JENTZSCH–SZEGÖ TYPE THEOREMS FOR THE ZEROS OF BEST APPROXIMANTS

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Dedicated to the memory of Walter Kaufmann-Bühler

ABSTRACT

Let E be a compact set in the complex plane $\mathbb{C}$ having connected and regular complement. For a non-entire function f analytic in the interior of E and continuous on E, we investigate the limiting distribution of the zeros of the sequence of polynomials $\{p_n^*\}_1^\infty$ of best uniform approximation to f on E. The zeros of best L_p polynomial approximants and best uniform rational approximants having a bounded number of free poles are also considered.

1. *Introduction*

The classical theorems of Jentzsch and Szegö to which we refer in the title concern the limiting behaviour of the zeros of the partial sums of a power series. More precisely, if

$$s_n(z) := \sum_{k=0}^{n} a_k z^k, \quad n = 0, 1, 2, \ldots,$$

are the partial sums of a power series $\sum_0^\infty a_k z^k$ having *finite* positive radius of convergence ρ, then Jentzsch proved [6] that each point on the circle of convergence $C_\rho: |z| = \rho$ is a limit point of the set of zeros of the polynomials $s_n(z)$, $n = 1, 2, \ldots$. In [10], Szegö substantially augmented this result by showing that there is a subsequence $\{n_k\}_{k=1}^\infty$ for which the zeros of the partial sums $s_{n_k}(z)$ are uniformly distributed in angle; that is, if $S(\alpha,\beta)$ is the sector

$$S(\alpha,\beta) := \{z \in \mathbb{C}: \alpha < \arg(z) < \beta\}, \quad \alpha < \beta < 2\pi + \alpha$$

and $Z_n(A)$ denotes the number of zeros of $s_n(z)$ in the set A, then

$$\lim_{k\to\infty} n_k^{-1} Z_{n_k}(S(\alpha,\beta)) = (\beta-\alpha)/2\pi$$

for all sectors $S(\alpha,\beta)$.

Since the partial sums $s_n(z)$ are least-squares approximants to the sum function $f(z) = \sum_0^\infty a_k z^k$ on any circle $C_r: |z| = r$, $r < \rho$, it is natural to ask if other sequences of approximating polynomials possess these Jentzsch–Szegö type properties. J. L. Walsh [14] studied this question for so-called *maximally convergent* polynomial approximants to a function $f(z)$ analytic on a compact set $E \subset \mathbb{C}$, and M. Fekete and J. L. Walsh [4] analysed the asymptotic behaviour of zeros for certain sequences of extremal polynomials, such as the Chebyshev polynomials associated with E.

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More recently, Blatt and Saff investigated the behaviour of zeros of polynomials of best uniform approximation to a function f that is continuous on the compact set E, analytic in its (2-dimensional) interior E°, but not analytic at every boundary point of E. In particular, they proved in [2] that if $f \in C[a, b]$, f not analytic on the interval $[a, b]$ (for example $f(x) = |x|$ on $[-1, 1]$), then every point of $[a, b]$ must be a limit point of the set of zeros of the polynomials $\{p_n^*\}_1^\infty$ of the best uniform approximation to f on $[a, b]$. This fact answers affirmatively a question raised by P. Borwein in [3].

To state the results of Blatt and Saff in full generality we need to introduce some terminology from potential theory. Throughout this paper E denotes a compact set in the complex plane $\mathbb{C}$ that has *positive* logarithmic capacity (transfinite diameter) (cf. [11]). We denote this capacity by $\operatorname{cap}(E)$. Let μ_E be the unique unit measure with $\operatorname{supp}(\mu_E) \subset E$ that minimizes the energy integral

$$I[\mu] := \iint \log|z-t|^{-1}\, d\mu(t)\, d\mu(z) \tag{1.1}$$

over all unit measures supported on E. Then μ_E is called the *equilibrium distribution* for E and

$$U(\mu_E; z) := \int \log|z-t|^{-1}\, d\mu_E(t) \tag{1.2}$$

is the *conductor potential* of E. The minimum energy $I[\mu_E]$ is related to the capacity of E via

$$\operatorname{cap}(E) = \exp(-I[\mu_E]). \tag{1.3}$$

The complement of E with respect to the extended plane $\bar{\mathbb{C}}$ consists of the union of at most countably many pairwise disjoint connected open sets (components) and we denote by K the unbounded component of $\bar{\mathbb{C}} \backslash E$. The *Green's function* $g_K(z, \infty)$ with pole at infinity for K is given by (cf. [11, p. 82])

$$g_K(z, \infty) = I[\mu_E] - U(\mu_E; z) = -\{\log[\operatorname{cap}(E)] + U(\mu_E; z)\}, \tag{1.4}$$

and is positive and harmonic in $K \backslash \{\infty\}$. We say that K is *regular* if for each point $z_0 \in \partial K$ (the boundary of K) we have

$$\lim_{z \to z_0} g_K(z, \infty) = 0, \qquad z \in K. \tag{1.5}$$

For example, if E is a continuum (not a single point), then K is regular.

Now let f be a function continuous on E and set

$$e_n(f) := \min\{\|f - p_n\|_E : p_n \in \Pi_n\}, \tag{1.6}$$

where $\|\cdot\|_E$ denotes the supremum norm on E and Π_n is the collection of all algebraic polynomials having degree at most n. We denote by $p_n^* = p_n^*(f, E)$ the polynomial in Π_n of best uniform approximation to f on E; that is,

$$\|f - p_n^*\|_E = e_n(f). \tag{1.7}$$

By the theorem of Mergelyan (cf. [8]), if the complement of E is connected, then

$$\lim_{n \to \infty} e_n(f) = 0$$

for any function f continuous on E and analytic in E°. On writing

$$p_n^*(z) = a_n^* z^n + \dots, \tag{1.8}$$

it is shown in [2] that the coefficients a_n^* 'carry the information' as to whether such a function f is analytic *on* E. Namely, we have the following.

THEOREM 1.1 [2]. *Let f be continuous on E and analytic in E°, where the complement of E is connected and regular. Then the following assertions are equivalent:*

(i) *f is not analytic on E;*

(ii) $(\limsup)_{n\to\infty} |a_n^*|^{1/n} = 1/\text{cap}(E)$.

Using this result Blatt and Saff established the following Jentzsch-type theorem.

THEOREM 1.2 [2]. *Suppose that f is continuous on E, analytic in E°, but not analytic on E, where the complement of E is connected and regular. Assume further that f does not vanish identically on any component of E°. Then every boundary point of E is a limit point of the set of zeros of the sequence of best approximants $\{p_n^*\}_1^\infty$.*

The analogue of Theorem 1.2 for the case when f is analytic on E, but not entire, appears in the previously cited work of Walsh [14]. (We remark, however, that Theorem 1.2 is considerably more delicate than these results of Walsh and cannot be directly deduced from them.)

The purpose of the present paper is to examine, in the spirit of Szegö, the limiting distribution of the zeros of the polynomials p_n^*. In [2], Blatt and Saff considered this question for the case when the complement of E is simply connected. Here we resolve this question for arbitrary connected regular complements. Roughly speaking, we shall show that if ν_n^* is the discrete measure having mass $1/n$ at each zero of p_n^*, then under the assumptions of Theorem 1.2 there is a subsequence $\nu_{n_k}^*$ that converges in the weak-star topology to the equilibrium distribution μ_E for E.

The outline of this paper is as follows. In Section 2 we state and discuss our main results, and in Section 3 we provide their proofs. Finally, in Section 4 we present some further applications of our results.

2. *Main results*

For a polynomial p_n of precise degree n, we denote by $\nu_n = \nu(p_n)$ the discrete unit measure defined on the Borel sets in $\mathbb{C}$ having mass $1/n$ at each zero of p_n, with the obvious modification in this definition for the case when p_n has multiple zeros. We say that ν_n *converges weakly*† to the measure μ as $n\to\infty$ if

$$\lim_{n\to\infty} \int \phi \, d\nu_n = \int \phi \, d\mu \tag{2.1}$$

for every continuous function ϕ on $\mathbb{C}$ having compact support.

With the above notation we state our key theorem from which the main results follow.

THEOREM 2.1. *Let E be a compact set in the complex plane $\mathbb{C}$ with* $\text{cap}(E) > 0$ *and set* $E^* := \text{supp}(\mu_E)$, *where μ_E is the equilibrium distribution for E. Let $\mathcal{N}$ be an infinite subset of positive integers and $\{p_n\}_{n\in\mathcal{N}}$ be a sequence of monic polynomials of respective*

† In the terminology of Landkof [7] this is called *vague* convergence.

degrees precisely n. Then $\nu_n = \nu(p_n)$ *converges weakly to* μ_E *as* $n \to \infty$, $n \in \mathcal{N}$, *if conditions* (a) *and* (b) *below are satisfied.*

(a) $(\limsup)_{n\to\infty} \|p_n\|_{E^*}^{1/n} \leq \operatorname{cap}(E)$, $n \in \mathcal{N}$;

(b) $\lim_{n\to\infty} \nu_n(A) = 0$, $n \in \mathcal{N}$, *for every closed set A contained in the union of the bounded components of* $\bar{\mathbb{C}} \setminus E^*$.

Conversely, suppose that the unbounded component of $\bar{\mathscr{C}} \setminus E^*$ *is regular and the zeros of the* p_n *are uniformly bounded. If* $\nu_n \to \mu_E$ *weakly, then conditions* (a) *and* (b) *hold.*

Since, for every monic polynomial $p_n(z) = z^n + \ldots \in \Pi_n$ it is known that

$$\|p_n\|_{E^*}^{1/n} \geq \operatorname{cap}(E^*) = \operatorname{cap}(E), \qquad n = 1, 2, \ldots, \tag{2.2}$$

condition (a) is equivalent to

(a′) $\lim_{n\to\infty} \|p_n\|_{E^*}^{1/n} = \operatorname{cap}(E)$, $n \in \mathcal{N}$.

We further note that condition (b) is vacuously satisfied if E has empty interior and a connected complement.

Theorem 2.1 has an immediate application to the *Chebyshev polynomials* $T_n(z) = T_n(E; z)$ corresponding to certain sets E. For each $n \geq 1$, the monic polynomial $T_n(z) = z^n + \ldots \in \Pi_n$ is defined by

$$\|T_n\|_E = \min\{\|z^n - P_{n-1}(z)\|_E : P_{n-1} \in \Pi_{n-1}\} \tag{2.3}$$

and is known to satisfy (cf. [11, p. 73])

$$\lim_{n\to\infty} \|T_n\|_E^{1/n} = \operatorname{cap}(E). \tag{2.4}$$

Hence, from Theorem 2.1, we obtain the following.

COROLLARY 2.1. *Suppose that E is a compact set with positive capacity that has empty interior and a connected complement. Then* $\nu(T_n)$ *converges weakly to* μ_E *as* $n \to \infty$.

We remark that the conclusion of Corollary 2.1 may not hold if E has either *non-empty* interior or a *disconnected* complement. The examples E: $|z| \leq 1$ and E: $|z| = 1$, for both of which $T_n(z) = z^n$, provide simple illustrations of this fact. On the other hand, if E is a proper subarc of the unit circle, say

$$E = \{z \in \mathbb{C} : |z| = 1,\ \varepsilon \leq |\arg(z)|\}, \qquad 0 < \varepsilon < \pi,$$

then Corollary 2.1 does apply and we have $\nu(T_n) \to \mu_E$ weakly.

For the situation of Theorem 1.2, where f is not analytic on E, Theorem 2.1 leads to the following.

THEOREM 2.2. *Let f and E be as in Theorem* 1.2 *and let* $\{p_n^*\}_1^\infty$ *be the polynomials of best uniform approximation to f on E. Then* $\nu_n = \nu(p_n^*)$ *converges weakly to* μ_E *as* $n \to \infty$ *through a sequence* $\Lambda = \Lambda(f)$ *of positive integers.*

Consequently, for any Borel set $B \subset \mathbb{C}$,

$$\mu_E(B^\circ) \leq \liminf_{n\to\infty} \nu_n(B) \leq \limsup_{n\to\infty} \nu_n(B) \leq \mu_E(\bar{B}), \qquad n \in \Lambda, \tag{2.5}$$

where B° *and* $\bar{B}$ *denote the interior and closure of B, respectively.*

In the above theorem, $\Lambda = \Lambda(f)$ is any sequence for which (see Theorem 1.1(ii))

$$\lim_{n\to\infty} |a_n^*|^{1/n} = 1/\mathrm{cap}(E), \qquad n\in\Lambda, \tag{2.6}$$

or, equivalently,

$$\lim_{n\to\infty} [e_{n-1}(f)-e_n(f)]^{1/n} = 1, \qquad n\in\Lambda. \tag{2.7}$$

Since E has connected regular complement in Theorem 2.2, we have $\mathrm{supp}(\mu_E) = \partial E$. Hence the assertion of Theorem 1.2 that each point of ∂E is a limit point of the set of zeros of $\{p_n^*\}_1^\infty$ is an immediate consequence of (2.5). We further remark that Theorem 2.2 also holds for sequences of polynomials of 'near best' approximation; that is, if $q_n \in \Pi_n$, $n = 1, 2, \ldots$, satisfy

$$\|f-q_n\|_E \leqslant e_n(f)+O(\tau^n), \qquad n = 1,2,\ldots \tag{2.8}$$

for some $\tau\in(0,1)$, then the conclusions of Theorem 2.2 remain valid if p_n^* is replaced by q_n. In contrast, if polynomials $q_n \in \Pi_n$ satisfy, for some $\varepsilon > 0$,

$$\|f-q_n\|_E \leqslant (1+\varepsilon)e_n(f), \qquad n = 1,2,\ldots,$$

then the limit points of the set of zeros of $\{q_n\}_1^\infty$ need not be dense on ∂E (cf. [**1**]) and hence Theorem 2.2 is no longer valid for such q_n.

To describe the analogue of Theorem 2.2 for the case when f is analytic *on* E, where E has a connected regular complement K, we define for each $\sigma > 1$ the region

$$E_\sigma := E \cup \{z\in K : 0 < g_K(z,\infty) < \log\sigma\}, \tag{2.9}$$

which has boundary

$$\Gamma_\sigma := \{z\in K : g_K(z,\infty) = \log\sigma\}. \tag{2.10}$$

If f is analytic on E, but not entire, then there exists a largest real number $\rho = \rho(f) > 1$ such that f is single-valued and analytic throughout E_ρ. Furthermore,

$$\limsup_{n\to\infty} [e_n(f)]^{1/n} = 1/\rho(f) \tag{2.11}$$

(cf. [**12**, §4.7]). The counterpart of Theorem 2.2 may now be formulated as follows.

THEOREM 2.3. *Let E be as in Theorem* 1.1 *and f be analytic on E, but not entire, and suppose that f does not vanish indentically on any component of E_ρ, where $\rho = \rho(f)$ is defined as above. Let $\{p_n^*\}_1^\infty$ denote the polynomials of best uniform approximation to f on E. Then $\nu_n = \nu(p_n^*)$ converges weakly to μ_{E_ρ} (the equilibrium distribution for $\bar{E}_\rho$) as $n\to\infty$ through a sequence $\Lambda = \Lambda(f)$ of positive integers.*

Consequently, (2.5) *holds with μ_E replaced by μ_{E_ρ}.*

In Theorem 2.3, $\Lambda = \Lambda(f)$ is any sequence for which

$$\lim_{n\to\infty} [e_{n-1}(f)-e_n(f)]^{1/n} = 1/\rho(f), \qquad n\in\Lambda. \tag{2.12}$$

Furthermore, Theorem 2.3 holds, more generally, if $\{p_n^*\}_1^\infty$ is replaced by a sequence $\{q_n\}_1^\infty$, $q_n \in \Pi_n$, that satisfies

$$\|f-q_n\|_E \leqslant e_n(f)+O(\tau^n), \qquad n = 1,2,\ldots$$

for some $\tau\in(0, 1/\rho(f))$.

3. *Proofs of results in Section* 2

A basic ingredient in the proof of Theorem 2.1 is the following lemma.

LEMMA 3.1. *Let E, E^*, and $\{p_n\}_{n\in\mathcal{N}}$ be as in Theorem* 2.1. *If condition* (a) *of Theorem* 2.1 *holds, then*

$$\lim_{n\to\infty} \nu_n(A) = 0, \qquad n\in\mathcal{N}, \tag{3.1}$$

for every closed set $A \subset K^*$, where K^* is the unbounded component of $\bar{\mathbb{C}}\backslash E^*$.

Proof. Although this lemma is essentially a consequence of Walsh's theory of exact harmonic majorants (cf. [13, 14]), we shall give a straightforward proof.

For each $\tau > 1$, set

$$K_\tau^* := \{z\in K^* : g_{K^*}(z,\infty) > \log\tau\}, \tag{3.2}$$

where $g_{K^*}(z,\infty)$ is the Green's function with pole at ∞ for K^*. It is not difficult to see that K_τ^* is connected (even if ∂K^* contains irregular points). Let $A\subset K^*$ be closed and choose $\rho > 1$ such that $A\subset K_\rho^*$. We denote by $\{z_{n,k}\}_{k\in I_n}$ the set of zeros of p_n that lie in K_ρ^* and define, for $n\in\mathcal{N}$,

$$t_n(z) := \log|p_n(z)| - ng_{K^*}(z,\infty) + \sum_{k\in I_n} g_{K^*}(z, z_{n,k}), \tag{3.3}$$

where $g_{K^*}(z, z_{n,k})$ is the Green's function with pole at $z_{n,k}$ for the domain K^*. Notice that each $t_n(z)$ is subharmonic in K^*.

By symmetry of the Green's function we have, for $k\in I_n$,

$$g_{K^*}(\infty, z_{n,k}) = g_{K^*}(z_{n,k},\infty) \geqslant \log\rho. \tag{3.4}$$

Hence, for $n\in\mathcal{N}$,

$$t_n(\infty) \geqslant n\log[\operatorname{cap}(E)] + n\nu_n(K_\rho^*)\log\rho. \tag{3.5}$$

On the other hand, it is clear from (3.3) that

$$\limsup_{\substack{z\to\zeta\\ z\in K^*}} t_n(z) \leqslant \log\|p_n\|_{E^*}$$

for all $\zeta\in\partial K^*$ except for a set of capacity zero. Thus, by the generalized maximum principle (cf. [11, Theorem III.28]), we have

$$t_n(\infty) \leqslant \log\|p_n\|_{E^*}, \qquad n\in\mathcal{N}. \tag{3.6}$$

Finally, combining (3.5) and (3.6) yields

$$\nu_n(A) \leqslant \nu_n(K_\rho^*) \leqslant \{\log\|p_n\|_{E^*}^{1/n} - \log[\operatorname{cap}(E)]\}/(\log\rho), \tag{3.7}$$

so that (3.1) follows from condition (a) of Theorem 2.1.

We can now give the proof of Theorem 2.1.

Proof of Theorem 2.1. First assume that conditions (a) and (b) hold. By Helly's selection theorem [7, p. 12], there is a subsequence of the unit measures $\nu_n = \nu(p_n)$ that converges weakly to a measure μ. We shall continue to denote this subsequence by ν_n. Since, for any open set $\Omega\subset\mathbb{C}$,

$$\mu(\Omega) \leqslant \liminf_{n\to\infty} \nu_n(\Omega), \tag{3.8}$$

it follows from condition (b) and Lemma 3.1 that μ is a unit measure with $\operatorname{supp}(\mu) \subset E^*$.

Now choose $\tau > 1$ so large that the boundary of K_τ^* in (3.2) is a simple closed curve in K^*, and let D be any open connected set contained in K^* such that $\bar{K}_\tau^* \subset D$ and $\delta := \operatorname{dist}(D, E^*) > 0$. Write

$$p_n(z) = q_n(z)\hat{p}_n(z), \tag{3.9}$$

where $q_n(z)$ is the monic polynomial of degree $n\nu_n(D)$ whose zeros are all the zeros of p_n in D. Setting

$$d_n := \deg(\hat{p}_n) = n\nu_n(\mathbb{C}\setminus D),$$

it follows that

$$\|p_n\|_{E^*} \geqslant \delta^{n\nu_n(D)}\|\hat{p}_n\|_{E^*}.$$

Since, by Lemma 3.1, $\nu_n(D) \to 0$ and $d_n/n = [1-\nu_n(D)] \to 1$ as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \|\hat{p}_n\|_{E^*}^{1/d_n} \leqslant \limsup_{n\to\infty} \|p_n\|_{E^*}^{1/n} \leqslant \operatorname{cap}(E). \tag{3.10}$$

Next, define

$$h_n(z) := U(\mu_E; z) + (1/d_n)\log|\hat{p}_n(z)|. \tag{3.11}$$

These functions are subharmonic in K^* and, by (3.10) and the Bernstein–Walsh lemma (cf. [12, pp. 77, 87]), we have

$$\limsup_{n\to\infty} h_n(z) \leqslant 0, \qquad z \in K^*. \tag{3.12}$$

The functions h_n are harmonic in D and uniformly bounded from above there. Thus $\{h_n\}$ forms a normal family in D. Any limit function h of this family is harmonic in D, and has $h(\infty) = 0$ as well as $h(z) \leqslant 0$ for $z \in \partial K_\tau^*$. Therefore $h(z) = 0$ in K_τ^* and hence $h \equiv 0$ in D. We have shown that the whole sequence $\{h_n\}$ converges to $h \equiv 0$ in D. Hence

$$\lim_{n\to\infty} U(\hat{\nu}_n; z) = U(\mu_E; z), \qquad z \in D, \tag{3.13}$$

where $U(\hat{\nu}_n; z)$ is the potential associated with $\hat{\nu}_n := \nu(\hat{p}_n)$.

Since $\nu_n \to \mu$ weakly, it is easy to see that $\hat{\nu}_n \to \mu$ weakly. Hence, by the principle of descent [7, p. 62],

$$U(\mu; z) \leqslant \liminf_{n\to\infty} U(\hat{\nu}_n; z), \qquad z \in \mathbb{C},$$

and so from (3.13) and the arbitrariness of D we get

$$U(\mu; z) \leqslant U(\mu_E; z), \qquad z \in K^*. \tag{3.14}$$

Finally, since $U(\mu_E; z) \leqslant I[\mu_E]$ for every $z \in \mathbb{C}$ and $U(\mu; z)$ is lower semicontinuous, we obtain from (3.14) that

$$U(\mu; z) \leqslant I[\mu_E], \quad z \in E^*. \tag{3.15}$$

On integrating (3.15) with respect to $d\mu$ over E^*, and recalling that $\mu(E^*) = 1$ and $\operatorname{supp}(\mu) \subset E^*$, we see that the energy $I[\mu]$ satisfies $I[\mu] \leqslant I[\mu_E]$. Hence both μ and μ_E are distributions that minimize energy over E^*. By uniqueness of the solution to the minimum energy problem (cf. [11, p. 80]), we get $\mu = \mu_E$ and so the whole sequence $\{\nu_n\}_{n\in\mathcal{N}}$ converges weakly to μ_E.

In the converse direction, we now assume that K^* is regular and $\nu_n = \nu(p_n)$

converges weakly to μ_E. Then clearly condition (b) holds, so it remains to show the validity of condition (a). For this purpose, choose $\mathcal{N}_1 \subset \mathcal{N}$ such that

$$\lim_{\substack{n\to\infty\\ n\in\mathcal{N}_1}} \|p_n\|_{E^*}^{1/n} = \limsup_{\substack{n\to\infty\\ n\in\mathcal{N}}} \|p_n\|_{E^*}^{1/n}, \tag{3.16}$$

and, for each $n\in\mathcal{N}_1$, let $z_n\in E^*$ satisfy $|p_n(z_n)| = \|p_n\|_{E^*}$. We can assume without loss of generality that $z_n \to z_0$ as $n\to\infty$, $n\in\mathcal{N}_1$. Since $\nu_n \to \mu_E$ weakly, and the supports of the ν_n are all contained in a fixed compact set, the generalized principle of descent [7, Theorem 1.3] yields

$$U(\mu_E; z_0) \leqslant \liminf_{n\to\infty} U(\nu_n; z_n)$$

or, equivalently,

$$\lim_{\substack{n\to\infty\\ n\in\mathcal{N}_1}} \frac{1}{n}\log\|p_n\|_{E^*} = \lim_{\substack{n\to\infty\\ n\in\mathcal{N}_1}} -U(\nu_n; z_n) \leqslant -U(\mu_E; z_0). \tag{3.17}$$

Since K^* is regular, $-U(\mu_E; z_0) = \log[\operatorname{cap}(E)]$ and so (3.16) and (3.17) imply condition (a).

Proof of Theorem 2.2. Let $T_n(z)$ be the Chebyshev polynomial of degree n for E and set

$$q_{n-1}(z) := p_n^*(z) - a_n^* T_n(z),$$

where $p_n^*(z) = a_n^* z^n + \ldots$. Since $q_{n-1}\in\Pi_{n-1}$, we have

$$e_{n-1}(f) \leqslant \|f - q_{n-1}\|_E \leqslant e_n(f) + |a_n^*|\,\|T_n\|_E,$$

that is,

$$[e_{n-1}(f) - e_n(f)]^{1/n} \leqslant |a_n^*|^{1/n}\,\|T_n\|_E^{1/n}, \qquad n = 1, 2, \ldots. \tag{3.18}$$

Moreover, the assumption that f is not analytic on E implies that

$$\limsup_{n\to\infty} [e_n(f)]^{1/n} = 1.$$

Hence there exists a sequence Λ of positive integers such that

$$\lim_{n\to\infty} [e_{n-1}(f) - e_n(f)]^{1/n} = 1, \qquad n\in\Lambda. \tag{3.19}$$

From (3.18), (3.19), and (2.4), we get

$$1/\operatorname{cap}(E) \leqslant \liminf_{n\to\infty} |a_n^*|^{1/n}, \qquad n\in\Lambda, \tag{3.20}$$

and since the polynomials p_n^* are uniformly bounded on E, the monic polynomials $p_n(z) := p_n^*(z)/a_n^*$, $n\in\Lambda$, satisfy condition (a) of Theorem 2.1. Finally, the assumption that f does not vanish in any component of E° together with Hurwitz's theorem imply that condition (b) of Theorem 2.1 also holds. Hence $\nu(p_n) = \nu(p_n^*)$ converges weakly to μ_E as $n\to\infty$, $n\in\Lambda$.

Since the proof of Theorem 2.3 is similar, we leave the details to the reader.

4. *Some further applications*

Theorem 2.2 can easily be extended to the case of best rational approximants having a bounded number of free poles. We say that $r_{n,N}$ is *of type* (n, N) if $r_{n,N} = p_n/q_N$ with $p_n\in\Pi_n$, $q_N(\not\equiv 0)\in\Pi_N$.

THEOREM 4.1. *Let E and f be as in Theorem 1.2, and let N be a fixed non-negative integer. For each $n = 0, 1, \ldots$, let $r^*_{n,N}$ denote a rational function of type (n, N) of best uniform approximation to f on E. Then the discrete unit measures $\nu(r^*_{n,N})$ associated with the zeros of $r^*_{n,N}(z)$ converge weakly to μ_E as $n \to \infty$ through a sequence $\Lambda = \Lambda(f)$ of positive integers.*

Proof. Choose $1 < \sigma < \infty$ such that the region E_σ defined in (2.9) contains the origin. Let $s_j^{(n)}$, $j = 1, 2, \ldots, k_n$ denote the poles of $r^*_{n,N}(z)$ in E_σ and let $t_j^{(n)}$, $j = 1, 2, \ldots, l_n$ denote the poles in the complement of E_σ. Write

$$r^*_{n,N}(z) = (a_n z^n + \ldots) \Big/ \left[\prod_{j=1}^{k_n} (z - s_j^{(n)})\right] \left[\prod_{j=1}^{l_n} (z - t_j^{(n)})/t_j^{(n)}\right], \tag{4.1}$$

where $\prod_{j=1}^{0} \psi(j)$ is defined to be 1. For each $n = 0, 1, \ldots$, let $P_n(z) = a_n z^n + \ldots$ denote the numerator polynomial in (4.1).

Let $T_n(z)$ be the Chebyshev polynomial of degree n for E and set

$$r_{n-1,N}(z) := r^*_{n,N}(z) - a_n T_{n-k_n}(z) \Big/ \left[\prod_{j=1}^{l_n} (z - t_j^{(n)})/t_j^{(n)}\right], \qquad n > N. \tag{4.2}$$

Since $r_{n-1,N}(z)$ is of type $(n-1, N)$, we have

$$\|f - r^*_{n-1,N}\|_E \leqslant \|f - r_{n-1,N}\|_E \leqslant \|f - r^*_{n,N}\|_E + L|a_n| \, \|T_{n-k_n}\|_E, \tag{4.3}$$

where L is a constant which depends only on N and the distance between E and Γ_σ.

The assumption that f is not analytic on E implies (cf. [9, 5]) that

$$\limsup_{n \to \infty} \|f - r^*_{n,N}\|^{1/n} = 1. \tag{4.4}$$

Hence there exists a sequence $\Lambda = \Lambda(f)$ of positive integers such that

$$\lim_{n \to \infty} (\|f - r^*_{n-1,N}\|_E - \|f - r^*_{n,N}\|_E)^{1/n} = 1, \qquad n \in \Lambda. \tag{4.5}$$

From (4.3), (4.5), (2.4), and the boundedness of k_n, we get

$$1/\text{cap}(E) \leqslant \liminf_{n \to \infty} |a_n|^{1/n}, \qquad n \in \Lambda, \tag{4.6}$$

and since the polynomials P_n are uniformly bounded on E, the monic polynomials $p_n(z) := P_n(z)/a_n$, $n \in \Lambda$, satisfy condition (a) of Theorem 2.1. As in the proof of Theorem 2.2, condition (b) of Theorem 2.1 also holds. Hence $\nu(p_n) = \nu(r^*_{n,N})$ converges weakly to μ_E as $n \to \infty$, $n \in \Lambda$.

We mention one further result that can be proved in a similar manner. Consider approximation on a rectifiable Jordan curve Γ as measured by the line integral

$$\int_\Gamma w(z) \, |f(z) - q(z)|^p \, |dz|, \qquad p > 0, \tag{4.7}$$

where w is a non-negative integrable function on Γ such that, for some $\alpha > 0$, $(w)^{-\alpha}$ is also integrable on Γ.

THEOREM 4.2. *Let Γ be a rectifiable Jordan curve and let f be a function which is analytic in the interior of Γ and continuous on its closure. For each $n = 0, 1, \ldots$, let $q^*_n \in \Pi_n$ denote a polynomial of best approximation to f on Γ as measured by (4.7). If*

f is not analytic on Γ, then $\nu(q_n^*)$ converges weakly to μ_Γ, the equilibrium distribution for Γ, as $n \to \infty$, through a sequence $\Lambda = \Lambda(f)$ of positive integers.

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