOREM A. *Let $f(z)=1+c_1z+c_2z^2+\cdots$ be a zero-free holomorphic function in D. Then there exists a sequence of C-polynomials $P_n(z)$ assuming the value one at $z=0$ which converges to $f(z)$ uniformly on every closed subset of D. If in addition $M\equiv\sup_{|z|<1}|f(z)|<\infty$, then the $P_n(z)$ converge to $f(z)$ boundedly.*

In the present paper we show that if $f(z)$ is bounded in D, then the C-polynomials $P_n(z)$ of Theorem A can be chosen so that for all n we have

$$|P_n(z)| < 2M, \qquad z \text{ in } D. \tag{1}$$

This result improves the bound of $2(M+1)$ obtained in [6]. Furthermore, we prove that if the function $f(z)$ is not a C-polynomial, then the constant $2M$ is the best possible uniform bound for any sequence of C-polynomials which converge to $f(z)$ in D.

We first establish

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THEOREM 1. *If the function $f(z)$ of Theorem A is bounded in D, then the C-polynomials $P_n(z)$ of Theorem A can be chosen so that inequality* (1) *holds for all n.*

PROOF. Clearly we may assume that $f(z)$ is nonconstant in D. Let r_k ($k=1, 2, \cdots$) be a strictly increasing sequence of positive numbers which converge to one, and put $M_k \equiv \max_{|z|\leq r_k} |f(z)|$. It follows from the Maximum Principle that $M-M_k>0$. Also we have $M-M_k \to 0$ as $k\to\infty$.

Let $s_n(z)\equiv 1+\sum_{i=1}^{n} c_i z^i$, and choose $s_{n_k}(z)$, $n_1<n_2<\cdots$, so that $s_{n_k}(z)\not\equiv 0$ and

$$|f(z) - s_{n_k}(z)| < M - M_k \tag{2}$$

for $|z|<r_k$. Put

$$t_k(z) \equiv s_{n_k}(r_k z), \qquad t_k^*(z) \equiv z^{d_k}\overline{t_k(1/\bar{z})},$$

where d_k is the degree of $t_k(z)$. Then from (2) we have

$$|t_k(z)| < M \quad \text{for } |z| < 1,$$

and hence the lemma in [6] implies that

$$P_k(z) \equiv t_k(z) + z^k t_k^*(z)$$

is a C-polynomial which satisfies $|P_k(z)|<2M$ for $|z|<1$.

The fact that the sequence $P_k(z)$ converges to $f(z)$ uniformly on every closed subset of D follows by the same reasoning used in the proof of Theorem A. This proves Theorem 1.

To show that the uniform bound in (1) is best possible we use the following result due to Ankeny and Rivlin [1]:

THEOREM B. *If $p(z)$ is a polynomial of degree n such that $\max_{|z|=1}|p(z)|=1$ and $p(z)$ has no zeros in D, then*

$$\max_{|z|=R} |p(z)| \leqq (1 + R^n)/2, \qquad R > 1,$$

with equality only for $p(z)=(\lambda+\mu z^n)/2$, where $|\lambda|=|\mu|=1$.

If $P(z)$ is a polynomial of degree n whose zeros all lie on C and if $\max_{|z|=1}|P(z)|=A$, then we can apply Theorem B to the polynomial $p(z)=z^n P(1/z)/A$. It follows that for $|z|\leqq\rho<1$ we have

$$|P(z)| \leqq A(1 + \rho^n)/2. \tag{3}$$

We can now prove

THEOREM 2. *Let* $\{Q_{n_k}(z)\}_{k=1}^{\infty}$ *be a sequence of C-polynomials converging pointwise to the function* $F(z)$ *in* D. *Suppose that the degree of* $Q_{n_k}(z)$ *is* n_k $(n_1<n_2<\cdots)$. *If* $\sup_{|z|<1}|F(z)|=M_0<\infty$, then $\sup_{k,|z|<1}|Q_{n_k}(z)|\geqq 2M_0$.

PROOF. If for all k and all z in D we have $|Q_{n_k}(z)|\leqq 2M_0-\eta$, where $\eta>0$ is independent of k and z, then by inequality (3) for each fixed z $(|z|=\rho)$ in D we have

$$|F(z)| = \lim_{k\to\infty} |Q_{n_k}(z)| \leqq (2M_0-\eta)\lim_{k\to\infty}\left(\frac{1+\rho^{n_k}}{2}\right) = M_0 - \frac{\eta}{2}.$$

Therefore we would have $\sup_{|z|<1}|F(z)|<M_0$ contrary to the hypothesis of the theorem. This completes the proof.

COROLLARY. *If the function* $f(z)$ *of Theorem* A *is bounded in* D *and is not a C-polynomial, then the constant* $2M$ *is the smallest possible uniform bound in* D *for any sequence of C-polynomials which converge to* $f(z)$ *in* D.

REMARK 1. Let $P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0$, $a_n\neq 0$, be a C-polynomial such that $\max_{|z|=1}|P(z)|=1$. In [**3**, problem 4.2] Hayman asks if

$$(4)\qquad |a_k| \leqq 1/2 \quad \text{for } k=0,1,\cdots,n.$$

From inequality (3) it is easy to see that $|a_0|=|a_n|\leqq 1/2$, with equality only for $P(z)=(\lambda+\mu z^n)/2$, where $|\lambda|=|\mu|=1$. Furthermore the inequalities (4) are readily verified for the special cases $n=1$, 2, and 3.

Indeed if $n=3$, it follows from a well-known result proved by Lax [**4**] that $|P'(z)|\leqq 3/2$, for $|z|\leqq 1$. Since the polynomial

$$Q(z) \equiv z^2\,\overline{P'(1/\bar{z})}$$

has no zeros in D, we also have

$$|Q'(z)| = |2\bar{a}_1 z + 2\bar{a}_2| \leqq 3/2 \quad \text{for } |z| \leqq 1.$$

But $|a_1|=|a_2|$, and so the last inequality implies that

$$(5)\qquad |a_i| \leqq 3/8 \quad \text{for } i=1,2.$$

Equality in (5) is attained for the polynomial $P(z)=(z+1)^3/8$.

It seems possible, therefore, that estimates sharper than (4) can be obtained for certain inside coefficients of $P(z)$.

REMARK 2. In [7], Maynard Thompson proved the following theorem:

THEOREM C. *Let $f(z)$ be a bounded analytic function in D. Then there exists a sequence of rational functions $R_n(z)$ of the form*

$$R_n(z) = \sum_{k=1}^{m_n} (z_{n,k} - z)^{-1}, \qquad |z_{n,k}| = 1, \tag{6}$$

which converge to $f(z)$ uniformly on every closed subset of D and which satisfy

$$|R_n(z)| \leqq K/(1 - |z|) \quad \textit{for } |z| < 1,$$

where K is a constant independent of n and z.

Theorem A enables us to prove a related result. Indeed if $f(z)$ is an arbitrary analytic function in D, then $g(z) \equiv \exp\,(\int_0^z f(t)dt)$ is analytic and different from zero in D. Furthermore $|g(z)|$ will be bounded in D if Re $(\int_0^z f(t)dt)$ is bounded above. Therefore, by Theorem A, we can find a sequence of C-polynomials $p_n(z)$ such that for z in D we have

$$\lim_{n\to\infty} \frac{p_n'(z)}{p_n(z)} = \frac{g'(z)}{g(z)} = f(z).$$

Since $p_n'(z)/p_n(z)$ is a rational function of the form (6), we have established

THEOREM 3. *Let $f(z)$ be an analytic function in D. Then there exists a sequence of rational functions $R_n(z)$ of the form* (6) *which converges to $f(z)$ uniformly on every closed subset of D. If in addition* Re $(\int_0^z f(t)dt)$ *is bounded above in D, then the sets of poles $\{z_{n,k}\}_{k=1}^{m_n}$ are such that the numbers $\prod_{k=1}^{m_n} |e^{i\theta} - z_{n,k}|$ are uniformly bounded for all $0 \leqq \theta \leqq 2\pi$ and for all n.*

A comparison of Theorem C and Theorem 3 leads to an interesting question concerning the mutual relationship between the two conditions

$$|p_n(z)| \leqq M_1 \quad \text{for } |z| < 1, \tag{7}$$

and

$$|p_n'(z)/p_n(z)| \leqq M_2/(1 - |z|) \quad \text{for } |z| < 1, \tag{8}$$

where the $p_n(z)$ are C-polynomials and the numbers M_1 and M_2 are independent of n and z. It is easy to see that neither condition alone implies the other. For if $p_n(z) = (z^n - 1)/(z - 1)$, then (8) holds with

$M_2=2$, but $\max_{|z|=1} |p_n(z)| = n \to \infty$. On the other hand, Theorem A implies that there exists a sequence of C-polynomials $p_n(z)$ which satisfy (7) and which converge in D to the function $\exp[(z-1)/(z+1)]$. Since $p_n'(z)/p_n(z) \to 2/(z+1)^2$, condition (8) does not hold for these $p_n(z)$.

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