SOME EXAMPLES IN APPROXIMATION ON THE UNIT DISK BY RECIPROCALS OF POLYNOMIALS

by

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Abstract. This paper is a continuation of the authors' study of approximation by reciprocals of polynomials. A Jackson-type theorem for such approximants is established for a certain class of functions f analytic and nonzero in the disk $|z| < 1$ and continuous on $|z| \leq 1$. Furthermore, we obtain the sharp degree of convergence for reciprocal polynomial approximation on $|z| \leq 1$ to functions f that are analytic on $|z| \leq 1$, nonzero in $|z| < 1$, and vanish somewhere on $|z| = 1$.

1. Statement of results.

In our papers [1], [2] we investigated the rate of approximation of real and complex-valued functions on $[-1,1]$ by reciprocals of polynomials. Here we extend some of these results to the case of

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approximation on the unit disk of the complex plane $\mathbb{C}$.

For any continuous function f in the closed unit disk $D = \{z : |z| \le 1\}$, let $E_{on}(f;D)$ denote the error in best uniform approximation of f on D by reciprocals of polynomials of degree $\le n$. J.L. Walsh [4] proved that $E_{on}(f;D) \to 0$ as $n \to \infty$ if and only if the continuous function f is analytic in the open disk $|z| < 1$ and does not vanish there. We denote the set of all such functions by $A_o(D)$. Under an additional assumption on f we can prove the following analogue of Jackson's theorem:

<u>Theorem</u> 1. <u>Let</u> $f \in A_o(D)$ <u>and suppose that the set</u> $\{f(z) : z \in D\}$ <u>lies in a half-plane</u> $\mathrm{Re}(z\bar{z}_o) \ge 0$, <u>for some</u> z_o, $|z_o| = 1$. <u>Then there exists a constant</u> M (<u>independent of</u> f <u>and</u> z_o) <u>such that</u>

$$(1.1) \qquad E_{on}(f;D) \le M\omega(f;n^{-1}), \qquad n = 1,2,3,\dots ,$$

<u>where</u> $\omega(f;\delta)$ <u>denotes the modulus of continuity of</u> $f(e^{i\theta})$ <u>on</u> $[-\pi,\pi]$.

<u>Example</u>. It is easy to see that any single-valued branch of the function $(1-z)^{\alpha}$ satisfies the assumptions of Theorem 1 provided $0 < \alpha \le 1$. It follows that there exists a constant c such that

$$(1.2) \qquad E_{on}((1-z)^{\alpha};D) \le cn^{-\alpha}, \quad 0 < \alpha \le 1; \quad n = 1,2,3,\dots .$$

It can be also shown that the estimate (1.2) is precise in the sense that there exists a constant $d > 0$ such that

$$(1.3) \qquad E_{on}((1-z)^{\alpha};D) \ge dn^{-\alpha}, \quad 0 < \alpha \le 1, \quad n = 1,2,3,\dots .$$

From (1.3) it follows that the estimate given in Theorem 1 is, in general, the best possible.

The asymptotic character of $E_{on}(f;D)$ can be described precisely if we assume that f is analytic in the <u>closed</u> unit disk.

<u>Theorem</u> 2. <u>Let</u> $f \in A_o(D)$ <u>be analytic in the closed unit disk</u> D <u>and assume</u> f <u>vanishes somewhere on</u> $|z| = 1$. <u>Denote by</u> r <u>the smallest order of zeros of</u> f <u>on</u> $|z| = 1$. <u>Then there exist positive</u>

<u>constants</u> $A(f)$, $B(f)$ <u>such that</u>

(1.4) $$A(f)n^{-r} \le E_{on}(f;D) \le B(f)n^{-r}, \qquad n = 1,2,3,\ldots .$$

<u>In particular, for any positive integer</u> r <u>there exist positive constants</u> A_r, B_r <u>such that</u>

(1.5) $$A_r n^{-r} \le E_{on}((1-z)^r;D) \le B_r n^{-r}, \qquad n = 1,2,3,\ldots .$$

Finally, we mention the result of Walsh [4, Theorem V] that describes completely the functions for which $E_{on}(f;D)$ decreases exponentially.

<u>Theorem</u> 3 (Walsh). <u>For any continuous function</u> $f(\not\equiv 0)$ <u>on</u> D <u>the following conditions are equivalent</u>:

(i) $\limsup_{n\to\infty}[E_{on}(f;D)]^{1/n} \le 1/R < 1$,

(ii) f <u>is analytic on</u> D <u>and meromorphic and different from zero in</u> $D_R := \{z: |z| < R\}$.

2. <u>Proof of Theorem 1</u>.

By the assumption on f there exists z_o, $|z_o| = 1$, such that

$$\mathrm{Re}(f(z)\overline{z}_o) \ge 0, \qquad z \in D.$$

Consider the function

(2.1) $$G(z) := f(z) + Az_o\omega(f;n^{-1}),$$

where $A > 0$ will be chosen later. Notice that

(2.2) $$|G(z)| = |G(z)\overline{z}_o| = |f(z)\overline{z}_o + A\omega(f;n^{-1})| \ge A\omega(f;n^{-1}), \ z \in D.$$

Now set

$$g(\theta) := G(e^{i\theta}), \qquad -\pi \le \theta \le \pi.$$

From (2.2) it follows that

$$(2.3) \qquad |g(\theta)| \geq A\omega(f;n^{-1}), \qquad -\pi \leq \theta \leq \pi.$$

Furthermore,

$$(2.4) \qquad \omega(g;n^{-1}) = \omega(f;n^{-1})$$

(recall that $\omega(f;n^{-1})$ denotes the modulus of continuity of the function $f(e^{i\theta})$ on $[-\pi,\pi]$).

Let $K_n(t)$ be the Jackson kernel (see Lorentz [3, p.55]). Since

$$\int_{-\pi}^{\pi} K_n(t)dt = 1, \quad \int_{-\pi}^{\pi} |t^k|K_n(t)dt = O(n^{-k}), \qquad k = 1,2,$$

we obtain for all θ that

$$(2.5) \qquad \int_{-\pi}^{\pi} |g(\theta+t) - g(\theta)|^k K_n(t)dt \leq c[\omega(g;n^{-1})]^k, \qquad k = 1,2,$$

where $c > 0$ is an absolute constant. Now define

$$(2.6) \qquad p_n(\theta) := \int_{-\pi}^{\pi} \frac{1}{g(\theta+t)} K_n(t)dt.$$

It is well-known that $p_n(\theta)$ has the form $\sum_{k=-n}^{n} \lambda_k c_k e^{ik\theta}$, where $\sum_{-\infty}^{\infty} c_k e^{ik\theta}$ is the Fourier series of $1/g(\theta)$. Since $G \neq 0$ in D (by (2.2)), $1/G$ is analytic in $|z| < 1$ and consequently $c_k = 0$ for $k < 0$. It follows that $p_n(\theta)$ is a polynomial in $e^{i\theta}$ of degree $\leq n$. We shall use the notation $P_n(z)$ for the corresponding algebraic polynomial in z, that is, $P_n(z) = \sum_{0}^{n} \lambda_k c_k z^k$. Now,

$$\left|\frac{1}{g(\theta)} - p_n(\theta)\right| = \left|\int_{-\pi}^{\pi} [1/g(\theta) - 1/g(\theta+t)]K_n(t)dt\right|$$

$$\leq \int_{-\pi}^{\pi} \frac{|g(\theta+t) - g(\theta)|}{|g(\theta)|\,|g(\theta+t)|} K_n(t)dt$$

$$\leq \frac{1}{|g(\theta)|A\omega(f;n^{-1})} \int_{-\pi}^{\pi} |g(\theta+t) - g(\theta)|K_n(t)dt \qquad \text{(by (2.3))}$$

$$\leq \frac{1}{|g(\theta)|A\omega(f;n^{-1})} c\omega(f;n^{-1}) \qquad \text{(by (2.5),(2.4))}$$

$$= \frac{c}{|g(\theta)|A} .$$

The choice

(2.7) $$A := 2c$$

therefore yields

(2.8) $$|1-g(\theta)p_n(\theta)| \leq 1/2 \quad , \quad -\pi \leq \theta \leq \pi,$$

which implies that

(2.9) $$|g(\theta)p_n(\theta)| \geq 1/2 \quad , \quad -\pi \leq \theta \leq \pi.$$

From (2.8) we deduce, by the maximum principle, that $|1-G(z)P_n(z)| \leq 1/2$ for $|z| \leq 1$ and therefore

$$|G(z)P_n(z)| \geq 1/2 \quad , \quad |z| \leq 1.$$

In particular, $P_n(z) \neq 0$ in D and applying the maximum principle again we conclude that

(2.10) $$\max_{|z|\leq 1} |G(z) - 1/P_n(z)| = \max_{-\pi\leq\theta\leq\pi} |g(\theta) - 1/p_n(\theta)|.$$

Now,

$$|g(\theta) - 1/p_n(\theta)| = \left|\int_{-\pi}^{\pi} \frac{g(\theta+t) - g(\theta)}{g(\theta)g(\theta+t)} \cdot \frac{g(\theta)}{p_n(\theta)} \cdot K_n(t)dt\right|$$

$$\le 2\int_{-\pi}^{\pi} |g(\theta+t) - g(\theta)| \cdot \left|\frac{g(\theta)}{g(\theta+t)}\right| \cdot K_n(t)dt \quad \text{(by (2.9))}$$

$$\le 2\int_{-\pi}^{\pi} |g(\theta+t)-g(\theta)| K_n(t)dt + 2\int_{-\pi}^{\pi} \frac{|g(\theta+t)-g(\theta)|^2}{|g(\theta+t)|} K_n(t)dt$$

$$\le 2c\omega(f;n^{-1}) + \frac{2}{2c\omega(f;n^{-1})} c[\omega(f;n^{-1})]^2$$
$$\text{(by (2.5), (2.4) and (2.7))}$$

$$= (2c + 1)\omega(f;n^{-1}).$$

From (2.10) and from the definition (2.1) of G it now follows (see also (2.7)) that

$$\max_{|z|\le 1} |f(z) - 1/P_n(z)| \le (4c + 1)\omega(f;n^{-1}). \qquad \square$$

3. Proof of Theorem 2.

To establish the upper bound in (1.4), we first prove that, for each positive integer r,

$$E_{0n}((1-z)^r;D) \le B_r n^{-r}, \qquad n = 1,2,3,\dots . \tag{3.1}$$

Define

$$p(z) := \left[\frac{1 - Q(z)^r}{1-z}\right]^r, \qquad n \ge 2, \tag{3.2}$$

where

$$Q(z) := \frac{1-z^n}{n(1-z)} .$$

Since

$$Q(z) = \frac{1-z^n}{n(1-z)} = 1 - \frac{n-1}{2}(1-z) + O((1-z)^2) ,$$

$p(z)$ is a polynomial (of degree $(n-1)r^2 - r$) satisfying $p(1) = (r(n-1)/2)^r$. Also,

$$(3.3)\qquad |Q(z)| = \frac{1}{n}\,|1+z+\cdots+z^{n-1}| < 1 \quad \text{for} \quad |z| \le 1,\ z \ne 1.$$

It follows that $p(z) \ne 0$ in D and consequently it suffices to estimate $|(1-z)^r - 1/p(z)|$ on $|z| = 1$. Since $p(z)$ has real coefficients we may restrict ourselves to the case $z = e^{i\theta}$, $0 \le \theta \le \pi$.

Case 1. $\pi^2/2n \le \theta \le \pi$.
In this case

$$n|1-z| = 2n\sin(\theta/2) \ge 2n\sin(\pi^2/4n) \ge 2n \cdot \frac{2}{\pi} \cdot \frac{\pi^2}{4n} = \pi,$$

so that

$$(3.4)\qquad |Q(z)| = \left|\frac{1-z^n}{n(1-z)}\right| \le \frac{2}{\pi}.$$

Now write

$$(3.5)\qquad (1-z)^r - \frac{1}{p(z)} = (1-z)^r\,\frac{[(1-Q(z)^r)^r - 1]}{[1-Q(z)^r]^r}$$

$$= \frac{-Q(z)^r(1-z)^r}{[1-Q(z)^r]^r}\sum_{k=0}^{r-1}[1-Q(z)^r]^k.$$

Using (3.3), (3.4) and the obvious inequality $|Q(z)(1-z)| \le 2/n$, we obtain

$$|(1-z)^r - \frac{1}{p(z)}| \le \frac{2^r n^{-r}\sum_{k=0}^{r-1} 2^k}{(1-(2/\pi)^r)^r} =: c_r n^{-r},$$

where c_r depends only on r.

Case 2. $2\epsilon/n \le \theta < \pi^2/2n$, for some $0 < \epsilon < 1$.
In this case $\theta < 2\pi/n$ and since the function $(\sin(n\theta/2))/\sin(\theta/2)$ is decreasing for $0 < \theta < 2\pi/n$, we obtain

$$|Q(z)| = \left|\frac{1-z^n}{n(1-z)}\right| = \left|\frac{\sin(n\theta/2)}{n\sin(\theta/2)}\right| \le \frac{\sin\epsilon}{n\sin(\epsilon/n)}.$$

Using the Maclaurin development for the sine function one can easily show that

$$\frac{\sin \epsilon}{n \sin(\epsilon/n)} < 1 - \epsilon^2/10 \quad \text{for} \quad 0 < \epsilon < 1, \quad n \geq 2,$$

and therefore

$$|Q(z)|^r \leq (1 - \epsilon^2/10)^r \leq 1 - \epsilon^2/10$$

which implies

$$|1 - Q(z)^r| \geq \epsilon^2/10 \quad \text{for} \quad 0 < \epsilon < 1, \quad n \geq 2.$$

Using this estimate together with (3.3) and $|Q(z)(1-z)| \leq 2/n$ we obtain from (3.5) that

$$\left|(1-z)^r - \frac{1}{p(z)}\right| \leq \frac{2^r n^{-r} \sum_{k=0}^{r-1} 2^k}{(\epsilon^2/10)^r} =: c_r \epsilon^{-2r} n^{-r} ,$$

where c_r depends only on r.

Case 3. $0 < \theta < 2\epsilon/n$, $\epsilon > 0$ is small enough.

In this case

$$(3.6) \qquad |1-z| < 2\epsilon/n$$

so that

$$(3.7) \qquad |1-z|^r < (2\epsilon/n)^r < n^{-r} \quad \text{if} \quad \epsilon < 1/2.$$

Next, we write $p(z)$ in (3.2) in the form

$$(3.8) \qquad p(z) = \left[\frac{1 - Q(z)}{1-z}\right]^r \left[\sum_{k=0}^{r-1} Q(z)^k\right]^r ,$$

where

$$Q(z) = \frac{1-z^n}{n(1-z)} = 1 + \frac{1}{n} \sum_{j=2}^{n} \binom{n}{j} (z-1)^{j-1}.$$

Since

$$\left|\frac{1}{n}\sum_{j=2}^{n}\binom{n}{j}(z-1)^{j-1}\right| \le \sum_{j=2}^{n}\frac{1}{j!}(2\epsilon)^{j-1} \qquad \text{(by (3.6))}$$

$$= 2\epsilon\sum_{j=2}^{n}\frac{1}{j!}(2\epsilon)^{j-2}$$

$$\le 2\epsilon e \quad , \quad \text{if } \epsilon < 1/2,$$

we obtain

$$\sum_{k=0}^{r-1}Q(z)^k = \begin{cases} 1 & \text{if } r = 1 \\ r + O(\epsilon) & \text{if } r \ge 2, \end{cases}$$

where $O(\epsilon)$ depends only on r. It follows that there exists ϵ_r, $0 < \epsilon_r < 1/2$ (that depends only on r), such that

(3.9) $$\left|\sum_{k=0}^{r-1}Q(z)^k\right| \ge 1 \quad , \text{ provided } 0 < \theta < 2\epsilon_r/n.$$

From (3.7), (3.8), and (3.9) we obtain for $z = e^{i\theta}$, $0 < \theta < 2\epsilon_r/n$,

(3.10) $$\left|(1-z)^r - 1/p(z)\right| \le |1-z|^r + |1/p(z)|$$

$$\le n^{-r} + \left|\frac{1-z}{1 - Q(z)}\right|^r.$$

It therefore suffices to show that

(3.11) $$\left|\frac{1-z}{1 - Q(z)}\right| \le cn^{-1} \quad , \quad n \ge 2,$$

or

$$\left|\frac{n^2(1-z)^2}{n(1-z) - (1-z^n)}\right|^2 \le c^2 \ , \quad n \ge 2,$$

where $c > 0$ is an absolute constant. Putting $z = e^{i\theta}$ we have

$$|n^2(1-z)^2|^2 = 16n^4\sin^4(\theta/2) \le \theta^4 n^4. \tag{3.12}$$

Next, for $0 < \theta < 2\epsilon_r/n$, we have

$$|n(1-z) - (1-z^n)|^2 = 2n(n-1) + 2 - 2n(n-1)\cos\theta + 2n\cos n\theta - 2\cos n\theta - 2n\cos(n-1)\theta$$

$$= 4n(n-1)\sin^2(\theta/2) + 4\sin^2(n\theta/2) - 4n\sin(\theta/2)\sin[(n-1/2)\theta]$$

$$= 4n^2\sin^2(\theta/2) + 4\sin^2(n\theta/2) - 8n\sin(\theta/2)\sin(n\theta/2)\cos[(n-1)\theta/2]$$

$$= 4\{[n\sin(\theta/2) - \sin(n\theta/2)]^2 + 4n\sin(\theta/2)\sin(n\theta/2)\sin^2[(n-1)\theta/4]\}$$

$$\ge 16n\sin(\theta/2)\sin(n\theta/2)\sin^2[(n-1)\theta/4]$$

$$\ge 16n(2/\pi)^4(\theta/2)(n\theta/2)[(n-1)\theta/4]^2.$$

Hence

$$|n(1-z) - (1-z^n)|^2 \ge (4/\pi^4)\theta^4 n^2(n-1)^2. \tag{3.13}$$

The inequalities (3.12), (3.13) yield (3.11) with $c = \pi^2$. Hence,

$$|(1-z)^r - 1/p(z)| \le (1+\pi^{2r})n^{-r}, \quad 0 < \theta < 2\epsilon_r/n.$$

On choosing $\epsilon = \epsilon_r$ in Case 2 we conclude that

$$\max_{|z|\le 1} |(1-z)^r - 1/p(z)| \le c_r n^{-r}, \quad n = 1,2,3,\dots ,$$

where $c_r > 0$ depends only on r. Using a standard technique, the last inequality implies (3.1) for some constant B_r depending only on r (recall that $p(z)$ is of degree $(n-1)r^2 - r$).

To prove the upper bound in (1.4) we write

$$f(z) = g(z)\prod_{j=1}^{\nu}(z-z_j)^{r_j},$$

where $z_1, z_2, \dots, z_\nu$ are the distinct zeros of f on $|z| = 1$ and g

is analytic in the closed disk $D:|z| \leq 1$ and different from zero there. We just proved that

$$E_{on}((z-z_j)^{r_j};D) \leq B_j n^{-r_j}.$$

Also, by Theorem 3, there exist constants $A > 0$ and $0 < \rho < 1$ such that

$$E_{on}(g;D) \leq A\rho^n \quad , \quad n = 1,2,\ldots .$$

Applying Lemma 4.2 in [1] we conclude that for some constants $A_o > 0$ and $0 < \rho_o < 1$,

$$\begin{aligned} E_{on}(f;D) &\leq \text{const}(f) \sum_{j=1}^{\nu} n^{-r_j} + A_o\rho_o^n \\ &\leq \text{const}(f)\cdot n^{-r} \quad , \quad n = 1,2,3,\ldots , \end{aligned}$$

where $r = \min_j r_j$.

Next we prove the lower bound in (1.5). Pick a polynomial $P_n(z)$ of degree $\leq n$ such that

$$(3.14) \qquad \|(1-z)^r - 1/P_n(z)\|_D = E_{on}((1-z)^r;D) =: E_n$$

and let $p_n(\theta)$ denote the trigonometric polynomial $P_n(e^{i\theta})$. Then

$$(3.15) \qquad \|(1-e^{i\theta})^r - 1/p_n(\theta)\|_{[-\pi,\pi]} = E_n$$

and therefore

$$(3.16) \qquad |p_n(0)| \geq 1/E_n.$$

For $|\theta| \geq (\pi/2)(3E_n)^{1/r} =: \delta$ we have

$$|1-e^{i\theta}|^r = |2\sin(\theta/2)|^r \geq |2\theta/\pi|^r \geq 3E_n.$$

Hence (by (3.15))

$$(3.17) \qquad |p_n(\theta)| \le 1/(2E_n) \quad \text{for} \quad |\theta| \ge \delta.$$

It follows (see (3.16)) that $|p_n(\theta)|$ attains its maximum at some point θ_0 in $[-\delta,\delta]$. Now,

$$|p_n(\theta_0) - p_n(\delta)| \ge |p_n(\theta_0)| - |p_n(\delta)| = \|p_n\| - |p_n(\delta)| \ge \|p_n\| - 1/(2E_n) \quad ,$$

where $\|\cdot\|$ denotes the sup norm on $[-\pi,\pi]$. Since $\|p_n\| \ge 1/E_n$ by (3.16), we obtain

$$(3.18) \qquad |p_n(\theta_0) - p_n(\delta)| \ge \|p_n\|/2.$$

On the other hand,

$$|p_n(\theta_0) - p_n(\delta)| \le |\theta_0 - \delta|\cdot\|p_n'\| \le 2\delta\|p_n'\| \le 2\delta n\|p_n\|$$

by Bernstein's inequality (see Lorentz [3, p.39]). Combining this with (3.18) we obtain that $\delta \ge 1/(4n)$. From the definition of δ it now follows that

$$E_n \ge c^r n^{-r},$$

where $0 < c < 1$ is an absolute constant. This proves the lower bound in (1.5).

For the general case, we pick a zero of f of the smallest order r ($z=1$, say) and write $f(z) = (1-z)^r(a+g(z))$, where $a \ne 0$ and $g(z) = O(1-z)$. We can find $\epsilon = \epsilon(f) > 0$ such that $|g(e^{i\theta})| < |a|/2$ for $|\theta| \le \epsilon$. Using the above argument (with obvious modifications) one can show that

$$\max_{-\epsilon\le\theta\le\epsilon} |(1-e^{i\theta})^r(a+g(e^{i\theta})) - 1/p_n(\theta)| \ge c(f)n^{-r}$$

which yields the lower bound in (1.4). □

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