

JACKSON TYPE THEOREMS IN APPROXIMATION BY RECIPROCAL OF POLYNOMIALS

A.L. LEVIN¹ AND E.B. SAFF²

ABSTRACT. It was previously shown by the authors that Jackson type theorems hold for the case of approximating a continuous real-valued function f on a real interval by the *reciprocals* of complex polynomials. In this paper we extend these results to the general case when f is complex-valued.

1. Statement of results. Let $C^*[-\pi, \pi]$ denote the set of 2π -periodic continuous complex-valued functions and let $C[-1, 1]$ denote the set of continuous complex-valued functions on $[-1, 1]$. For any $f \in C^*[-\pi, \pi]$ (resp. $f \in C[-1, 1]$) we denote by $E_{0n}^*(f)$ (resp. by $E_{0n}(f)$) the error in best uniform approximation of f on $[-\pi, \pi]$ (resp. on $[-1, 1]$) by *reciprocals* of trigonometric (resp. algebraic) polynomials of degree $\leq n$ with complex coefficients.

Our goal is to prove the following Jackson type theorems.

THEOREM 1. *There exists a constant M such that for any $f \in C^*[-\pi, \pi]$,*

$$E_{0n}^*(f) \leq M\omega(f; n^{-1}), \quad n = 1, 2, 3, \dots,$$

where $\omega(f; \delta)$ denotes the modulus of continuity of f on $[-\pi, \pi]$.

THEOREM 2. *There exists a constant M such that, for any $f \in C[-1, 1]$,*

$$E_{0n}(f) \leq M\omega(f; n^{-1}), \quad n = 1, 2, 3, \dots,$$

where $\omega(f; \delta)$ denotes the modulus of continuity of f on $[-1, 1]$.

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For the case of real-valued f , these theorems (with slightly different notation) were proved in our paper [1]. Although the idea of the proof remains the same, the passage to a complex-valued f is not straightforward (in contrast with polynomial approximation). It requires a preliminary construction (see Lemma 1 below) that is trivial in the case of real f but rather complicated in general.

2. Proofs. We first formulate two lemmas. In these results, $\|\cdot\|$ denotes the sup norm on $[-\pi, \pi]$ and ω is the modulus of continuity on $[-\pi, \pi]$.

LEMMA 1. *For any $f \in C^*[-\pi, \pi]$, for any positive integer n , and for any $A > 0$, there exists a function $g \in C^*[-\pi, \pi]$ such that*

- (1) $\|f - g\| \leq 4A\omega(f; n^{-1})$,
- (2) $|g(x)| \geq \frac{1}{2}A\omega(f; n^{-1})$, $-\pi \leq x \leq \pi$, and
- (3) $\omega(g; n^{-1}) \leq (1 + 8\pi)\omega(f; n^{-1})$.

Also, if f is even, then g may be chosen even as well.

LEMMA 2. *There exist absolute constants $A_0 > 0$, $A_1 > 0$ such that, for any $g \in C^*[-\pi, \pi]$ that satisfies (2) with $A = A_0$ and (3), one can find a trigonometric polynomial P_n of degree $\leq n$ such that*

$$\|g - 1/P_n\| \leq A_1\omega(f; n^{-1}).$$

Also, if g is even, then P_n may be chosen even as well.

Theorem 1 is an immediate consequence of these lemmas. Indeed, applying Lemma 1 with $A = A_0$ and Lemma 2 we obtain that

$$E_{0n}^*(f) \leq \|f - g\| + \|g - 1/P_n\| \leq M\omega(f; n^{-1}),$$

where $M := 4A_0 + A_1$. Theorem 2 follows from Theorem 1 by a standard argument (notice the last assertions of the lemmas).

PROOF OF LEMMA 1. Set

$$(4) \quad K_1 := \{x \in [-\pi, \pi] : |f(x)| \geq A\omega(f; n^{-1})\},$$

$$(5) \quad K_2 := \{x \in [-\pi, \pi] : |f(x)| < A\omega(f; n^{-1})\}.$$

We assume first that $\pm\pi \in K_1$. In this case we can represent K_2 as a union $\cup(a_k, b_k)$ of disjoint open intervals in $(-\pi, \pi)$ with

$$(6) \quad |f(a_k)| = |f(b_k)| = A\omega(f; n^{-1}).$$

Further, we write K_2 as a union $K'_2 \cup K''_2$, where

$$(7) \quad K'_2 := \cup \left\{ (a_k, b_k) : |f(x)| \geq 1/2 A\omega(f; n^{-1}), \text{ all } x \in (a_k, b_k) \right\},$$

$$(8) \quad K''_2 := \cup \left\{ (a_k, b_k) : |f(x)| < 1/2 A\omega(f; n^{-1}) \text{ for some } x \in (a_k, b_k) \right\}.$$

Then, for the length $\Delta_k := b_k - a_k$ of any interval (a_k, b_k) in K''_2 , we have the estimate

$$(9) \quad \omega(f; \Delta_k) \geq ||f(b_k)| - \min_{(a_k, b_k)} |f| \geq \frac{1}{2} A\omega(f; n^{-1}),$$

by (6) and (8).

For every interval (a_k, b_k) in K''_2 , write (cf. (6)) $f(a_k) = A\omega(f; n^{-1}) \exp(i\alpha_k)$, $f(b_k) = A\omega(f; n^{-1}) \exp(i\beta_k)$, with $|\beta_k - \alpha_k| \leq \pi$ and let $L_k(x)$ be the linear function that satisfies

$$L_k(a_k) = \alpha_k, \quad L_k(b_k) = \beta_k.$$

Then, for any $h > 0$,

$$(10) \quad |L_k(x+h) - L_k(x)| \leq \frac{\pi}{\Delta_k} h, \quad \text{where } \Delta_k := b_k - a_k.$$

Now define the function g on $[-\pi, \pi]$ by

$$(11) \quad g(x) := f(x), \quad x \in K_1 \cup K'_2,$$

$$(12) \quad g(x) := A\omega(f; n^{-1}) \exp(iL_k(x)), \quad x \in (a_k, b_k) \subset K''_2.$$

From the construction of g it follows that $g \in C^*[-\pi, \pi]$ and satisfies

$$(13) \quad \|f - g\| \leq 2A\omega(f; n^{-1}),$$

$$(14) \quad |g(x)| \geq \frac{1}{2}A\omega(f; n^{-1}), \quad -\pi \leq x \leq \pi.$$

To estimate the modulus of continuity of g we make use of the well-known inequality

$$(15) \quad \frac{\omega(f; h)}{h} \leq 2 \frac{\omega(f; h')}{h'}, \quad \text{for } h \geq h' > 0.$$

Let $x, x+h$ ($h > 0$) be any two points in $[-\pi, \pi]$.

Case 1. $x, x+h \in K_1 \cup K'_2$. Then (cf. (11)) $|g(x+h) - g(x)| \leq \omega(f; h)$.

Case 2. $x, x+h \in (a_k, b_k) \subset K''_2$. Since $|\exp(it) - \exp(is)| \leq |t - s|$, we obtain, from (12) and (10):

$$\begin{aligned} |g(x+h) - g(x)| &\leq A\omega(f; n^{-1}) \frac{\pi}{\Delta_k} h \\ &\leq 2\pi \frac{\omega(f; \Delta_k)}{\Delta_k} h \quad (\text{by (9)}) \\ &\leq 4\pi \frac{\omega(f; h)}{h} h \quad (\text{by (15), since } \Delta_k \geq h) \\ &= 4\pi\omega(f; h). \end{aligned}$$

Case 3. $x \in (a_k, b_k) \subset K''_2$, $x+h \in K_1 \cup K'_2$. Write

$$\begin{aligned} |g(x+h) - g(x)| &\leq |g(b_k) - g(x)| + |g(x+h) - g(b_k)| \\ &= |g(b_k) - g(x)| + |f(x+h) - f(b_k)| \\ &\leq |g(b_k) - g(x)| + \omega(f; h). \end{aligned}$$

Since $|b_k - x| < \Delta_k$, we obtain as in Case 2, that

$$\begin{aligned} |g(b_k) - g(x)| &\leq 2\pi \frac{\omega(f; \Delta_k)}{\Delta_k} |b_k - x| \\ &\leq 4\pi \frac{\omega(f; |b_k - x|)}{|b_k - x|} \cdot |b_k - x|, \quad (\text{by (15)}) \\ &= 4\pi\omega(f; |b_k - x|) \leq 4\pi\omega(f; h). \end{aligned}$$

Hence

$$(16) \quad |g(x+h) - g(x)| \leq (1+4\pi)\omega(f; h).$$

Case 4. $x \in K_1 \cup K_2'$, $x+h \in K_2''$. Just as in Case 3, it can be shown that inequality (16) holds.

Case 5. $x \in (a_k, b_k) \subset K_2''$, $x+h \in (a_l, b_l) \subset K_2''$, with $k \neq l$. In this case we write (assume $b_k \leq a_l$)

$$|g(x+h) - g(x)| \leq |g(b_k) - g(x)| + |g(a_l) - g(b_k)| + |g(x+h) - g(a_l)|,$$

and proceeding as in Case 3 we conclude that

$$|g(x+h) - g(x)| \leq (1+8\pi)\omega(f; h).$$

Putting all the cases together we obtain

$$(17) \quad \omega(g; h) \leq (1+8\pi)\omega(f; h), \quad h > 0.$$

The inequalities (13), (14) and (17) prove Lemma 1 for the case $\pm\pi \in K_1$. If $\pm\pi \in K_2$, that is if $|f(\pm\pi)| < A\omega(f; n^{-1})$, we replace f by $\tilde{f} := f + 2A\omega(f; n^{-1})$ and apply the above argument to construct the function g that satisfies (13), (14), and (17) with \tilde{f} instead of f . Since $\omega(\tilde{f}; h) = \omega(f; h)$ and $\|f - \tilde{f}\| \leq 2A\omega(f; n^{-1})$, the same function g will satisfy the requirements (1), (2), and (3) of Lemma 1.

Finally, if f is even, then each of the sets K_1 , K_2' , and K_2'' is symmetric with respect to the origin. From this and from the definition (11), (12) of g it follows easily that g is also even. \square

REMARK. If f is real, the function g can be constructed in a much simpler way, namely we can set $g(x) := f(x) + iA\omega(f; n^{-1})$.

PROOF OF LEMMA 2. The proof is essentially contained in our paper [1]. For the reader's convenience we reproduce it briefly.

Let $K_n(t)$ be the Jackson kernel (cf. Lorentz [2, p. 55]). Then, for any $g \in C^*[-\pi, \pi]$,

$$(18) \quad \int_{-\pi}^{\pi} |g(x+t) - g(x)|^j K_n(t) dt \leq c[\omega(g; n^{-1})]^j, \quad j = 1, 2,$$

where c is an absolute constant. Define

$$(19) \quad A_0 := 4c(1 + 8\pi)$$

and let g be the function from Lemma 1 with $A = A_0$. Further, define the trigonometric polynomial P_n of degree $\leq n$ by

$$(20) \quad P_n(x) := \int_{-\pi}^{\pi} \frac{1}{g(x+t)} K_n(t) dt.$$

Then

$$\begin{aligned} |1 - P_n(x)g(x)| &= \left| \int_{-\pi}^{\pi} \frac{g(x+t) - g(x)}{g(x+t)} K_n(t) dt \right| \\ &\leq \frac{2}{A_0\omega(f; n^{-1})} \cdot c\omega(g; n^{-1}), \quad (\text{by (2), (18)}) \\ &\leq \frac{2c(1 + 8\pi)}{A_0} = \frac{1}{2}, \quad (\text{by (3), (19)}). \end{aligned}$$

Hence,

$$(21) \quad |P_n(x)g(x)| \geq 1/2, \quad -\pi \leq x \leq \pi.$$

Now,

$$\begin{aligned} &|g(x) - 1/P_n(x)| \\ &\leq \int_{-\pi}^{\pi} \left| \frac{g(x+t) - g(x)}{g(x)g(x+t)} \right| \cdot \left| \frac{g(x)}{P_n(x)} \right| \cdot K_n(t) dt \\ &\leq 2 \int_{-\pi}^{\pi} |g(x+t) - g(x)| \cdot \left| \frac{g(x)}{g(x+t)} \right| \cdot K_n(t) dt \quad (\text{by (21)}) \\ &\leq 2 \int_{-\pi}^{\pi} |g(x+t) - g(x)| K_n(t) dt + 2 \int_{-\pi}^{\pi} \frac{|g(x+t) - g(x)|^2}{|g(x+t)|} K_n(t) dt \\ &\leq 2c\omega(g; n^{-1}) + 4c(\omega(g; n^{-1}))^2/A_0\omega(f; n^{-1}) \quad (\text{by (2), (18)}) \\ &\leq (2c + 1)(1 + 8\pi)\omega(f; n^{-1}) =: A_1\omega(f; n^{-1}) \quad (\text{by (2), (3), and (19)}). \end{aligned}$$

Finally, if $g \in C^*[-\pi, \pi]$ is even, then (cf. (20)) P_n is an even trigonometric polynomial. \square

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DEPARTMENT OF MATHEMATICS, EVERYMAN'S UNIVERSITY, 16 KLAUSNER ST.,
POB 39328, TEL-AVIV 61392, ISRAEL

INSTITUTE FOR CONSTRUCTIVE MATHEMATICS, DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF SOUTH FLORIDA, TAMPA, FL 33620

