ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 1, Winter 1989

SUFFICIENT CONDITIONS FOR ASYMPTOTICS ASSOCIATED WITH WEIGHTED EXTREMAL PROBLEMS ON R

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DEDICATED TO PROFESSOR A. SHARMA

ABSTRACT. We derive a sufficient condition for asymptotics as $n \to \infty$, for

$$E_{np}(W) := \inf\{\|(x^n + P(x))W(x)\|_{L_p(\mathbf{R})} : \deg(P) < n\},$$

where $1 < p < \infty$, and $W(x)$ is a weight function supported on $\mathbf{R}$. This will be used in a forthcoming paper to show that if $W_\alpha(x) := \exp(-|x|^\alpha)$, $x \in \mathbf{R}, \alpha > 0$, then, for $1 < p < \infty$,

$$\lim_{n\to\infty} E_{np}(W_\alpha)/\{(\beta_\alpha n^{1/\alpha}/2)^{n+1/p} e^{-n/\alpha}\} = 2K_p,$$

where β_α and K_p are constants depending only on α and p respectively.

1. Introduction. Let $W(x)$ be a measurable function, non-negative in $\mathbf{R}$, with all power moments finite, positive on a set of positive measure, and let

$$p_n(W^2; x) = \gamma_n x^n + \cdots, \ \gamma_n > 0,$$

denote the n^{th} orthonormal polynomial for $W^2(x)$ so that, for $m, n = 0, 1, 2, \ldots$,

$$\int_{-\infty}^{\infty} p_m(W^2; x) p_n(W^2; x) W^2(x) dx = \delta_{mn}.$$

Recently, Freud's conjecture concerning the asymptotic behaviour of γ_{n-1}/γ_n as $n \to \infty$ for the weight $\exp(-|x|^\alpha)$ was proved in full

[1] Research completed while the author was visiting the University of South Florida.

[2] Research supported, in part, by the National Science Foundation.
Received by the editors on September 19, 1986.

generality - see [**5**] and also [**3,6,7,8**]. In this paper, we derive a sufficient condition for asymptotics for γ_n as $n \to \infty$. This will be applied in a subsequent paper to a subclass of the weights considered in [**5**]

One characteristic of γ_n is the following extremal property:

$$(1.1)\qquad 1/\gamma_n = \min\{\|(x^n + P(x))W(x)\|_{L_2(\mathbf{R})} : \deg(P) < n\}.$$

Here we consider also the L_p analogue of (1.1), namely

$$(1.2)\qquad E_{np}(W) := \min\{\|(x^n + P(x))W(x)\|_{L_p(\mathbf{R})} : \deg(P) < n\},$$

when $1 < p < \infty$. Our main tool is a formula due to Bernstein (see [**1**, pp. 250-254]), which states that if $S(x)$ is a polynomial of degree at most $2n$, positive in $(-1,1)$ and possibly having simple zeros at ± 1, then for $1 \leq p < \infty$,

(1.3)

$$\min\{\|(x^n + P(x))(1-x^2)^{(1-1/p)/2} S(x)^{-1/2}\|_{L_p[-1,1]} : \deg(P) < n\}$$
$$= K_p 2^{-n}\{G[S(x)]\}^{-1/2},$$

where

$$(1.4)\qquad K_p := \{\Gamma(1/2)\Gamma((p+1)/2)/\Gamma(p/2+1)\}^{1/p},$$

and $G[S(x)]$ is the *weighted geometric mean* of $S(x)$,

$$(1.5)\qquad G[S(x)] := \exp\left(\pi^{-1}\int_{-1}^{1} \log S(x)dx/\sqrt{1-x^2}\right).$$

Our results are stated in §2 and proved in §3. We should especially like to thank Paul Nevai for his encouragement, and also wish to acknowledge the encouragement of, and discussions with A.L. Levin, Al. Magnus, H.N. Mhaskar and V. Totik.

2. Statement of results. We shall state separately the conditions for asymptotic upper and lower bounds for $E_{np}(W)$. Throughout, $\mathcal{P}_n$ denotes the class of real polynomials of degree at most n. Further, given a non-negative measurable function $f(x)$ and $a > 0$, we set, as in (1.5),

$$G[f(ax)] := \exp\left(\pi^{-1}\int_{-1}^{1}\log f(ax)dx/\sqrt{1-x^2}\right).$$

PROPOSITION 2.1. *Let $1 \le p < \infty$, and let $W(x) \in L_p(\mathbf{R})$ be a non-negative function such that, for all positive a,*

$$\int_{-1}^{1}\log W(ax)dx/\sqrt{1-x^2} > -\infty. \tag{2.1}$$

Assume that, for every $q \in [p,\infty)$,

$$x^n W(x) \in L_q(\mathbf{R}),\ n = 0,1,2,\dots .$$

Assume further that there exist respectively increasing and decreasing sequences $\{c_n\}_1^\infty$ and $\{\delta_n\}_1^\infty$ of positive numbers such that

$$\lim_{n\to\infty}\delta_n = 0, \tag{2.2}$$

and, for $n = 1,2,3,\dots$, and each $P \in \mathcal{P}_n$,

$$||PW||_{L_p(\mathbf{R})} \le (1+\delta_n)||PW||_{L_p[-c_n,c_n]}. \tag{2.3}$$

Finally, assume that, for every $q \in [p,\infty)$ and each $g(x)$ positive and continuous in $[-,1,1]$, there exists $P_{2n-2}(x) \in \mathcal{P}_{2n-2}$, positive in $[-1,1]$, $n = 1,2,3,\dots$ such that,

$$\liminf_{n\to\infty}\int_{-1}^{1}\log\{\sqrt{P_{2n-2}(x)}\,W(c_n x)g(x)\}dx/\sqrt{1-x^2} \ge 0, \tag{2.4}$$

and

$$\lim_{n\to\infty}\sup||\sqrt{P_{2n-2}(x)}\,W(c_n x)g(x)||_{L_q[-1,1]} \le 2^{1/q}. \tag{2.5}$$

Then, if K_p is given by (1.4),

$$\limsup_{n\to\infty} E_{np}(W)/\{(c_n/2)^{n+1/p}G[W(c_n x)]\} \le 2K_p. \tag{2.6}$$

Note that (2.3) is an "infinite-finite range" inequality of the type investigated in [**5**] and that (2.4) and (2.5) essentially require that $\sqrt{P_{2n-2}(x)}\, W(c_n x)g(x)$ approximates 1 in a suitable sense. We remark that (2.1) is redundant, and included only for clarity.

PROPOSITION 2.2. *Let $1 < p < \infty$, and let $W(x)$ be a non-negative function such that*

$$x^n W(x) \in L_p(\mathbf{R}), \quad n = 0, 1, 2, \ldots,$$

such that (2.1) holds for all positive a, and such that, for each $q \in (p, \infty)$ and $a > 0$,

$$W(x)^{-1} \in L_q[-a, a]. \tag{2.7}$$

Assume further that $\{d_n\}_1^\infty$ is an increasing sequence of positive numbers with the following property: For every $q \in (p, \infty)$ and each $g(x)$ even, positive and continuous in $[-1, 1]$, there exists $P_{2n}(x) \in \mathcal{P}_{2n}$, positive in $[-1, 1]$, $n = 1, 2, 3, \ldots$, such that

$$\limsup_{n\to\infty} \int_{-1}^{1} \log\{\sqrt{P_{2n}(x)}\, W(d_n x)g(x)\}dx/\sqrt{1-x^2} \leq 0, \tag{2.8}$$

and

$$\limsup_{n\to\infty} \|\{\sqrt{P_{2n}(x)}\, W(d_n x)g(x)\}^{-1}\|_{L_q[-1,1]} \leq 2^{1/q}. \tag{2.9}$$

Then, if K_p is given by (1.4),

$$\liminf_{n\to\infty} E_{np}(W)/\{(d_n/2)^{n+1/p} G[W(d_n x)\} \geq 2K_p. \tag{2.10}$$

In applications (see [**4**]) the sequences $\{d_n\}_1^\infty$ and $\{c_n\}_1^\infty$ are different, but are sufficiently close to deduce asymptotics for $E_{np}(W)$ from (2.6) and (2.10), with the aid of the following lemma:

LEMMA 2.3. *Let $W(x) := e^{-Q(x)}$, where $Q(x)$ is even, continuous in $\mathbf{R}$ and $Q''(x)$ exists for $x > 0$, while $xQ'(x)$ is positive and increasing in $(0, \infty)$, with*

$$\lim_{x\to+\infty} xQ'(x) = +\infty. \tag{2.11}$$

Assume further that there exist $C_1, C_2 > 0$ *such that*

$$x|Q''(x)|/Q'(x) \le C_1, \; x \in (0,\infty), \tag{2.12}$$

and

$$Q'(2x)/Q'(x) \le C_2, \; x \in (0,\infty). \tag{2.13}$$

Let $a_n = a_n(W)$ *be the positive root of the equation*

$$n = 2\pi^{-1} \int_0^1 a_n t Q'(a_n t) dt / \sqrt{1-t^2}\,, \tag{2.14}$$

for n *large enough, and let* $\{e_n\}_1^\infty$ *be a sequence of positive numbers satisfying*

$$\lim_{n\to\infty} n^{1/2}(e_n/a_n - 1) = 0. \tag{2.15}$$

Then

$$\lim_{n\to\infty} e_n^{n+1/p} G[W(e_n x)] / \{a_n^{n+1/p} G[W(a_n x)]\} = 1. \tag{2.16}$$

Propositions 2.1 and 2.2 will be used in a forthcoming paper [**4**] to show that, for a large class of weights $W(x) := e^{-Q(x)}$, there holds, for $1 < p < \infty$,

$$\lim_{n\to\infty} E_{np}(W)/\{(a_n/2)^{n+1/p} G[W(a_n x)]\} = 2K_p. \tag{2.17}$$

In particular, the result applies to $W(x) := W_\alpha(x) := \exp(-|x|^\alpha)$, $\alpha > 0$.

3. Proofs. Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n and x.

PROOF OF PROPOSITION 2.1. By the infinite-finite range inequality (2.3) and by the definition (1.2) of $E_{np}(W)$,

$$\begin{aligned} E_{np}(W) &\le (1+\delta_n) \inf_{P\in\mathcal{P}_{n-1}} \|\{u^n + P(u)\}W(u)\|_{L_p[-c_n,c_n]} \\ &= (1+\delta_n) c_n^{n+1/p} \inf_{P\in\mathcal{P}_{n-1}} \|\{x^n + P(x)\}W(c_n x)\|_{L_p[-1,1]}, \end{aligned} \tag{3.1}$$

by the substitution $u = c_n x$. Next, let $g(x)$ be positive and continuous in $[-1,1]$, and let $P_{2n-2}(x) \in \mathcal{P}_{2n-2}$ be positive in $[-1,1], n = 1,2,3,\ldots$. Further, let $1 < r,\ s < \infty$ satisfy $r^{-1} + s^{-1} = 1$. By Hölder's inequality,

(3.2)
$$\begin{aligned}
&\inf_{P\in\mathcal{P}_{n-1}} \|\{x^n + P(x)\}W(c_n x)\|_{L_p[-1,1]} \\
&\leq \inf_{P\in\mathcal{P}_{n-1}} \|\{x^n + P(x)\}(1-x^2)^{(1-1/(ps))/2} \\
&\qquad\qquad \{P_{2n-2}(x)(1-x^2)\}^{-1/2}\|_{L_{ps}[-1,1]} \\
&\qquad \times \|(1-x^2)^{1/(2ps)}P_{2n-2}(x)^{1/2}W(c_n x)\|_{L_{pr}[-1,1]} \\
&\leq K_{ps}2^{-n}G[P_{2n-2}(x)(1-x^2)]^{-1/2}\|(1-x^2)^{1/(2ps)}g(x)^{-1}\|_{L_{prs}[-1,1]} \\
&\qquad \times \|g(x)P_{2n-2}(x)^{1/2}W(c_n x)\|_{L_{pr^2}[-1,1]},
\end{aligned}$$

by Bernstein's formula (1.3) in the L_{ps} norm applied to $S(x) := P_{2n-2}(x)(1-x^2)$ and by Hölder's inequality, with parameters r and s.

Note next that $G[1] = 1$ and, for any $a, b \in \mathbf{R}$,

$$G[S(x)^a T(x)^b] = G[S(x)]^a G[T(x)]^b.$$

Let $\varepsilon > 0$, and choose $g(x)$ positive and continuous in $[-1,1]$ such that $(1-x^2)^{1/(2ps)}g(x)^{-1}$ approximates 1 in the sense that

$$\|(1-x^2)^{1/(2ps)}g(x)^{-1}\|_{L_{prs}[-1,1]} \leq 2^{1/(prs)}(1+\varepsilon), \tag{3.3}$$

and

$$G[(1-x^2)^{1/(2ps)}g(x)^{-1}]^{-1} \leq 1+\varepsilon. \tag{3.4}$$

Further, let $\{P_{2n-2}(x)\}_1^\infty$ satisfy (2.4) and (2.5) with $q = pr^2$. Then

$$\begin{aligned}
&G[P_{2n-2}(x)(1-x^2)]^{-1/2} \\
&\quad = G[\sqrt{P_{2n-2}(x)}\, W(c_n x)g(x)]^{-1}G[W(c_n x)] \\
&\qquad \times G[(1-x^2)^{1/(2ps)}g(x)^{-1}]^{-1}G[1-x^2]^{-(1-1/(ps))/2} \\
&\quad \leq (1+\varepsilon)G[W(c_n x)](1+\varepsilon)2^{1-1/(ps)}, \quad n \text{ large enough},
\end{aligned} \tag{3.5}$$

by (2.4), (3.4) and a standard integral [**2**, p. 243, numbers 864.31 and 864.32], which shows that

$$G[1-x^2] = 1/4. \tag{3.6}$$

Combining (3.1), (3.2), (3.3), (3.5) and (2.5), we obtain for n large enough,

$$\begin{aligned} E_{np}(W) \leq & (1+\varepsilon)^5 c_n^{n+1/p} K_{ps} 2^{-n} G[W(c_n x)] 2^{1-1/ps} \\ & \times 2^{1/(prs)} 2^{1/(pr^2)}. \end{aligned}$$

Hence, since $\varepsilon > 0$ is arbitrary,

$$\begin{aligned} & \limsup_{n\to\infty} E_{np}(W)/\{(c_n/2)^{n+1/p} G[W(c_n x)]\} \\ & \leq K_{ps} 2^{1-1/ps+1/p+1/(prs)+1/(pr^2)}. \end{aligned} \tag{3.7}$$

Letting $s \to 1$ so that $r \to \infty$, we obtain (2.6), noting that K_q is continuous in q for $q \in [1, \infty)$. □

LEMMA 3.1. *Let $0 < p < \infty$ and $1 < r, s < \infty$ satisfy $r^{-1} + s^{-1} = 1$. If J, H are measurable functions such that $H^{-1} \in L_{pr/s}[-1,1]$ and $JH \in L_p[-1,1]$, then*

$$||JH||_{L_p[-1,1]} \geq ||J||_{L_p[-1,1]} ||H^{-1}||^{-1}_{L_{pr/s}[-1,1]}. \tag{3.8}$$

PROOF. By Hölder's inequality, with parameters r, s,

$$\begin{aligned} ||J||_{L_{p/s}[-1,1]} &= ||JH \cdot H^{-1}||_{L_{p/s}[-1,1]} \\ &\leq ||JH||_{L_{(p/s)s}[-1,1]} ||H^{-1}||_{L_{(p/s)r}[-1,1]}. \end{aligned}$$

□

PROOF OF PROPOSITION 2.2. Let $1 < r < s < \infty$ satisfy $r^{-1} + s^{-1} = 1$ and $1 < s < p$. Let $\varepsilon > 0$, and choose $g(x)$ even, continuous and positive in $[-1, 1]$ so that $(1-x^2)^{(1-s/p)/2} g(x)$ approximates 1 in the following sense:

$$||(1-x^2)^{(1-s/p)/2} g(x)||_{L_{pr}[-1,1]} \leq 2^{1/(pr)}(1+\varepsilon) \tag{3.9}$$

and

$$G[(1-x^2)^{(1-s/p)/2}g(x)] \geq 1-\varepsilon. \tag{3.10}$$

Further, choose $\{P_{2n}(x)\}_1^\infty$ to satisfy (2.8) and (2.9) with $q = pr^2/s$. We have

(3.11)
$$\begin{aligned}
E_{np}(W) &\geq \inf_{P\in\mathcal{P}_{n-1}} \|\{u^n + P(u)\}W(u)\|_{L_p[-d_n,d_n]} \\
&= d_n^{n+1/p} \inf_{P\in\mathcal{P}_{n-1}} \|\{x^n + P(x)\}W(d_n x)\|_{L_p[-1,1]} \\
&\geq d_n^{n+1/p} \inf_{P\in\mathcal{P}_{n-1}} \|\{x^n + P(x)\}(1-x^2)^{(1-s/p)/2}P_{2n}(x)^{-1/2}\|_{L_{p/s}[-1,1]} \\
&\quad \times \|\{(1-x^2)^{-(1-s/p)/2}P_{2n}(x)^{1/2}W(d_n x)\}^{-1}\|_{L_{pr/s}[-1,1]}^{-1},
\end{aligned}$$

by Lemma 3.1. Next, using Bernstein's formula (1.3), and Hölder's inequality again, we obtain

(3.12)
$$\begin{aligned}
E_{np}(W) &\geq d_n^{n+1/p} K_{p/s} 2^{-n}\{G[P_{2n}(x)]\}^{-1/2} \\
&\qquad \times \|(1-x^2)^{(1-s/p)/2}g(x)\|_{L_{pr}[-1,1]}^{-1} \\
&\qquad\quad \|\{g(x)P_{2n}(x)^{1/2}W(d_n x)\}^{-1}\|_{L_{pr^2/s}[-1,1]}^{-1} \\
&\geq d_n^{n+1/p} K_{p/s} 2^{-n}\{G[P_{2n}(x)]\}^{-1/2} 2^{-1/(pr)}(1+\varepsilon)^{-1}2^{-s/(pr^2)},
\end{aligned}$$

for n large enough, by (3.9) and (2.9). Here,

$$\begin{aligned}
&G[P_{2n}(x)]^{-1/2} \\
&= G[\sqrt{P_{2n}(x)}\, W(d_n x)g(x)]^{-1}G[W(d_n x)] \\
&\quad \times G[(1-x^2)^{(1-s/p)/2}g(x)]G[1-x^2]^{-(1-s/p)/2} \\
&\geq (1-\varepsilon)G[W(d_n x)](1-\varepsilon)2^{1-s/p}, \quad n \text{ large enough},
\end{aligned} \tag{3.13}$$

by (2.8), (3.6) and (3.10). Combining (3.12) and (3.13), letting $\varepsilon \to 0$ and then letting $s \to 1$ so that $r \to \infty$, we obtain (2.10). □

PROOF OF LEMMA 2.3. By (2.15), we can write

$$e_n = a_n(1+\eta_n), \quad n = 1, 2, 3, \ldots,$$

where

$$\lim_{n\to\infty} n^{1/2}\eta_n = 0. \tag{3.14}$$

Now, given $0 < t < \infty$, there exists ξ between 1 and $1+\eta_n$ such that

$$\begin{aligned}\log W(e_n t) &= -Q(e_n t)\\ &= -\{Q(a_n t) + a_n\eta_n t Q'(a_n t) + (a_n\eta_n t)^2 Q''(\xi a_n t)/2\}.\end{aligned}$$

Here, by monotonicity of $tQ'(t)$, and by (2.12) and (2.13),

$$\begin{aligned}(a_n t)^2|Q''(\xi a_n t)| &\le C_1 a_n t Q'(\xi a_n t)\\ &\le C_1 2a_n t Q'(2a_n t) \le 2C_1C_2 a_n t Q'(a_n t).\end{aligned}$$

Then, using evenness of W, we see that

$$\begin{aligned}&\pi^{-1}\int_{-1}^{1} \log W(e_n t)dt/\sqrt{1-t^2}\\ &= -2\pi^{-1}\int_0^1 Q(a_n t)dt/\sqrt{1-t^2}\\ &\quad - \eta_n 2\pi^{-1}\int_0^1 a_n t Q'(a_n t)dt/\sqrt{1-t^2}\,(1+O(\eta_n))\\ &= -2\pi^{-1}\int_0^1 Q(a_n t)dt/\sqrt{1-t^2} - n\eta_n(1+O(\eta_n)),\end{aligned}$$

by (2.14). Hence we see that

$$\begin{aligned}&e_n^{n+1/p}G[W(e_n x)]\\ &= \{a_n(1+\eta_n)\}^{n+1/p}G[W(a_n x)]\exp(-n\eta_n + O(n\eta_n^2))\\ &= a_n^{n+1/p}G[W(a_n x)]\exp((n+1/p)\eta_n - n\eta_n + O(n\eta_n^2)).\end{aligned}$$

Then (3.14) yields (2.16). □

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