ON POLYNOMIALS OF MINIMAL L_q-DEVIATION, $0 < q < 1$

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ABSTRACT

It is shown that the monic polynomial of minimal weighted L_q-deviation is unique when $0 < q < 1$ and the weight satisfies certain properties.

A classical problem in approximation theory consists of finding a monic polynomial $p_n^*(x) = x^n + a_{n-1}^* x^{n-1} + \ldots + a_1^* x + a_0^*$ which satisfies

$$\int_{-1}^{1} \omega(x)\,|p_n^*(x)|^q\,dx = \min_{p_{n-1}\in P_{n-1}} \int_{-1}^{1} \omega(x)\,|x^n + p_{n-1}(x)|^q\,dx, \tag{1}$$

where P_{n-1} denotes the collection of polynomials of degree at most $n-1$, ω is a given non-negative weight and $1 \leqslant q \leqslant \infty$. (The case in which $q = \infty$ corresponds to minimizing the supremum norm.) Under mild conditions on ω, this problem has a unique solution for $1 < q < \infty$, because of the strict convexity of the L_q-norm. Uniqueness for $q = 1, \infty$ follows from somewhat more delicate considerations.

For the Chebyshev weight $\omega(x) = (1-x^2)^{-\frac{1}{2}}$ the explicit solution of (1) for all $1 \leqslant q \leqslant \infty$ is given by the Chebyshev polynomial of first kind,

$$\tilde{T}_n(x) := 2^{1-n} \cos(n \arccos x)$$

(see [6, p. 81]). However, even for the weight $\omega(x) \equiv 1$, the solution of (1) is known explicitly only for $q = 1, 2, \infty$.

The purpose of this paper is to investigate the extremal problem (1) in the 'non-classical case' when $0 < q < 1$. Our interest in this problem was inspired by a recent extension of Bernstein's inequality to the case where $0 < q < 1$. The extension was first given by Arestov [1] with a simple proof presented by von Golitschek and Lorentz [2]. (The first proof with an extra constant factor appeared in [4].) According to the extended version of Bernstein's inequality, for every $q > 0$ and every trigonometric polynomial $S_n \in T_n$,

$$\int_0^{2\pi} \left|\frac{S_n'(\theta)}{n}\right|^q d\theta \leqslant \int_0^{2\pi} |S_n(\theta)|^q\,d\theta. \tag{2}$$

Moreover, equality in (2) holds only for $S_n(\theta) = a\cos n\theta + b \sin n\theta$. (Here and in what follows T_n and P_n denote the real trigonometric and algebraic polynomials of degree at most n, respectively.)

Let us show that Arestov's result easily implies that for $\omega(x) = (1-x^2)^{-\frac{1}{2}}$ the

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Chebyshev polynomial $\tilde{T}_n(x)$ is the unique solution of (1) for every $q > 0$. For every $x^n + p_{n-1}(x) \in P_n \backslash \{\tilde{T}_n\}$ we have

$$\int_{-1}^{1} |x^n + p_{n-1}(x)|^q \frac{dx}{(1-x^2)^{\frac{1}{2}}} = 2^{(1-n)q-1} \int_0^{2\pi} |\cos n\theta + S_{n-1}(\theta)|^q \, d\theta,$$

where $S_{n-1}(\theta) \in T_{n-1} \backslash \{0\}$. From the above version of Bernstein's inequality it follows that the sequence of numbers

$$a_k := 2^{(1-n)q-1} \int_0^{2\pi} |\cos n\theta + n^{-4k} S_{n-1}^{(4k)}(\theta)|^q \, d\theta, \qquad k = 0, 1, \ldots$$

satisfies $a_0 > a_1$ and $a_k \geqslant a_{k+1}$ if $k \geqslant 1$. Furthermore, it is easily seen that $n^{-4k} S_{n-1}^{(4k)}(\theta)$ tends to zero uniformly on $[0, 2\pi]$ as $k \to \infty$. Hence, the a_k monotonically decrease to

$$2^{(1-n)q-1} \int_0^{2\pi} |\cos n\theta|^q \, d\theta = \int_{-1}^{1} |\tilde{T}_n(x)|^q \frac{dx}{(1-x^2)^{\frac{1}{2}}}.$$

This verifies that

$$a_0 = \int_{-1}^{1} |x^n + p_{n-1}(x)|^q \frac{dx}{(1-x^2)^{\frac{1}{2}}} > \int_{-1}^{1} |\tilde{T}_n(x)|^q \frac{dx}{(1-x^2)^{\frac{1}{2}}}.$$

The main goal of the present paper is to address the following question. Is there a unique solution to (1) if $0 < q < 1$ for weights different from $(1-x^2)^{-\frac{1}{2}}$? We shall develop a method of investigating the extremal problem (1) based on the implicit function theorem which implies uniqueness of the solution of (1) for a certain class of weights including Jacobi weights $\omega(x) = (1-x)^\alpha (1+x)^\beta$ with $\alpha \geqslant 0$ and $\beta \geqslant 0$. Thus, in particular, we obtain the uniqueness for $\omega(x) \equiv 1$. In order to achieve this goal we extend some well-known properties of the solutions of (1) for $1 \leqslant q < \infty$, such as non-coalescence of zeros and orthogonality relations, to the case when $0 < q < 1$. Even these questions, which can be easily answered if $q \geqslant 1$, pose difficulties for $0 < q < 1$ in view of the singularity of the integrals arising in the process of solution.

Let us now formulate our main results. We denote by W_q the set of all weights ω with the following properties.

(i) For some $\alpha_0 = -1 < \alpha_1 < \ldots < \alpha_l < 1 = \alpha_{l+1}$ we have $\omega \in C^1(\alpha_j, \alpha_{j+1})$ and $\omega > 0$ on (α_j, α_{j+1}), $0 \leqslant j \leqslant l$.

(ii) For every $0 \leqslant j \leqslant l+1$ there exists β_j, with $\beta_j > -q$ if $1 \leqslant j \leqslant l$ and $\beta_j > -\frac{1}{2}(q+1)$ if $j = 0, l+1$, such that

$$\lim_{x \to \alpha_j^+} |x - \alpha_j|^{-\beta_j} \omega(x)$$

exists and is positive. The same property (with possibly different β_j) is assumed to hold with respect to left-hand limits.

THEOREM 1. *Let* $\omega \in W_q$ *and* $0 < q < 1$. *Then any solution of* (1) *has the form*

$$p_n^*(x) = \prod_{i=1}^{n} (x - x_i^*),$$

where $-1 < x_1^* < \ldots < x_n^* < 1$.

Moreover, if $\omega \in W_q \cap C(-1, 1)$, *then*

$$\int_{-1}^{1} \omega(x) \frac{|p_n^*(x)|^q}{x - x_j^*} dx = 0, \qquad 1 \leqslant j \leqslant n, \tag{3}$$

or, equivalently,

$$\int_{-1}^{1} \omega(x)\, x^k |p_n^*(x)|^{q-1} \operatorname{sgn} p_n^*(x)\, dx = 0, \qquad 0 \leqslant k \leqslant n-1. \tag{4}$$

To prove uniqueness of the extremal polynomial, our method requires some further assumptions on ω including either
(a) $\omega(1) = \omega(-1) = 0$ and ω'/ω is monotone, or
(b) $\omega^2(1) + \omega^2(-1) > 0$ and ω'/ω is decreasing.

THEOREM 2. *Let* $\omega \in C[-1, 1] \cap C^1(-1, 1) \cap W_q$ *be positive on* $(-1, 1)$ *and satisfy either* (a) *or* (b). *Then, for every* $0 < q < 1$, *problem* (1) *has a unique solution.*

Note that if $\omega \in C^1(-1, 1)$ is positive on $(-1, 1)$, then the requirement that $\omega \in W_q$ restricts only the behaviour of ω at the end points of $(-1, 1)$.

The proof of Theorem 1 uses some standard variational arguments which, however, lead to singular integrals that cause technical difficulties. The proof of Theorem 2 is based on the simple observation that it is sufficient to verify that for every given $0 < q < 1$ the nonlinear system of equations (3) has a unique solution $-1 < x_1^* < \ldots < x_n^* < 1$. Furthermore, it turns out that under the assumptions of Theorem 2, the Jacobian of the system (3) does not vanish. Some additional observations and the implicit function theorem then imply that any solution of (3) for a given $0 < q^* < 1$ can be extended to a C^1-function of q providing a solution of (3) for each $0 < q \leqslant 1$. Finally, taking into account the classical result that the solution of (3) for $q = 1$ (the L_1-extremal problem) is unique, we can conclude that its solution for q^* must also be unique.

Before embarking on the proof of Theorem 1, we establish three lemmas which are of independent interest.

LEMMA 1. *Let* $q > 0$ *and* $p_m(x) = \prod_{i=1}^{m} (x - x_i)$, *where* $|x_i| \leqslant 1$ *for* $i = 1, \ldots, m$. *(The* x_i *may be real or complex.) Then, for every* $\alpha > \max\{-1, -\frac{1}{2}(m+2)q\}$, *we have*

$$\int_{-\infty}^{\infty} |x|^{\alpha} \frac{|(x^2-1)p_m(x)|^q}{x^2 - 1} dx > 0. \tag{5}$$

Furthermore, for any $\beta > \max\{-1, -\frac{1}{2}(q+1)\}$,

$$\int_{0}^{\infty} x^{\beta} \frac{|x-1|^q}{x-1} dx > 0. \tag{6}$$

Proof. First we observe that the only possible singularities of the integrals in (5) and (6) are 0, ± 1, $\pm\infty$. The assumptions on α, β imply that these integrals are convergent if taken over any finite interval. Thus, if the integral in (5) or (6) is divergent, then it diverges to $+\infty$ since the integrands are positive for $|x| > 1$. Therefore the statement of the lemma is trivial in this situation, that is, we may assume that the integrals in (5) and (6) are convergent.

Since $|1-ac| \geqslant |a-c|$ for any real a with $|a| \leqslant 1$ and any complex number c with $|c| \leqslant 1$, it follows that $|p_m(1/x)| \geqslant |x|^{-m}|p_m(x)|$ for $|x| \leqslant 1$. Thus

$$\begin{aligned}\int_{-\infty}^{\infty} |x|^{\alpha} \frac{|(x^2-1)p_m(x)|^q}{x^2-1}\,dx &= \int_{-1}^{1} |x|^{\alpha} \frac{|(x^2-1)p_m(x)|^q}{x^2-1}\,dx + \int_{-1}^{1} |x|^{-\alpha} \frac{|(x^2-1)p_m(1/x)|^q}{|x|^{2q}(1-x^2)}\,dx \\ &= \int_{-1}^{1} |x|^{-\alpha-2q}\,(1-x^2)^{q-1} \left\{|p_m(1/x)|^q - |x|^{2\alpha+2q+mq}\frac{|p_m(x)|^q}{|x|^{mq}}\right\} dx \\ &> 0.\end{aligned}$$

Inequality (6) can be similarly verified.

As usual, we say that a function f is piecewise C^1 on $[a,b]$ $(-\infty \leqslant a < b \leqslant \infty)$ if there is a finite decomposition of $[a,b]$ to subintervals such that f is C^1 on each open component. (We shall refer to the end points of these subintervals as the *singularities* of f.) The next technical lemma provides a sufficient condition for the differentiability of the convolution of two functions. It will be a frequently used tool to justify the differentiations that occur throughout the paper.

LEMMA 2. *Let*

$$\phi(x) := \int_a^b f(t)\,g(t-x)\,dt,$$

where $f \in L_1[a,b]$ is piecewise C^1 on $[a,b]$ and g is piecewise C^1 on $(-\infty,\infty)$ and absolutely continuous on any finite interval. Assume that for a given $x_0 \in \mathbb{R}$ we have $x_0+\eta_i \neq \xi_j$, a, b $(1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant r)$, where $\{\xi_j\}_{j=1}^r$ and $\{\eta_i\}_{i=1}^s$ are the singularities of f and g, respectively. Then $\phi'(x_0)$ exists and

$$\phi'(x_0) = -\int_a^b f(t)g'(t-x_0)\,dt.$$

Proof. We may assume without loss of generality that there is a partition

$$a =: \beta_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_s < \beta_s < \alpha_{s+1} := b$$

such that $x_0+\eta_i \in (\alpha_i, \beta_i)$, $1 \leqslant i \leqslant s$, and $\xi_j \notin \bigcup_1^s (\alpha_i, \beta_i)$, $1 \leqslant j \leqslant r$. Then we have

$$\phi(x) = \int_a^b f(t)g(t-x)\,dt = \sum_{i=1}^{s} a_i(x) + \sum_{i=1}^{s+1} b_i(x),$$

where

$$a_i(x) := \int_{\alpha_i}^{\beta_i} f(t)g(t-x)\,dt, \qquad b_i(x) := \int_{\beta_{i-1}}^{\alpha_i} f(t)g(t-x)\,dt.$$

Consider first the function $b_i(x)$. For any x_1 sufficiently close to x_0, the function $g(t-x_1)$ is C^1 in a neighbourhood of $[\beta_{i-1}, \alpha_i]$. Therefore, by the Lebesgue dominated convergence theorem,

$$b_i'(x_0) = -\int_{\beta_{i-1}}^{\alpha_i} f(t)g'(t-x_0)\,dt.$$

Similarly, using the fact that f is C^1 in a neighbourhood of $[\alpha_i, \beta_i]$, we obtain that

$$a_i(x) = \int_{\alpha_i}^{\beta_i} f(t)g(t-x)\,dt = \int_{\alpha_i-x}^{\beta_i-x} f(t+x)g(t)\,dt$$

is differentiable at x_0 and

$$a_i'(x_0)=f(\alpha_i)\,g(\alpha_i-x_0)-f(\beta_i)\,g(\beta_i-x_0)+\int_{\alpha_i-x_0}^{\beta_i-x_0} f'(t+x_0)\,g(t)\,dt.$$

Finally, integrating by parts (see [5, p. 266]) we arrive at

$$a_i'(x_0)=-\int_{\alpha_i-x_0}^{\beta_i-x_0} f(t+x_0)\,g'(t)\,dt=-\int_{\alpha_i}^{\beta_i} f(t)\,g'(t-x_0)\,dt.$$

LEMMA 3. *Suppose that the points $x_k\in[a,b]$ converge to x_0 and the real numbers p_k converge to p_0, where $p_0>0$. Further, assume that continuous functions $\omega_k(x)$ converge uniformly to $\omega_0(x)$ on $[a,b]$. Then*

$$\lim_{k\to\infty}\int_a^b \frac{\omega_k(x)\,dx}{|x-x_k|^{1-p_k}}=\int_a^b \frac{\omega_0(x)\,dx}{|x-x_0|^{1-p_0}}.$$

Proof. First we observe that

$$\lim_{k\to\infty}\int_a^b \frac{\omega_0(x)\,dx}{|x-x_0|^{1-p_k}}=\int_a^b \frac{\omega_0(x)\,dx}{|x-x_0|^{1-p_0}}$$

by the Lebesgue dominated convergence theorem. Furthermore, it can be easily seen from the uniform convergence assumption that

$$\begin{aligned}&\lim_{k\to\infty}\left\{\int_a^b \frac{\omega_k(x)\,dx}{|x-x_k|^{1-p_k}}-\int_a^b \frac{\omega_0(x)\,dx}{|x-x_0|^{1-p_k}}\right\}\\ &=\lim_{k\to\infty}\left\{\int_{A_k}\frac{(\omega_k(x+x_k)-\omega_0(x+x_0))\,dx}{|x|^{1-p_k}}+\int_{B_k}\frac{\omega_k(x+x_k)\,dx}{|x|^{1-p_k}}-\int_{C_k}\frac{\omega_0(x+x_0)\,dx}{|x|^{1-p_k}}\right\}\\ &=0,\end{aligned}$$

where $A_k:=[a-x_k,\ b-x_k]\cap[a-x_0,\ b-x_0]$, $B_k:=[a-x_k,\ b-x_k]\backslash[a-x_0,\ b-x_0]$, $C_k:=[a-x_0,\ b-x_0]\backslash[a-x_k,\ b-x_k]$.

We can now give the following.

Proof of Theorem 1. Since $|x-z|>|x-\operatorname{Re} z|$ for every $x\in\mathbb{R}$ and $z\in\mathbb{C}\backslash\mathbb{R}$, a solution of (1) must have only real zeros which obviously belong to $[-1,1]$. Let us show that if p_n^* is a solution of (1), then $p_n^*(1)\neq 0$ and $p_n^*(-1)\neq 0$.

Assume, for example, that $p_n^*(1)=0$ and consider

$$\psi(\varepsilon):=\int_{-1}^{1}\omega(x)\,|p_n^*(x)+\varepsilon q_{n-1}(x)|^q\,dx=\int_{-1}^{1}\omega(x)\,|q_{n-1}(x)|^q|x-1+\varepsilon|^q\,dx,$$

where $q_{n-1}(x):=p_n^*(x)/(x-1)$. Then, by Lemma 2, the function $\psi(\varepsilon)$ is differentiable for sufficiently small ε and

$$\psi'(\varepsilon)=q\int_{-1}^{1}\omega(x)\,|q_{n-1}(x)|^q\,\frac{|x-1+\varepsilon|^q}{x-1+\varepsilon}\,dx.$$

We shall show that $\psi'(\varepsilon)<0$ for small $\varepsilon>0$. Clearly, for $1<\gamma<2/\varepsilon$,

$$\begin{aligned}\psi'(\varepsilon)&<q\int_{1-\gamma\varepsilon}^{1}\omega(x)\,|q_{n-1}(x)|^q\,\frac{|x-1+\varepsilon|^q}{x-1+\varepsilon}\,dx\\ &=-q\varepsilon^q\int_0^{\gamma}\omega(1-\varepsilon u)\,|q_{n-1}(1-\varepsilon u)|^q\,\frac{|u-1|^q}{u-1}\,du.\end{aligned}$$

Furthermore, from the definition of W_q, we have

$$\omega(1-\varepsilon u)\,|q_{n-1}(1-\varepsilon u)|^q = C\varepsilon^\beta u^\beta + o(\varepsilon^\beta)\,u^\beta$$

for some $C > 0$ and $\beta > -\frac{1}{2}(q+1)$, uniformly with respect to $u \in [0, \gamma]$. Hence

$$\psi'(\varepsilon) < -Cq\varepsilon^{q+\beta}\int_0^\gamma u^\beta \frac{|u-1|^q}{u-1}\,du + o(\varepsilon^{q+\beta}).$$

Now using (6) it can be seen that with a proper choice of γ we have $\psi'(\varepsilon) < 0$ $(0 < \varepsilon \leqslant \varepsilon_0)$. But, on the other hand, $\psi(\varepsilon) \geqslant \psi(0)$ must hold for every ε, which gives the desired contradiction.

The next step in the proof is to show that $p_n^*(x)$ has no multiple (repeated) zeros. Assume to the contrary that $p_n^*(x) = (x-x_0)^2\, q_{n-2}(x)$ for some $-1 < x_0 < 1$, $q_{n-2} \in P_{n-2}$. Set

$$\begin{aligned}\phi(\varepsilon) :&= \int_{-1}^{1} \omega(x)\,|q_{n-2}(x)|^q |(x-x_0)^2 - \varepsilon|^q\,dx \\ &= \tfrac{1}{2}\int_0^{(1+x_0)^2} u^{-\frac{1}{2}}\omega(x_0-\sqrt{u})\,|q_{n-2}(x_0-\sqrt{u})|^q|u-\varepsilon|^q\,du \\ &\quad + \tfrac{1}{2}\int_0^{(1-x_0)^2} u^{-\frac{1}{2}}\,\omega(x_0+\sqrt{u})\,|q_{n-2}(x_0+\sqrt{u})|^q|u-\varepsilon|^q\,du \\ &= \tfrac{1}{2}\int_0^{(1+x_0)^2} \tilde{\omega}_1(u)\,|u-\varepsilon|^q\,du + \tfrac{1}{2}\int_0^{(1-x_0)^2} \tilde{\omega}_2(u)\,|u-\varepsilon|^q\,du,\end{aligned}$$

where $\tilde{\omega}_j(u) := u^{-\frac{1}{2}}\,\omega(x_0+(-1)^j\sqrt{u})\,|q_{n-2}(x_0+(-1)^j\sqrt{u})|^q$, $j = 1, 2$. Then $\phi'(\varepsilon)$ exists for small $\varepsilon > 0$ and

$$\phi'(\varepsilon) = -\frac{q}{2}\left\{\int_0^{(1+x_0)^2} \tilde{\omega}_1(u)\,\frac{|u-\varepsilon|^q}{u-\varepsilon}\,du + \int_0^{(1-x_0)^2} \tilde{\omega}_2(u)\,\frac{|u-\varepsilon|^q}{u-\varepsilon}\,du\right\}.$$

We now verify that

$$J_j(\varepsilon) := \int_0^a \tilde{\omega}_j(u)\,\frac{|u-\varepsilon|^q}{u-\varepsilon}\,du > 0, \qquad j = 1, 2,$$

for any $a > 0$, if ε is chosen to be sufficiently small. Indeed, using the fact that $\tilde{\omega}_j(u) = c_j u^{\beta_j} + o(u^{\beta_j})$ for some $c_j > 0$ and $\beta_j > -\frac{1}{2}(q+1)$ $(j = 1, 2)$, we have, for every $1 < \gamma < a/\varepsilon$,

$$\begin{aligned}J_j(\varepsilon) &> \int_0^{\gamma\varepsilon} \tilde{\omega}_j(u)\,\frac{|u-\varepsilon|^q}{u-\varepsilon}\,du = \varepsilon^q\int_0^\gamma \tilde{\omega}_j(\varepsilon u)\,\frac{|u-1|^q}{u-1}\,du \\ &= c_j\,\varepsilon^{q+\beta_j}\int_0^\gamma u^{\beta_j}\frac{|u-1|^q}{u-1}\,du + o\,(\varepsilon^{q+\beta_j}), \qquad j = 1, 2.\end{aligned}$$

It now follows from (6) that, for a suitable $\gamma > 0$, we have $J_j(\varepsilon) > 0$ for $\varepsilon > 0$ small enough. This yields $\phi'(\varepsilon) < 0$ $(0 < \varepsilon \leqslant \varepsilon^*)$ which again contradicts the fact that $\phi(0) \leqslant \phi(\varepsilon)$, $\varepsilon \in \mathbb{R}$.

Thus we have shown that a solution $p_n^*(x)$ of (1) has the form

$$p_n^*(x) = \prod_{i=1}^{n}(x - x_i^*)$$

where $-1 < x_1^* < \ldots < x_n^* < 1$. Assume now that $\omega \in W_q \cap C(-1, 1)$ and set

$$\Gamma_j(\varepsilon) := \int_{-1}^{1} \omega(x)\,|x - x_j^* + \varepsilon|^q \left|\prod_{\substack{i=1\\ i\neq j}}^{n}(x - x_i^*)\right|^q dx.$$

Clearly, $\Gamma_j(\varepsilon) \geqslant \Gamma_j(0)$, $1 \leqslant j \leqslant n$, $\varepsilon \in \mathbb{R}$. Moreover, by Lemma 2 we have for sufficiently small $\varepsilon \neq 0$

$$\Gamma_j'(\varepsilon) = q \int_{-1}^{1} \omega(x) \frac{|x-x_j^*+\varepsilon|^q}{x-x_j^*+\varepsilon} \left| \prod_{\substack{i=1\\ i\neq j}}^{n} (x-x_i^*) \right|^q dx.$$

Since $\omega \in C(-1, 1)$ it follows from Lemma 3 that $\Gamma_j'(0)$ also exists, and

$$\Gamma_j'(0) = q \int_{-1}^{1} \omega(x) \frac{|p_n^*(x)|^q}{x-x_j^*}\, dx = 0, \qquad 1 \leqslant j \leqslant n.$$

This verifies relations (3) which are equivalent to (4).

Now in view of Theorem 1 we can verify Theorem 2 by showing that the system of nonlinear equations (3) has a unique solution of the form $-1 < x_1^* < \ldots < x_n^* < 1$ for every $q > 0$. For this purpose we are going to use the implicit function theorem and the strategy outlined in the introduction. This requires that when $1 \geqslant q \geqslant q^* > 0$ (that is, q is bounded away from zero), the solutions $-1 < x_1^* < \ldots < x_n^* < 1$ of (3) are uniformly separated from each other and from the end points of the interval. Our next lemma establishes this fact.

LEMMA 4. *Let $0 < q^* < 1$ be given and assume that $\omega \in C[-1, 1]$ and $\omega > 0$ on $(-1, 1)$. Then there exists a positive constant C, depending only on q^*, n, and ω, such that for every $q^* \leqslant q \leqslant 1$ and every solution $x_0^* := -1 < x_1^* < \ldots < x_n^* < 1 =: x_{n+1}^*$ of (3) we have $x_{i+1}^* - x_i^* \geqslant C$ for $0 \leqslant i \leqslant n$.*

Proof. Assume the contrary. Then there exist $q_k \to q_0$ $(q^* \leqslant q_k \leqslant 1)$ and $x_i^{(k)} \to x_i^{(0)}$ $(k \to \infty,\ 0 \leqslant i \leqslant n+1;\ -1 = x_0^{(k)} = x_0^{(0)},\ 1 = x_{n+1}^{(k)} = x_{n+1}^{(0)})$ such that

$$\int_{-1}^{1} \omega(x) \frac{|p_{n,k}(x)|^{q_k}}{x-x_i^{(k)}}\, dx = 0, \qquad 1 \leqslant i \leqslant n,\ k = 1, 2, \ldots, \tag{7}$$

where $p_{n,k}(x) = \prod_{i=1}^{n} (x-x_i^{(k)})$, $k = 0, 1, \ldots$, and not all the $x_i^{(0)}$ are distinct.

Assume at first that $x_1^{(0)} = -1$. Then using (7) for $i = 1$ and taking the limit as $k \to \infty$ (Lemma 3) we obtain

$$\int_{-1}^{1} \omega(x) \frac{|p_{n,0}(x)|^{q_0}}{x+1}\, dx = 0;$$

a contradiction. Analogously, we can show that $x_n^{(0)} < 1$; that is, $-1 < x_i^{(0)} < 1$ for $1 \leqslant i \leqslant n$.

Now assume that $x_{r-1}^{(0)} < x_r^{(0)} = \ldots = x_s^{(0)} < x_{s+1}^{(0)}$ for some $1 \leqslant r < s \leqslant n$. Using (7) with $i = r$ and $i = s$ and subtracting one from another we get

$$\int_{-1}^{1} \omega(x) \frac{|p_{n,k}(x)|^{q_k}}{(x-x_r^{(k)})(x-x_s^{(k)})}\, dx = 0, \qquad k = 1, 2, \ldots. \tag{8}$$

Setting

$$\varepsilon_k := \frac{x_s^{(k)} - x_r^{(k)}}{2}, \qquad \bar{x}^{(k)} := \frac{x_r^{(k)} + x_s^{(k)}}{2},$$

$$u := \frac{x - \bar{x}^{(k)}}{\varepsilon_k}, \qquad u_i^{(k)} := \frac{x_i^{(k)} - \bar{x}^{(k)}}{\varepsilon_k},$$

($k = 1, 2, \ldots$, $1 \leqslant i \leqslant n$) we have $\varepsilon_k \to 0$, $\tilde{x}^{(k)} \to x_r^{(0)}$ ($k \to \infty$) and, without loss of generality, $u_i^{(k)} \to u_i^{(0)}$ as $k \to \infty$ ($r \leqslant i \leqslant s$). Note that $|u_i^{(0)}| \leqslant 1$, $r \leqslant i \leqslant s$, and $-1 = u_r^{(k)} = u_r^{(0)}$, $1 = u_s^{(k)} = u_s^{(0)}$. Set

$$T(u) := \prod_{i=r+1}^{s-1} (u - u_i^{(0)}).$$

By (5) we can choose $\gamma > 1$ such that

$$\int_{-\gamma}^{\gamma} \frac{|(u^2-1)T(u)|^{q_0}}{u^2-1}\, du > 0. \tag{9}$$

Furthermore, with the above substitutions, (8) transforms to

$$\int_{(-1-\tilde{x}^{(k)})/\varepsilon_k}^{(1-\tilde{x}^{(k)})/\varepsilon_k} \omega(\varepsilon_k\, u + \tilde{x}^{(k)}) \frac{|p_{n,k}(\varepsilon_k u + \tilde{x}^{(k)})|^{q_k}}{u^2-1}\, du = 0.$$

Thus, if ε_k is small enough,

$$\int_{-\gamma}^{\gamma} \omega(\varepsilon_k\, u + \tilde{x}^{(k)}) \frac{|p_{n,k}(\varepsilon_k u + \tilde{x}^{(k)})|^{q_k}}{u^2-1}\, du \leqslant 0. \tag{10}$$

Now note that

$$\begin{aligned} p_{n,k}(\varepsilon_k\, u + \tilde{x}^{(k)}) &= \prod_{i=1}^{n} (\varepsilon_k\, u + \tilde{x}^{(k)} - x_i^{(k)}) \\ &= \prod_{i=1}^{n} (\varepsilon_k\, u - \varepsilon_k\, u_i^{(k)}) = \varepsilon_k^{s-r+1} (u^2-1)\, \tilde{T}_k(u) \prod_{i=r+1}^{s-1} (u - u_i^{(k)}), \end{aligned}$$

where

$$\tilde{T}_k(u) := \prod_{\substack{i>s \\ i<r}} (\varepsilon_k\, u - \varepsilon_k\, u_i^{(k)}).$$

Since

$$\prod_{i=r+1}^{s-1} (u - u_i^{(k)}) \to \prod_{i=r+1}^{s-1} (u - u_i^{(0)}) = T(u)$$

and

$$\tilde{T}_k(u) \to \prod_{\substack{i>s \\ i<r}} (x_r^{(0)} - x_i^{(0)}) =: \eta \neq 0$$

as $k \to \infty$ uniformly on $[-\gamma, \gamma]$, it follows that

$$p_{n,k}(\varepsilon_k\, u + \tilde{x}^{(k)}) = \varepsilon_k^{s-r+1} (u^2-1) \{\eta\, T(u) + \eta_k(u)\}, \tag{11}$$

with $\eta_k(u) \to 0$ ($k \to \infty$) uniformly on $[-\gamma, \gamma]$. Using the fact that $|x_r^{(0)}| < 1$, we have $\omega(x_r^{(0)}) \neq 0$ and hence $\omega(\varepsilon_k u + \tilde{x}^{(k)}) = \omega(x_r^{(0)}) + \tilde{\eta}_k(u)$, with $\tilde{\eta}_k(u)$ tending to zero uniformly on $[-\gamma, \gamma]$. Substituting this and (11) into (10) we get

$$\int_{-\gamma}^{\gamma} \frac{|(u^2-1)(\eta\, T(u) + \eta_k(u))|^{q_k}}{u^2-1} (\omega(x_r^{(0)}) + \tilde{\eta}_k(u))\, du \leqslant 0.$$

Taking the limit as $k \to \infty$ we arrive at a contradiction with (9).

For $0 < q \leqslant 1$ and $-1 < x_1 < \ldots < x_n < 1$, we set

$$F_k(x_1, \ldots, x_n, q) := \int_{-1}^{1} \frac{|\prod_{i=1}^{n} (x - x_i)|^q}{x - x_k}\, \omega(x)\, dx, \qquad 1 \leqslant k \leqslant n.$$

Then the system (3) reduces to

$$F_k(x_1^*, \ldots, x_n^*, q) = 0, \qquad 1 \leqslant k \leqslant n. \tag{12}$$

Let $J(x_1, \ldots, x_n) := \partial(F_1, \ldots, F_n)/\partial(x_1, \ldots, x_n)$ be the Jacobian of this system.

LEMMA 5. *Let ω be as in Theorem 2. Then for any $0 < q \leqslant 1$ and any solution $-1 < x_1^* < \ldots < x_n^* < 1$ of (12) we have $J(x_1^*, \ldots, x_n^*) \neq 0$.*

Proof. For $j \neq k$ we have (Lemma 2)

$$\frac{\partial F_k}{\partial x_j} = q \int_{-1}^{1} \frac{\omega(x)}{x - x_k} |p_n(x)|^{q-1} \operatorname{sgn} p_n(x) \frac{\partial p_n}{\partial x_j}(x)\, dx, \tag{13}$$

where

$$p_n(x) = p_n(x, x_1, \ldots, x_n) := \prod_{i=1}^{n} (x - x_i).$$

Since $(\partial p_n/\partial x_j)/(x - x_k) \in P_{n-2}$ if $j \neq k$ and p_n satisfies the orthogonality relations (4) when $x_i = x_i^*$, $1 \leqslant i \leqslant n$, it follows that

$$\frac{\partial F_k}{\partial x_j}(x_1^*, \ldots, x_n^*) = 0, \qquad k \neq j. \tag{14}$$

Furthermore, again using Lemma 2,

$$\begin{aligned}
\frac{\partial F_k}{\partial x_k} &= \frac{\partial}{\partial x_k}\left(\int_{-1-x_k}^{1-x_k} x^{-1} |p_n(x + x_k)|^q \omega(x + x_k)\, dx\right) \\
&= -\frac{|p_n(1)|^q}{1 - x_k} \omega(1) - \frac{|p_n(-1)|^q}{1 + x_k} \omega(-1) + \int_{-1}^{1} \frac{|p_n(x)|^q}{x - x_k} \omega'(x)\, dx \\
&\quad + q \int_{-1-x_k}^{1-x_k} x^{-1} \frac{\partial p_n(x + x_k)}{\partial x_k} |p_n(x + x_k)|^{q-1} \operatorname{sgn} p_n(x + x_k)\, \omega(x + x_k)\, dx. \qquad (15)
\end{aligned}$$

Observing that $x^{-1} \partial p_n(x + x_k)/\partial x_k \in P_{n-2}$ and using once more (4) (which holds if $x_i = x_i^*$) we obtain that the last integral in (15) vanishes if $x_i = x_i^*$, $1 \leqslant i \leqslant n$. Hence

$$\begin{aligned}
\frac{\partial F_k}{\partial x_k}(x_1^*, \ldots, x_n^*) &= -\frac{|p_n^*(1)|^q}{1 - x_k^*} \omega(1) - \frac{|p_n^*(-1)|^q}{1 + x_k^*} \omega(-1) + \int_{-1}^{1} \frac{|p_n^*(x)|^q}{x - x_k^*} \omega'(x)\, dx \\
&= -\frac{|p_n^*(1)|^q}{1 - x_k^*} \omega(1) - \frac{|p_n^*(-1)|^q}{1 + x_k^*} \omega(-1) + \int_{-1}^{1} \frac{|p_n^*(x)|^q}{x - x_k^*} \left\{\frac{\omega'(x)}{\omega(x)} - \frac{\omega'(x_k^*)}{\omega(x_k^*)}\right\} \omega(x)\, dx,
\end{aligned}$$

with $p_n^*(x) = p_n(x, x_1^*, \ldots, x_n^*)$. Finally, our assumptions (a) or (b) on ω easily imply that

$$\frac{\partial F_k}{\partial x_k}(x_1^*, \ldots, x_n^*) \neq 0, \qquad 1 \leqslant k \leqslant n.$$

This together with (14) completes the proof of lemma.

REMARK. Let $-1 < x_1^* < \ldots < x_n^* < 1$ and $q^* > 0$. Then the partial derivatives $\partial F_k/\partial x_j$, $1 \leqslant j \leqslant n$, and $\partial F_k/\partial q$ $(1 \leqslant k \leqslant n)$ are continuous in a neighbourhood of $(x_1^*, \ldots, x_n^*, q^*) \in \mathbb{R}^{n+1}$. The continuity of $\partial F_k/\partial x_j$, $1 \leqslant j \leqslant n$, can be derived from relations (13) and (15) with the help of Lemma 3. For $\partial F_k/\partial q$ we have, by the Lebesgue dominated convergence theorem,

$$\frac{\partial F_k}{\partial q} = \int_{-1}^{1} \frac{|p_n(x)|^q \log|p_n(x)|}{x-x_k}\,\omega(x)\,dx, \tag{16}$$

and the continuity of this derivative can be verified similarly.

Proof of Theorem 2. Assume that for some $0 < q^* < 1$ we have two sets of points $-1 < x_1 < \dots < x_n < 1$ and $-1 < \tilde{x}_1 < \dots < \tilde{x}_n < 1$ such that $p_n(x) = \prod_{i=1}^{n} (x - x_i)$ and $\tilde{p}_n(x) = \prod_{i=1}^{n} (x - \tilde{x}_i)$ both minimize (1) and $x_i \neq \tilde{x}_i$ for some $1 \leqslant i \leqslant n$. Then, by Theorem 1,

$$F_k(x_1, \dots, x_n, q^*) = F_k(\tilde{x}_1, \dots, \tilde{x}_n, q^*) = 0, \qquad 1 \leqslant k \leqslant n,$$

that is, the system (12) has two distinct solutions for $q = q^*$. By Lemma 5 and the remark following it, the implicit function theorem is applicable to (12) at $(x_1, \dots, x_n, q^*)$ and $(\tilde{x}_1, \dots, \tilde{x}_n, q^*)$. It implies the existence of C^1-functions

$$-1 < x_1(q) < \dots < x_n(q) < 1$$

in a neighbourhood of q^* such that $x_i(q^*) = x_i$, $1 \leqslant i \leqslant n$, and $x_i(q)$, $1 \leqslant i \leqslant n$, are solutions of (12) in some neighbourhood of q^*. We shall show that the functions $x_i(q)$, $1 \leqslant i \leqslant n$, can be extended to $[q^*, 1]$ preserving all the above properties.

Let

$\delta^* := \sup\{0 \leqslant \delta < 1 - q^*$: there exist C^1-functions $-1 < x_1(q) < \dots < x_n(q) < 1$ on $[q^*, q^* + \delta)$ such that $x_i(q^*) = x_i$ and $x_i(q)$ is solution of (12) for each $q \in [q^*, q^* + \delta)$, $1 \leqslant i \leqslant n\}$.

We have shown above that $\delta^* > 0$. Now we claim that $\delta^* = 1 - q^*$. Assume that $\delta^* < 1 - q^*$. First we show the existence, for each $1 \leqslant i \leqslant n$, of $\lim x_i(q)$ as q tends to $q^* + \delta^*$ from the left. If this limit does not exist for some $1 \leqslant j \leqslant n$, then the function $x_j(q)$ has a non-degenerate interval $[\alpha_j, \beta_j]$ of cluster points as $q \to (q^* + \delta^*)^-$. For any choice ξ_i of cluster points of $x_i(q)$ as $q \to (q^* + \delta^*)^-$, $1 \leqslant i \leqslant n$, we have by Lemma 4 that $-1 < \xi_1 < \dots < \xi_n < 1$, and, clearly, the ξ_i are solutions of (12) for $q = q^* + \delta^* < 1$. Applying the implicit function theorem to (12) at $\xi_1, \dots, \xi_n, q^* + \delta^*$, we obtain a contradiction to local uniqueness of the solution to (12). Thus we may set $x_i^* := \lim x_i(q)$, as q tends to $q^* + \delta^*$ from the left, where by Lemma 4, $-1 < x_1^* < \dots < x_n^* < 1$ and $\{x_1^*, \dots, x_n^*, q^* + \delta^*\}$ is a solution of (12). Obviously, we can apply the implicit function theorem again to extend the $x_i(q)$ further to the right from $q^* + \delta^*$.

We have verified the existence of a C^1-vector function $X(q) = \{x_i(q)\}_{i=1}^{n}$ such that $X(q^*) = \{x_i\}_{i=1}^{n}$ and $X(q)$ is a solution of (12) for every $q^* \leqslant q \leqslant 1$. Analogously, there exists a C^1-vector function $\tilde{X}(q) = \{\tilde{x}_i(q)\}_{i=1}^{n}$ such that $\tilde{X}(q^*) = \{\tilde{x}_i\}_{i=1}^{n}$ and $\tilde{X}(q)$ is also a solution of (12) for every $q^* \leqslant q \leqslant 1$. Note that $X(q^*) \neq \tilde{X}(q^*)$. On the other hand, we know from the L_1-theory that (12) has a unique solution for $q = 1$, that is, we must have $X(1) = \tilde{X}(1)$. Thus there is q_0, $q^* < q_0 \leqslant 1$, such that $X(q_0) = \tilde{X}(q_0)$ but $\tilde{X}(q) \neq X(q)$ for every $q^* < q < q_0$. But this contradicts the implicit function theorem applied to (12) at $q = q_0$.

EXAMPLE. The Jacobi weights $\omega_{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$ satisfy the conditions of Theorem 1 if $\alpha, \beta > -\frac{1}{2}(q+1)$. The assumptions of Theorem 2 are satisfied if $\alpha, \beta \geqslant 0$.

Let us make now some final remarks. By adding some technical details it is possible to ease the smoothness conditions on the weight ω in Theorems 1 and 2. However, the main obstacle in extending Theorem 2 to a more general class of weights (including $\omega_{\alpha,\beta}(x)$ with $0 > \alpha,\beta \geqslant -\frac{1}{2}$) are the conditions (a) and (b) imposed in Theorem 2. These conditions are used only in Lemma 5 for verifying that the Jacobian of the system (12) does not vanish, namely that $\partial F_k/\partial x_k \neq 0$. This raises the question of whether Lemma 5 can be proved under more general conditions on ω. It would be of special interest to verify it for $\omega_{\alpha,\beta}$ when $-\frac{1}{2} < \alpha,\beta < 0$, which would open the possibility of comparing the location of zeros of minimal polynomials for $\omega(x) \equiv 1$ and $0 < q < 1$, with say the zeros of Chebyshev polynomials. For $1 \leqslant q \leqslant \infty$ this problem was considered in [3].

Denote by $x_i(q)$, $1 \leqslant i \leqslant n$, the zeros of the polynomial $p_n(x) = x^n + \dots$ that minimizes (1) for $\omega(x) \equiv 1$ and $0 < q \leqslant \infty$. It was shown in [3] that

$$|x_i(1)| < |x_i(q)| < |x_i(\infty)|, \qquad 1 \leqslant i \leqslant n,$$

if $1 < q < \infty$. An affirmative answer to the above question combined with the approach used in [3] would allow one to complete the picture by showing that $0 < |x_i(q)| < |x_i(1)|$ if $0 < q < 1$. Zero is a natural lower bound for $|x_i(q)|$, $q > 0$, because as $q \to 0^+$ the extremal problem (1) transforms to minimizing

$$\int_{-1}^{1} \log |p_n(x)| \, dx$$

and the unique solution to this problem is $p_n(x) = x^n$, that is, $x_i(0) = 0$, $1 \leqslant i \leqslant n$. Moreover, we conjecture that even a stronger statement is true; namely, that $x_i(q)$ is a strictly increasing function of q, $[\frac{1}{2}n] \leqslant i \leqslant n$, as q varies from 0 to $+\infty$. The key to the solution of this problem consists in verifying that the integral on the right-hand side of (16) does not vanish if $p_n(x)$ is the solution of (1) with $\omega(x) \equiv 1$.

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