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Row convergence theorems for generalised inverse vector-valued Padé approximants

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Abstract: The approximants mentioned in the title are related to vector-valued continued fractions and the vector ϵ-algorithm devised by Wynn in 1963. Here we establish a unitary invariance property of these approximants and describe how the classical (1-dimensional) Padé approximants can be obtained as a special case. The main results of the paper consist of De Montessus–De Ballore type convergence theorems for row sequences (having fixed denominator degree) of vector-valued approximants to meromorphic vector functions.

1. Introduction

In this paper, we discuss a natural extension of ordinary Padé approximation to approximation of a vector-valued function $\boldsymbol{f}(z)$ which is analytic at the origin. We may then expand $\boldsymbol{f}(z)$ as

$$\boldsymbol{f}(z) = \boldsymbol{c}_0 + \boldsymbol{c}_1 z + \cdots + \boldsymbol{c}_n z^n + \cdots, \tag{1.1}$$

and this series converges in some neighbourhood of the origin. Formally, we assume that $\boldsymbol{c}_i \in \mathbb{C}^d$, $i = 0, 1, \ldots,$ and $\boldsymbol{f}$: $\mathbb{C} \to \mathbb{C}^d$. Rational fractions of the form

$$\boldsymbol{r}(z) = \boldsymbol{p}(z)/q(z)$$

normally exist and approximate $\boldsymbol{f}(z)$ in the sense that

$$\boldsymbol{p}(z)/q(z) = \boldsymbol{f}(z) + \mathrm{O}(z^{n+1}), \tag{1.2}$$

$$\partial\{\boldsymbol{p}\} \leqslant n, \qquad \partial\{q\} = 2k, \tag{1.3}$$

$$q(z) \mid \boldsymbol{p}(z) \cdot \boldsymbol{p}^*(z), \tag{1.4}$$

and that $\boldsymbol{r}(z)$ is normally uniquely defined by (1.1)–(1.4) [3]. (The notation of (1.2)–(1.4) will be defined later in this section.) The rational form of $\boldsymbol{r}(z)$ can be obtained by using the ϵ-algorithm

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to extrapolate the partial sums of (1.1) and using generalised (Moore–Penrose) inverses for the vector valued quantities. Consequently, such fractions are called *generalised inverse, vector-valued Padé approximants*, or GIPAs for short.

The following requirements seem natural for an interesting system of vector-valued Padé approximants of $\boldsymbol{f}(z)$:

(i) If all the coefficients $\{\boldsymbol{c}_i\}$ have the same direction, i.e. $\boldsymbol{c}_i = \boldsymbol{\lambda} d_i$, $i = 0, 1, 2, \ldots, \boldsymbol{\lambda}$ fixed, $\boldsymbol{\lambda} \in \mathbb{C}^d$, and $R(z)$ is the (ordinary) Padé approximant of $\sum_{i=0}^{\infty} d_i z^i$, then $\boldsymbol{\lambda} R(z)$ is the vector-valued Padé approximant of $\boldsymbol{f}(z)$.

(ii) There is some sense in which the vector-valued Padé approximant is unique.

(iii) The poles of the vector valued Padé approximant normally occur at common positions in the z-plane.

Two distinct systems of vector-valued Padé approximants, which satisfy these requirements, have been established. Simultaneous Padé approximants, also known as solutions of the German polynomial approximation problem, were investigated by Hermite, and they satisfy these requirements. Row convergence theorems for simultaneous Padé approximants (analogous to those in this paper for GIPAs) have been obtained by Mall [8] and Graves-Morris and Saff [6]. The latter paper contains the relevant up-to-date references.

GIRI and GIPA methods grew out of Wynn's [11] vector-valued continued fractions, and McCleod's [7] analysis of their connection with the vector ϵ-algorithm.

The system of GIPAs also satisfies the requirements (i)–(iii) above. The equivalents of GIPAs for rational interpolation on distinct points are called GIRIs. A practical application of these has been made by Roberts and Graves-Morris [10] to the modal analysis of vibrating structures. The method enables data from different ports of the structure (corresponding to the different components of $\boldsymbol{f}(z)$) to be handled simultaneously, leading to uniquely determined values for the modal parameters. The data are supplied at distinct frequency points, but otherwise the model used is equivalent to that treated in Section 3 of this paper.

The issues of existence and degeneracy of GIPAs were ignored in (1.2)–(1.4). These equations need to be replaced by the precise formulation of Graves-Morris and Jenkins [4], which we give in (1.8)–(1.11).

Notation. We say that a function $\xi(z) = \mathrm{O}(z^{n+1})$ if

$$\xi(0) = \xi'(0) = \cdots = \xi^{(n)}(0) = 0.$$

No bounding properties on $\xi(z)$ are implied.

$\boldsymbol{p}(z)$ denotes a d-dimensional vector of polynomials, i.e.

$$\boldsymbol{p}(z) = (p_1(z), p_2(z), \ldots, p_d(z)).$$

$$\partial\{\boldsymbol{p}\} := \max_{1 \leqslant i \leqslant d} \partial\{p_i\}. \tag{1.6}$$

The order of magnitude of a sequence $\{x_n\}$ is denoted by the symbol $\theta(\alpha^n)$. The statement that

$$\{x_n\} = \theta(\alpha^n)$$

means that

$$\limsup_{n \to \infty} |x_n|^{1/n} \leqslant |\alpha|.$$

A superscript * denotes complex conjugate. In (1.4), for example, * denotes functional complex conjugate. If $\boldsymbol{p}(z)=\boldsymbol{a}+\boldsymbol{b}z$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant complex vectors, $\boldsymbol{p}^*(z)=\boldsymbol{a}^*+\boldsymbol{b}^*z$. We also reserve the notation Re $\boldsymbol{f}(z)$, Im $\boldsymbol{f}(z)$ for the real and imaginary functional parts. For example,

$$\operatorname{Re} \boldsymbol{f}(z)=\tfrac{1}{2}(\boldsymbol{f}(z)+\boldsymbol{f}^*(z)). \tag{1.7}$$

Truncation of Maclaurin series between orders l, m inclusively is denoted by

$$\left[\sum_{j=0}^{\infty} c_j z^j\right]_l^m := \sum_{j=l}^{m} c_j z^j.$$

Definition. $(\boldsymbol{p}(z), q(z))$ are said to be generalised inverse, vector-valued Padé polynomials (GIPPS) of type $[n/2k]$ for $\sum_{j=0}^{\infty}\boldsymbol{c}_j z^j$ if $\boldsymbol{p}(z)$, $q(z)$ are polynomials satisfying

(i) $$\left.\begin{aligned}\partial\{\boldsymbol{p}\} &\leqslant n-\alpha\\ \partial\{q\} &= 2k-2\alpha\end{aligned}\right\}\quad \text{for some } \alpha \geqslant 0, \tag{1.8}$$

(ii) $$q(z) \mid \boldsymbol{p}(z)\cdot\boldsymbol{p}^*(z), \tag{1.9}$$

(iii) $$q(z)=q^*(z), \tag{1.10}$$

(iv) either $q(0)\neq 0$ or $q(z)$ has a zero of even order 2β (precisely) at the origin,

(v) $$\boldsymbol{p}(z)-q(z)\sum_{i=0}^{\infty}\boldsymbol{c}_i z^i = \mathrm{O}(z^{n+\beta+1}). \tag{1.11}$$

Graves-Morris and Jenkins [5] showed that (non-trivial) GIPPs $(\boldsymbol{p}(z), q(z))$ exist for the problem expressed by (1.8)–(1.11). A solution can be found as follows: a matrix M has its elements (row index $i=0, 1,\ldots,2k-1$; column index $j=0, 1,\ldots,2k$) given by

$$M_{ii} := 0, \tag{1.12a}$$

$$M_{ij} := \sum_{l=0}^{j-i-1} \boldsymbol{c}_{l+i+n-2k+1}\cdot\boldsymbol{c}^*_{j-l+n-2k}, \quad j>i, \tag{1.12b}$$

$$M_{ij} := -\sum_{l=0}^{i-j-1} \boldsymbol{c}_{l+j+n-2k+1}\cdot\boldsymbol{c}^*_{i-l+n-2k}, \quad i>j. \tag{1.12c}$$

Let

$$\boldsymbol{q} := (q_{2k}, q_{2k-1},\ldots, q_0). \tag{1.13}$$

Then any non-trivial solution of the equations

$$M\boldsymbol{q}=0 \tag{1.14}$$

defines a denominator polynomial

$$q(z)=q_0+q_1 z+\cdots+q_{2k}z^{2k}. \tag{1.15}$$

Its corresponding numerator polynomial $\boldsymbol{p}(z)$ can be found from (1.11). If $q_0\neq 0$, $(\boldsymbol{p}(z), q(z))$ are GIPPs of type $[n/2k]$ for $\boldsymbol{f}(z)$; if $q_0=0$, certain common factors of $\boldsymbol{p}(z)$, $q(z)$ may have to be removed to produce GIPPs. The details are given by Graves-Morris and Jenkins [5], and their conclusions are summarised in Theorem 2.3.

Definition. If $(\boldsymbol{p}(z), q(z))$ are GIPPs of type $[n/2k]$ for $\boldsymbol{f}(z)$, and if

$$q(0) \neq 0, \tag{1.16}$$

then

$$\boldsymbol{r}(z) := \boldsymbol{p}(z)/q(z) \tag{1.17a}$$

is called a generalised inverse, vector-valued Padé approximant of type $[n/2k]$ for $\boldsymbol{f}(z)$, or GIPA for short.

Definition. If $(\boldsymbol{p}(z), q(z))$ are GIPPs of type $[n/2k]$ for $\boldsymbol{f}(z)$, then

$$\boldsymbol{r}(z) := \boldsymbol{p}(z)/q(z) \tag{1.17b}$$

is called a generalised inverse Padé form of type $[n/2k]$ for $\boldsymbol{f}(z)$, or GIPF for short.

The GIPF defined by (1.17b) is unique [5], and a fortiori the GIPA defined by (1.17a) is unique too. From (1.12)–(1.15), we obtain the generalised inverse Padé denominator as

$$q(z) = \begin{vmatrix} 0 & M_{01} & M_{02} & \cdots & M_{0,2k-1} & M_{0,2k} \\ -M_{01} & 0 & M_{12} & \cdots & M_{1,2k-1} & M_{1,2k} \\ -M_{02} & -M_{12} & 0 & \cdots & M_{2,2k-1} & M_{2,2k} \\ \vdots & \vdots & \vdots & & \vdots & \\ -M_{0,2k-1} & -M_{1,2k-1} & -M_{2,2k-1} & \cdots & 0 & M_{2k-1,2k} \\ z^{2k} & z^{2k-1} & z^{2k-2} & \cdots & z & 1 \end{vmatrix} \tag{1.18}$$

provided that $q(0) \neq 0$. Again, the corresponding numerator polynomial follows from (1.11) and $\boldsymbol{r}(z) = \boldsymbol{p}(z)/q(z)$ is the GIPA for $\boldsymbol{f}(z)$. A specific instance of (1.18) for the [1/2] type GIPA is given in Example 3.1.

In Section 2 of this paper, we establish the connection between a Padé approximant of $f(z)$ and the corresponding GIPA of $(f(z), f^*(z))$. We derive the unitary invariance property connecting GIPAs of $(\boldsymbol{f}(z), \boldsymbol{f}^*(z))$ and (Re $\boldsymbol{f}(z)$, Im $\boldsymbol{f}(z)$). An amusing consequence of this analysis is that our methods allow the construction of an (ordinary) Padé approximant of $f(z)$ from a GIPA of (Re $f(z)$, Im $f(z)$) using real arithmetic only.

In Sections 3 and 4, we establish two theorems about row sequences of GIPAs. Under conditions on $\boldsymbol{f}(z)$ similar to those of De Montessus' theorem [9], we show that all GIPAs of sufficiently high order in the appropriate row exist and converge to $\boldsymbol{f}(z)$ in the familiar domain.

2. Further properties of generalised inverse Padé forms

We establish here the remarkable relationship between a Padé approximant of a function $f(z)$ and the corresponding GIPA of $(f(z), f^*(z))$. For a vector function $\boldsymbol{f}(z)$, we derive a unitary property of each GIPF of the extended vector function

$$\boldsymbol{f}^{\mathrm{E}}(z) = (\boldsymbol{f}(z), \boldsymbol{f}^*(z)). \tag{2.1}$$

Moreover, the row convergence theorem to be proved in Section 4 shows how natural it is to form GIPFs to $\boldsymbol{f}^{\mathrm{E}}(z)$. We also derive a result about degenerate approximants in Theorem 2.3 which we need for the proofs of the convergence theorems of Sections 3 and 4.

Let $\boldsymbol{f}: \mathbb{C} \to \mathbb{C}^d$ denote the vector-valued function to be approximated, and consider $\boldsymbol{f}^{\mathrm{E}}: \mathbb{C} \to \mathbb{C}^{2d}$ defined by (2.1). Let $(\boldsymbol{p}^{\mathrm{E}}(z), q(z))$ be GIPPs of type $[n/2k]$ for $\boldsymbol{f}^{\mathrm{E}}(z)$. We define $\boldsymbol{p}(z)$ componentwise by

$$p_i(z) := p_i^{\mathrm{E}}(z), \quad i = 1, 2, \ldots, d. \tag{2.2}$$

By definition, therefore,

$$\boldsymbol{f}(z)q(z) - \boldsymbol{p}(z) = \mathrm{O}(z^{n+1}). \tag{2.3}$$

From (2.3) and (2.3*),

$$\boldsymbol{p}^{\mathrm{E}}(z) = (\boldsymbol{p}(z), \boldsymbol{p}^*(z)). \tag{2.4}$$

Next, we prove that the process of formation of GIPPs is invariant under unitary transformations.

Theorem 2.1 (Unitary Invariance). *Let U be a $d \times d$ unitary matrix, let c be an arbitrary complex constant and let $\boldsymbol{f}(z): \mathbb{C} \to \mathbb{C}^d$. If $(\boldsymbol{p}(z), q(z))$ are GIPPs of type $[n/2k]$ for $\boldsymbol{f}(z)$, then $(cU\boldsymbol{p}(z), q(z))$ are GIPPs of type $[n/2k]$ for $cU\boldsymbol{f}(z)$.*

Proof. The accuracy-through-order property,

$$cU\boldsymbol{f}(z)q(z) - cU\boldsymbol{p}(z) = \mathrm{O}(z^{n+\beta+1})$$

follows from (1.11). The divisibility property holds because

$$(U\boldsymbol{p})^* \cdot (U\boldsymbol{p}) = \boldsymbol{p}^* \cdot \boldsymbol{p}. \quad \square$$

Corollary 2.1. *The unitary property holds also for generalised inverse rational interpolants defined on real interpolation points by the equations*

$$\boldsymbol{p}(x_i) = \boldsymbol{f}(x_i)q(x_i), \quad i = 0, 1, \ldots, n. \tag{2.5}$$

Proof. The proof is virtually the same, except that the modified interpolatory condition (2.5) replaces the accuracy-through-order condition (1.11).

Example 1. Let I denote the $d \times d$ identity matrix and let

$$U := \begin{pmatrix} I/\sqrt{2} & I/\sqrt{2} \\ -iI/\sqrt{2} & iI/\sqrt{2} \end{pmatrix} \tag{2.6}$$

define a $2d \times 2d$ unitary matrix. Then

$$(1/\sqrt{2})U\boldsymbol{f}^{\mathrm{E}} = (\mathrm{Re}\ \boldsymbol{f}, \mathrm{Im}\ \boldsymbol{f}), \tag{2.7}$$

where Re, Im denote the real and imaginary parts of the functional form, as in (1.7).

For the case of $d = 1$, we consider the particular function

$$\boldsymbol{f}(z) = f(z) = \gamma/(z - \zeta). \tag{2.8}$$

Let $\gamma = \alpha + i\beta$, with $\alpha, \beta \in \mathbb{R}$. From (2.8) we find that

$$\operatorname{Re} f(z) = \frac{g_1(z)}{(z-\zeta)(z-\zeta^*)}, \qquad \operatorname{Im} f(z) = \frac{g_2(z)}{(z-\zeta)(z-\zeta^*)} \tag{2.9}$$

where

$$g_1(z) := \alpha z - \operatorname{Re}(\zeta^*\gamma), \qquad g_2(z) = \beta z - \operatorname{Im}(\zeta^*\gamma). \tag{2.10}$$

We define $\boldsymbol{g}(z) := (g_1(z), g_2(z))$, so that

$$(\operatorname{Re} f(z), \operatorname{Im} f(z)) = \frac{\boldsymbol{g}(z)}{(z-\zeta)(z-\zeta^*)}. \tag{2.11}$$

From (2.10),

$$\boldsymbol{g}(z) \cdot \boldsymbol{g}^*(z) = |\gamma|^2 (z-\zeta)(z-\zeta^*) \tag{2.12}$$

and we observe that $\boldsymbol{g}(z) \cdot \boldsymbol{g}^*(z)$ vanishes at the pole of f. Had we first considered

$$\boldsymbol{f}^{\mathrm{E}}(z) = (f(z), f^*(z)) = \frac{(h_1(z), h_2(z))}{(z-\zeta)(z-\zeta^*)} \tag{2.13}$$

with $h_1(z) = \gamma(z-\zeta^*)$, $h_2(z) = \gamma^*(z-\zeta)$, $\boldsymbol{h}(z) := (h_1(z), h_2(z))$, we would have observed directly that

$$\boldsymbol{h}(z) \cdot \boldsymbol{h}^*(z) = 2|\gamma|^2 (z-\zeta)(z-\zeta^*).$$

This form makes the vanishing of $\boldsymbol{h} \cdot \boldsymbol{h}^*$ at the pole of f more obvious.

One conclusion to be drawn from the previous Theorem and demonstrated by Example 1 is that formation of GIPPs to (f, f^*) and to $(\operatorname{Re} f, \operatorname{Im} f)$ are unitarily equivalent processes.

Theorem 2.2 (Reduction to the Padé case). *Let $f(z)$ be analytic at the origin, so that $f(z) = \sum_{j=0}^{\infty} c_j z^j$, and let $\boldsymbol{f}^{\mathrm{E}}(z) := (f(z), f^*(z))$. Suppose that $f(z)$ has an $[l/m]$ type Padé approximant (in the classical or Frobenius sense) with $a(z)$, $b(z)$ as its numerator and denominator polynomials. Define*

$$n := l + m,$$
$$\boldsymbol{A}(z) := (a(z)b^*(z), a^*(z)b(z)), \qquad B(z) := b(z)b^*(z). \tag{2.14}$$

Then $(\boldsymbol{A}(z), B(z))$ are $[n/2m]$ type GIPPs for $\boldsymbol{f}^{\mathrm{E}}(z)$.

Proof. We assume that $a(z)$ and $b(z)$ have the properties

$$\partial\{a\} = \lambda \leqslant l, \qquad \partial\{b\} = \mu \leqslant m,$$
$$b(z)f(z) - a(z) = \mathrm{O}(z^{l+m+1}). \tag{2.15}$$

If $b(0) = 0$, let the multiplicity of the zero of $b(z)$ at the origin be β precisely. Otherwise, let $\beta := 0$. From (2.14), (2.15),

$$\boldsymbol{A}(z) - B(z)\boldsymbol{f}^{\mathrm{E}}(z) = \mathrm{O}(z^{n+\beta+1}). \tag{2.16}$$

The divisibility condition, that $B(z) \mid \boldsymbol{A}(z) \cdot \boldsymbol{A}^*(z)$ is easily verified. For the degree checks, the value $\alpha := m - \mu$ is non-negative and we get

$$\partial\{B\} = 2m - 2\alpha, \qquad \partial\{\boldsymbol{A}\} \leqslant n - \alpha$$

as required for GIPPs. □

Corollary 2.2. *Let* $f(z)=\Sigma_{j=0}^{\infty} c_j z^j$, *and suppose that* $((p_1(z),\ p_2(z)),\ q(z))$ *are GIPPs of type* $[n/2m]$ *for* $(\mathrm{Re}\, f(z),\ \mathrm{Im}\, f(z))$. *Then*

$$r(z) := (p_1(z) + \mathrm{i} p_2(z))/q(z) \tag{2.17}$$

is the (classical) Padé approximant of type $[n-m/m]$ *for* $f(z)$.

Proof. The rational fractional forms of a Padé approximant and of a GIPF are unique, and so the corollary follows from (2.7) and Theorem 2.2. □

Corollary 2.2 implies that $f(z)$, as expressed by the right-hand side of (2.17), is reducible from type $[n/2m]$ to type $[n-m/m]$. This property is exhibited by the following example.

Example 2.1. From the determinantal representation (1.18), we have

$$q^{[1/2]}(z) = \begin{vmatrix} 0 & |\boldsymbol{c}_0|^2 & \boldsymbol{c}_0^* \cdot \boldsymbol{c}_1 + \boldsymbol{c}_1^* \cdot \boldsymbol{c}_0 \\ -|\boldsymbol{c}_0|^2 & 0 & |\boldsymbol{c}_1|^2 \\ z^2 & z & 1 \end{vmatrix}$$

$$= |\boldsymbol{c}_0|^2 \{ |\boldsymbol{c}_0|^2 - (\boldsymbol{c}_0^* \cdot \boldsymbol{c}_1 + \boldsymbol{c}_1^* \cdot \boldsymbol{c}_0) z + |\boldsymbol{c}_1|^2 z^2 \}$$

and by cross-multiplication

$$\boldsymbol{p}^{[1/2]}(z) = |\boldsymbol{c}_0|^2 \{ \boldsymbol{c}_0 |\boldsymbol{c}_0|^2 + z [\boldsymbol{c}_1 |\boldsymbol{c}_0|^2 - \boldsymbol{c}_0 (\boldsymbol{c}_0^* \cdot \boldsymbol{c}_1 + \boldsymbol{c}_1^* \cdot \boldsymbol{c}_0)] \}$$

which form the [1/2] type GIPPs of $\Sigma_{j=0}^{\infty} \boldsymbol{c}_j z^j$. If we now consider the case when $\boldsymbol{c}_0,\ \boldsymbol{c}_1 \in \mathbb{R}^2$, and set

$$(\alpha_0, \beta_0) := \boldsymbol{c}_0, \quad (\alpha_1, \beta_1) := \boldsymbol{c}_1, \qquad d_0 := \alpha_0 + \mathrm{i}\beta_0, \quad d_1 := \alpha_1 + \mathrm{i}\beta_1$$

$$q(z) := q^{[1/2]}(z)/|\boldsymbol{c}_0|^2, \qquad p(z) := \{ p_1^{[1/2]}(z) + \mathrm{i} p_2^{[1/2]}(z) \}/|\boldsymbol{c}_0|^2,$$

then we readily find that

$$q(z) = (d_0 - z d_1)(d_0^* - z d_1^*), \qquad p(z) = d_0^2 (d_0^* - z d_1^*).$$

The polynomials $p(z)$ and $q(z)$ share a common factor of $(d_0^* - z d_1^*)$. We notice that $p(z)/q(z)$ is the [0/1] type Padé approximant of $\Sigma_{j=0}^{\infty} d_j z^j$, and so Corollary 2.2 is verified in this instance.

It is worth remarking that this method provides Padé approximants of power series with complex-valued coefficients using real arithmetic only.

Were Theorem 2.2 to have the stronger hypothesis that $f(z)$ have an $[l/m]$ type Padé approximant according to the Baker definition with $b(0)=1$, and the same constructive definitions (2.14), the conclusion would be that $A(z)/B(z)$ is an $[n/2m]$ type GIPA for $\boldsymbol{f}^{\mathrm{E}}(z)$. The converse of these remarks applies to Corollary 2.2.

The next theorem concerns degeneracies of GIPPs. It is used primarily as a lemma for Theorems 3.1 and 4.1. The proof of the theorem is given by Graves-Morris and Jenkins [5].

Theorem 2.3. *Let* $(\boldsymbol{p}(z),\ q(z))$ *be the polynomials constructed from a solution of the homogeneous equations* (1.14), *plus* (1.15) *and* (1.11). *Let* λ *be the least non-negative integer for which*

$$[z^{-\lambda} q(z)]_{z=0} \neq 0, \tag{2.18}$$

and let

$$\sigma := \left[\tfrac{1}{2}(\lambda+1)\right]. \tag{2.19}$$

Then GIPPs $(\hat{\boldsymbol{p}}(z), \hat{q}(z))$ *of type* $[n-\sigma/2k-2\sigma]$ *for* $\boldsymbol{f}(z)$ *exist such that* $\hat{q}(0) \neq 0$ *and*

$$\hat{\boldsymbol{p}}(z)/\hat{q}(z) = \boldsymbol{r}(z) := \boldsymbol{p}(z)/q(z).$$

Thus $\boldsymbol{r}(z)$ *is the unique GIPA of type* $[n-\sigma/2k-2\sigma]$ *for* $\boldsymbol{f}(z)$.

Moreover, $(z^{2\sigma}\hat{\boldsymbol{p}}(z), z^{2\sigma}\hat{q}(z))$ *are GIPPs of type* $[n/2k]$ *for* $\boldsymbol{f}(z)$ *satisfying the extended accuracy-through-order condition*

$$z^{2\sigma}\hat{\boldsymbol{p}}(z) - z^{2\sigma}\hat{q}(z)\boldsymbol{f}(z) = \mathrm{O}(z^{n+\sigma+1}). \tag{2.20}$$

Notice that the accuracy-through-order result in equation (2.20) is stronger than one might expect a priori.

3. First convergence theorem for rows

In this section, we consider approximation of a vector function $\boldsymbol{f}(z)$, which is analytic in a disk except for poles of total multiplicity k, using GIPAs of type $[n/2k]$. For our main theorem, we suppose that the given function may be expressed as

$$\boldsymbol{f}(z) = \boldsymbol{g}(z)/Q(z) \tag{3.1}$$

where

(i) Q is a monic *real* polynomial of degree k with roots $\{z_1, \ldots, z_k\}$ and $0 < |z_i| < \rho, \quad i = 1, 2, \ldots, k;$ (3.2)

(ii) $\boldsymbol{g}(z)$ is analytic in $|z| < \rho$; (3.3)

(iii) $\boldsymbol{g}(z_i) \cdot \boldsymbol{g}^*(z_i) \neq 0, \quad i = 1, 2, \ldots, k.$ (3.4)

If z_i is real, then (3.4) is equivalent to $\boldsymbol{g}(z_i) \neq \boldsymbol{0}$. We remark that, for k even, the convergence of GIPAs of type $[n/k]$ to $\boldsymbol{f}(z)$ should not be expected: suppose that $\{(\boldsymbol{P}^{[n/k]}(z), Q^{[n/k]}(z)), n = k/2, k/2+1, \ldots,\}$ is a sequence of GIPAs converging to $\boldsymbol{f}(z)$ in $|z| < \rho$ with

$$\lim_{n\to\infty} Q^{[n/k]}(z) = Q(z) \quad \text{and} \quad \lim_{n\to\infty} \boldsymbol{P}^{[n/k]}(z) = \boldsymbol{g}(z).$$

The property that

$$Q(z) \mid \boldsymbol{g}(z) \cdot \boldsymbol{g}^*(z)$$

would follow from the divisibility condition for GIPAs, and this would falsify (3.4).

To identify the domains of convergence needed in Theorems 3.1 and 4.1, we make the following definitions. For any positive r, let

$$D_r := \{z: |z| < r\}. \tag{3.5}$$

we define a disk with the poles deleted by

$$D_\rho^- := D_\rho - \bigcup_{i=1}^{k} \{z_i\}. \tag{3.6}$$

For any positive $\mu < \rho$, let K be any compact subset of $D_\rho^- \cap D_\mu$. Also, let E be any compact subset of $\mathbb{C}$. We establish convergence of the row sequence of GIPAs in the domain D_ρ^-.

Theorem 3.1. *Let $\boldsymbol{f}(z)$ be a vector function such that* (3.1)–(3.4) *are satisfied. Let $(\boldsymbol{P}_n(z), Q_n(z))$ be GIPPs of type $[n/2k]$ in which $Q_n(z)$ has leading coefficient equal to unity. Then*

$$\lim_{n\to\infty} \boldsymbol{P}_n(z)/Q_n(z) = \boldsymbol{f}(z), \quad z \in D_\rho^-, \tag{3.7}$$

and the rate of convergence is governed by

$$\lim_{n\to\infty} \sup \| \boldsymbol{f} - \boldsymbol{P}_n/Q_n \|_K^{1/n} \leqslant \mu/\rho. \tag{3.8}$$

The denominators converge according to

$$\lim_{n\to\infty} Q_n(z) = Q^2(z) \tag{3.9}$$

and the rate is given by

$$\| Q_n - Q^2 \|_E^{1/n} \leqslant \max_{1\leqslant i\leqslant k} \frac{|z_i|}{\rho}. \tag{3.10}$$

Proof. We use the non-trivial polynomials $\boldsymbol{p}_n(z)$ and $q_n(z)$ which constitute the $[n/2k]$ type GIPPs to $\boldsymbol{f}(z)$. Following Theorem 2.3, equations (2.18)–(2.20), we assign non-negative integers σ_n, λ_n to the GIPPs and obtain the properties

$$\boldsymbol{p}_n(z) - \boldsymbol{f}(z) q_n(z) = \mathrm{O}(z^{n+\sigma_n+1}) \tag{3.11}$$

and

$$\boldsymbol{p}_n(z) \cdot \boldsymbol{p}_n^*(z) = \pi_{2n-2k}(z) q_n(z) \tag{3.12}$$

for some polynomial $\pi_{2n-2k}(z)$ of degree $2n-2k$ at most.

From (3.11) and (3.12),

$$q_n(z)\{\pi_{2n-2k}(z) - \boldsymbol{p}_n(z)\cdot\boldsymbol{f}^*(z) - \boldsymbol{p}_n^*(z)\cdot\boldsymbol{f}(z) + q_n(z)\boldsymbol{f}(z)\cdot\boldsymbol{f}^*(z)\}$$
$$= \mathrm{O}(z^{2(n+\lambda_n-\sigma_n+1)}). \tag{3.13}$$

Multiplying (3.13) by $Q^2(z)/q_n(z)$ and using (2.19), we get

$$Q^2(z)\pi_{2n-2k}(z) - Q(z)\boldsymbol{p}_n(z)\cdot\boldsymbol{g}^*(z) - Q(z)\boldsymbol{p}_n^*(z)\cdot\boldsymbol{g}(z) + q_n(z)\boldsymbol{g}(z)\cdot\boldsymbol{g}^*(z)$$
$$= \mathrm{O}(z^{2n+1}). \tag{3.14}$$

We use Hermite's formula to give precise form to the right-hand side of (3.14). For any $\rho' < \rho$, we define

$$A_n(z) := -\frac{z^{2n+1}}{2\pi \mathrm{i}} \int_{|t|=\rho'} \{\boldsymbol{p}_n(t)\cdot\boldsymbol{g}^*(t) + \boldsymbol{p}_n^*(t)\cdot\boldsymbol{g}(t)\} \frac{Q(t)\,\mathrm{d}t}{(t-z)t^{2n+1}}$$
$$= -\frac{z^{2n+1}}{2\pi \mathrm{i}} \int_{|t|=\rho'} \{[\boldsymbol{f}(t)q_n(t)]_0^n\cdot\boldsymbol{g}^*(t) + [\boldsymbol{f}^*(t)q_n(t)]_0^n\cdot\boldsymbol{g}(t)\} \frac{Q(t)\,\mathrm{d}t}{(t-z)t^{2n+1}} \tag{3.15}$$

and

$$C_n(z) := \frac{z^{2n+1}}{2\pi \mathrm{i}} \int_{|t|=\rho'} q_n(t)\boldsymbol{g}(t)\cdot\boldsymbol{g}^*(t) \frac{\mathrm{d}t}{(t-z)t^{2n+1}}. \tag{3.16}$$

Then, for $z \in D_{\rho'}$, (3.14) becomes

$$Q^2(z)\pi_{2n-2k}(z) - Q(z)\boldsymbol{p}_n(z)\cdot\boldsymbol{g}^*(z) - Q(z)\boldsymbol{p}_n^*(z)\cdot\boldsymbol{g}(z) + q_n(z)\boldsymbol{g}(z)\cdot\boldsymbol{g}^*(z) = A_n(z) + C_n(z). \tag{3.17}$$

From (3.16), we estimate the sth derivative of $C_n(z)$ by

$$C_n^{(s)}(z) = \theta(|z|/\rho)^{2n}, \quad s = 0, 1, 2, \ldots . \tag{3.18}$$

(We have prematurely taken $\{q_n(t)\}$ to be uniformly bounded in $|t| < \rho$ in equation (3.16), but see (3.22).) To obtain a similar result for $A_n(z)$, we need to interpolate $q_n(z)$ at the zeros of $Q(z)$. We express

$$Q(z) = \prod_{j=1}^{\nu} (z - \zeta_j)^{m_j} \tag{3.19}$$

where the $\{\zeta_j,\ j = 1, 2, \ldots, \nu\}$ are distinct and satisfy

$$|\zeta_1| \leqslant |\zeta_2| \leqslant \cdots \leqslant |\zeta_\nu| < \rho. \tag{3.20}$$

We have $\sum_{j=1}^{\nu} m_j = k$ for compatibility with (3.2). We use a Hermite–Lagrange basis,

$$B := \{B_{j,s}(z),\ j = 1, 2, \ldots, \nu;\ s = 0, 1, \ldots, 2m_j - 1\}$$

of polynomials for $\mathbb{P}_{2k-1}$ with the properties that

$$\left[(\mathrm{d}/\mathrm{d}z)^i B_{j,s}(z)\right]_{z=\zeta_l} = \delta_{jl}\delta_{is}, \qquad 1 \leqslant l \leqslant \nu, \quad 0 \leqslant i \leqslant 2m_l - 1.$$

Then we can express $q_n(z)$ in interpolated form by

$$q_n(z) = \sum_{j=1}^{\nu} \sum_{s=0}^{2m_j-1} q_n^{(s)}(\zeta_j) B_{j,s}(z) + c_n Q^2(z). \tag{3.21}$$

We assume a normalization for $q_n(z)$ in which $c_n \geqslant 0$ and

$$c_n + \sum_{j=1}^{\nu} \sum_{s=0}^{2m_j-1} \left|q_n^{(s)}(\zeta_j)\right| = 1. \tag{3.22}$$

An important consequence of this normalisation is that $\{q_n(z)\}$ are uniformly bounded in $|z| < \rho$.

We claim that

$$q_n^{(s)}(\zeta_j) = \theta(|\zeta_j|/\rho)^n, \qquad j = 1, 2, \ldots, \nu, \quad s = 0, 1, \ldots, 2m_j - 1. \tag{3.23}$$

First, we consider ζ_1, a zero of $Q(z)$ closest to the origin, and our immediate aim is to estimate $q_n(\zeta_1)$ using (3.17) evaluated at ζ_1. Since $\boldsymbol{f}(z)$ is analytic in $|z| < |\zeta_1|$,

$$[\boldsymbol{f}(t)q_n(t)]_0^n = \theta(t/\zeta_1)^n \quad \text{for } |t| \geqslant |\zeta_1|. \tag{3.24}$$

Suppose that $0 \leqslant s \leqslant m_1 - 1$. Taking the sth order derivative of (3.17), and setting $z = \zeta_1$, we obtain

$$\left[(\mathrm{d}/\mathrm{d}z)^s\{q_n(z)\boldsymbol{g}(z)\cdot\boldsymbol{g}^*(z)\}\right]_{z=\zeta_1} = A_n^{(s)}(\zeta_1) + C_n^{(s)}(\zeta_1). \tag{3.25}$$

From (3.24) and (3.15), we have

$$A_n^{(s)}(\zeta_1) = \theta(\zeta_1/\rho)^n, \quad s = 0, 1, \ldots . \tag{3.26}$$

Combining this with (3.18), we have

$$\left[(\mathrm{d}/\mathrm{d}z)^s \{q_n(z)\boldsymbol{g}(z)\cdot\boldsymbol{g}^*(z)\}\right]_{z=\zeta_1} = \theta(\zeta_1/\rho)^n. \tag{3.27}$$

Since $\boldsymbol{g}(\zeta_1)\cdot\boldsymbol{g}^*(\zeta_1) \neq 0$, a simple induction on s shows that

$$q_n^{(s)}(\zeta_1) = \theta(\zeta_1/\rho)^n, \quad 0 \leqslant s \leqslant m_1 - 1. \tag{3.28}$$

Now suppose that

$$|\zeta_1| = |\zeta_2| = \cdots = |\zeta_\tau| < |\zeta_{\tau+1}|.$$

We have just shown in (3.28) that

$$q_n^{(s)}(\zeta_j) = \theta(\zeta_j/\rho)^n, \qquad j = 1, 2, \ldots, \tau, \quad 0 \leqslant s \leqslant m_j - 1. \tag{3.29}$$

Next we prove that

$$q_n^{(s)}(\zeta_{\tau+1}) = \theta(\zeta_{t+1}/\rho)^n, \quad 0 \leqslant s \leqslant m_\tau - 1. \tag{3.30}$$

To estimate $A_n^{(s)}(\zeta_{\tau+1})$, observe that we can rewrite (3.21) as

$$q_n(t) = \left\{ \sum_{j=1}^{\tau} \sum_{s=0}^{m_j-1} + \sum_{j=1}^{\tau} \sum_{s=m_j}^{2m_j-1} + \sum_{j=\tau+1}^{\nu} \sum_{s=0}^{2m_j-1} \right\} q_n^{(s)}(\zeta_j) B_{j,s}(t) + c_n Q^2(t). \tag{3.31}$$

Because $B_{j,s}(t)\,\boldsymbol{f}(t)$ is analytic in $|t| < |\zeta_j|$,

$$q_n^{(s)}(\zeta_j)\left[B_{j,s}(t)\,\boldsymbol{f}(t)\right]_0^n = \theta(t/\rho)^n, \quad |t| \geqslant |\zeta_1|$$

for $1 \leqslant j \leqslant \tau$, $0 \leqslant s \leqslant m_j - 1$, and so the contribution of these terms (in the splitting of (3.31)) to $A_n(\zeta_{\tau+1})$ as given by (3.15) is $\theta(\zeta_{\tau+1}/\rho)^{2n}$. For the remaining values of j, s, the function $B_{j,s}(t)\,\boldsymbol{f}(t)$ is analytic in $|z| < |\zeta_{\tau+1}|$. Thus, recalling (3.22),

$$q_n^{(s)}(\zeta_j)\left[B_{j,s}(t)\,\boldsymbol{f}(t)\right]_0^n = \theta(t/\zeta_{\tau+1})^n, \quad |t| \geqslant |\zeta_{\tau+1}|$$

and

$$c_n\left[Q^2(t)\,\boldsymbol{f}(t)\right]_0^n = \theta(1)^n \quad \text{for } |t| < \rho.$$

Therefore

$$A_n^{(s)}(\zeta_{\tau+1}) = \theta(\zeta_{\tau+1}/\rho)^n, \quad 0 \leqslant s \leqslant m_{\tau+1} - 1$$

and so from (3.17),

$$(\mathrm{d}/\mathrm{d}z)^s\{q_n(z)\boldsymbol{g}^*(z)\}\big|_{z=\zeta_{\tau+1}} = \theta(\zeta_{\tau+1}/\rho)^n.$$

from which (3.30) follows.

Continuing in a step-by-step fashion, we deduce that

$$q_n^{(s)}(\zeta_j) = \theta(\zeta_j/\rho)^n, \qquad 1 \leqslant j \leqslant \nu, \quad 0 \leqslant s \leqslant m_j - 1 \tag{3.32}$$

and

$$A_n^{(s)}(\zeta_j) = \theta(\zeta_j/\rho)^n, \quad s = 0, 1, \ldots .$$

We now consider the case of the sth order derivatives of (3.17) for $m_j \leqslant s \leqslant 2m_j - 1$. Initially, we put $z = \zeta_1$ in the sth derivative of (3.17), and obtain

$$\left[(\mathrm{d}/\mathrm{d}z)^s\{q_n(z)\boldsymbol{g}(z)\cdot\boldsymbol{g}^*(z)\}\right]_{z=\zeta_1} = A_n^{(s)}(\zeta_1) + C_n^{(s)}(\zeta_1) + D_n^{(s)}(\zeta_1), \quad m_1 \leqslant s \leqslant 2m_1 - 1 \tag{3.33}$$

with

$$D_n(z) := Q(z)\{\boldsymbol{g}(z)\cdot[q_n(z)\boldsymbol{f}^*(z)]_0^n + \boldsymbol{g}^*(z)\cdot[q_n(z)\boldsymbol{f}(z)]_0^n\}. \tag{3.34}$$

Consequently,

$$D_n^{(m_1)}(\zeta_1) = Q^{(m_1)}(\zeta_1)\{\boldsymbol{g}(z)\cdot[q_n(z)\boldsymbol{f}^*(z)]_0^n + \boldsymbol{g}^*(z)\cdot[q_n(z)\boldsymbol{f}(z)]_0^n\}\big|_{z=\zeta_1}. \tag{3.35}$$

From (3.32), and since $B_{j,s}(t)\boldsymbol{f}(t)$ is analytic in $|z| < |\zeta_j|$,

$$q_n^{(s)}(\zeta_j)\left[B_{j,s}(t)\boldsymbol{f}(t)\right]_0^n = \theta(t/\rho)^n, \quad |t| \geqslant |\zeta_j|, \tag{3.36}$$

for $1 \leqslant j \leqslant \nu$, $0 \leqslant s \leqslant m_j - 1$. For certain other values of these indices, namely $j = 1$, $m_1 + 1 \leqslant s \leqslant 2m_1 - 1$ and $2 \leqslant j \leqslant \nu$, $m_j \leqslant s \leqslant 2m_j - 1$,

$$(z - \zeta_1)Q(z) \mid B_{j,s}(z).$$

Therefore, for these values of j, s, the function $B_{j,s}\boldsymbol{f}$ is analytic in $|z| < \rho$,

$$\left[B_{j,s}(z)\boldsymbol{f}(z)\right]_{z=\zeta_1} = 0, \tag{3.37}$$

and incidentally

$$\left[Q^2(z)\boldsymbol{f}(z)\right]_{z=\zeta_1} = 0. \tag{3.38}$$

Referring back to (3.34), we need the interpolated form of $q_n(z)$ with the $B_{1,m_1}(z)$ term displayed explicitly. From (3.21), we obtain

$$\begin{aligned}[q_n\boldsymbol{f}]_0^n = {} & q_n^{(m_1)}(\zeta_1)\left[B_{1,m_1}\boldsymbol{f}\right]_0^n + \sum_{j=1}^{\nu}\sum_{s=0}^{m_j-1} q_n^{(s)}(\zeta_j)\left[B_{j,s}\boldsymbol{f}\right]_0^n \\ & + \sum_{s=m_1+1}^{2m_1-1} q_n^{(s)}(\zeta_1)[B_{1,s}\boldsymbol{f}]_0^n + \sum_{j=2}^{\nu}\sum_{s=m_j}^{2m_j-1} q_n^{(s)}(\zeta_j)\left[B_{j,s}\boldsymbol{f}\right]_0^n + c_n\left[Q^2\boldsymbol{f}\right]_0^n,\end{aligned} \tag{3.39}$$

where the variable z has been suppressed for brevity. For $z = \zeta_1$, equations (3.36), (3.37) and (3.38) provide estimates for all the terms in the right-hand side of (3.39) except the first. Therefore

$$[q_n\boldsymbol{f}]_0^n(\zeta_1) = q_n^{(m_1)}(\zeta_1)\left[B_{1,m_1}\boldsymbol{f}\right]_0^n(\zeta_1) + \theta(\zeta_1/\rho)^n. \tag{3.40}$$

Similarly, we find that

$$[q_n\boldsymbol{f}^*]_0^n(\zeta_1) = q_n^{(m_1)}(\zeta_1)\left[B_{1,m_1}\boldsymbol{f}^*\right]_0^n(\zeta_1) + \theta(\zeta_1/\rho)^n. \tag{3.41}$$

Because $B_{1,m_1}(z)\boldsymbol{f}(z)$ is analytic at ζ_1,

$$\left[B_{1,m_1}\boldsymbol{f}\right]_0^n(\zeta_1) \to \left[B_{1,m_1}\boldsymbol{f}\right](\zeta_1) \quad \text{as } n \to \infty.$$

From (3.1),

$$\left[B_{1,m_1}\boldsymbol{f}\right](\zeta_1) = \left[B_{1,m_1}/Q\right](\zeta_1)\boldsymbol{g}(\zeta_1) = \boldsymbol{g}(\zeta_1)/Q^{(m_1)}(\zeta_1).$$

Hence we find that

$$\left[B_{1,m_1} \boldsymbol{f}\right]_0^n(\zeta_1) \to \boldsymbol{g}(\zeta_1)/Q^{(m_1)}(\zeta_1) \tag{3.42}$$

and similarly that

$$\left[B_{1,m_1} \boldsymbol{f}^*\right]_0^n(\zeta_1) \to \boldsymbol{g}^*(\zeta_1)/Q^{(m_1)}(\zeta_1). \tag{3.43}$$

Substituting (3.42), (3.43) into (3.40), (3.40) into (3.35), and then (3.35) into (3.33) with $s = m_1$ yields with (3.32)

$$-\boldsymbol{g}(\zeta_1) \cdot \boldsymbol{g}^*(\zeta_1) q_n^{(m_1)}(\zeta_1) = \theta(\zeta_1/\rho)^n,$$

and hence

$$q_n^{(m_1)}(\zeta_1) = \theta(\zeta_1/\rho)^n.$$

The remainder of the proof that

$$q_n^{(s)}(\zeta_j) = \theta(\zeta_j/\rho)^n, \qquad 1 \leqslant j \leqslant \nu, \quad 0 \leqslant s \leqslant 2m_j - 1 \tag{3.44}$$

follows in a step-wise fashion similar to that preceding (3.32). From (3.21) it now follows that $c_n \to 1$ as $n \to \infty$ and that we may define

$$Q_n(z) = q_n(z)/c_n \tag{3.45}$$

for n sufficiently large. The results (3.9) and (3.10) now follow from (3.44) and (3.21). We again exploit the remainder formula to prove convergence of the approximants. For sufficiently small $\epsilon > 0$, and $z \in K$, we have

$$\boldsymbol{f}(z)Q_n(z) - \boldsymbol{P}_n(z) = \frac{z^{n+1}}{2\pi i}\left\{\int_{|t|=\mu} \frac{\boldsymbol{f}(t)Q_n(t)\,\mathrm{d}t}{t^{n+1}(t-z)} - \sum_{j=1}^{\nu} \int_{|t-\zeta_j|=\epsilon} \frac{\boldsymbol{f}(t)Q_n(t)\,\mathrm{d}t}{t^{n+1}(t-z)}\right\}. \tag{3.46}$$

Using (3.21), (3.44) and (3.45), we estimate the right-hand side of (3.46) and get

$$\boldsymbol{f}(z)Q_n(z) - \boldsymbol{P}_n(z) = \theta(|z|/\rho)^n, \quad z \in K.$$

Using (3.9), it follows that the approximants converge at the rate governed by (3.8).

4. Second convergence theorem for rows

As an alternative way to approximate a vector function $\boldsymbol{f}(z)$ which is analytic in a disk except for k simple poles, we consider in this section formation of GIPAs of type $[n/2k]$ to $(\boldsymbol{f}(z), \boldsymbol{f}^*(z))$. In Section 2 we explained that this procedure is unitarily equivalent to formation of GIPAs to (Re $\boldsymbol{f}(z)$, Im $\boldsymbol{f}(z)$), which involves real-valued coefficients only. Our main result is another row convergence theorem for a vector-valued function having the following properties.

Let

$$\boldsymbol{f}(z) := \boldsymbol{a}(z)/b(z) \tag{4.1}$$

where

$$b(z) := \prod_{i=1}^{k} (z - \zeta_i), \tag{4.2}$$

$S := \{\zeta_1, \zeta_1^*, \zeta_2, \zeta_2^*, \ldots, \zeta_k, \zeta_k^*\}$ is a set of $2k$ distinct points with $0 < |\zeta_j| < \rho$ for all j, and let $\boldsymbol{a}(z)$ be a vector function analytic in $|z| < \rho$, for which

$$\boldsymbol{a}(\zeta_i) \neq \mathbf{0}, \quad i = 1, 2, \ldots, k. \tag{4.3}$$

Let

$$\boldsymbol{f}^{\mathrm{E}}(z) := (\boldsymbol{f}(z), \boldsymbol{f}^*(z)) = (\boldsymbol{a}(z)/b(z), \boldsymbol{a}^*(z)/b^*(z)) \tag{4.4}$$

and so $\boldsymbol{f}^{\mathrm{E}}(z)$ has precisely $2k$ distinct poles in $|z| < \rho$. We are led to define

$$\boldsymbol{g}(z) := \boldsymbol{a}(z) b^*(z), \tag{4.5}$$

$$Q(z) := b(z) b^*(z). \tag{4.6}$$

so that

$$\boldsymbol{f}^{\mathrm{E}}(z) = (\boldsymbol{g}(z), \boldsymbol{g}^*(z))/Q(z). \tag{4.7}$$

Because the elements of S are distinct,

$$\boldsymbol{g}(\zeta_i) \neq \mathbf{0}, \quad i = 1, 2, \ldots, k, \tag{4.8}$$

but

$$\boldsymbol{g}(\zeta_i) \cdot \boldsymbol{g}^*(\zeta_i) = \boldsymbol{g}(\zeta_i^*) \cdot \boldsymbol{g}^*(\zeta_i^*) = 0, \quad i = 1, 2, \ldots, k. \tag{4.9}$$

Equation (4.9) shows that $\boldsymbol{g}(z) \cdot \boldsymbol{g}^*(z)$ and $Q(z)$ have S as a set of common zeros, and so (4.7) suggests that convergence of the row sequence of type $[n/2k]$ to $\boldsymbol{f}^E(z)$ should be a natural occurrence. This contrasts with the hypothesis (3.4) for convergence of the row sequence of type $[n/4k]$ in Section 3. Our main result involves the definitions (3.5) for D_r, $D_\rho^- := D_\rho - \bigcup\{\zeta_i, \zeta_i^*\}$ and the associated definitions of K and E.

Theorem 4.1. *Let* $(\boldsymbol{P}_n^{\mathrm{E}}(z), Q_n(z))$ *be a GIPF of type* $[n/2k]$ *to* $\boldsymbol{f}^{\mathrm{E}}(z) := (\boldsymbol{f}(z), \boldsymbol{f}^*(z))$ *as specified by* (4.1)–(4.3), *and let* $\boldsymbol{P}_n^{\mathrm{E}}(z) = (\boldsymbol{P}_n(z), \boldsymbol{P}_n^*(z))$ *as required by* (2.4). *Then, as* $n \to \infty$,

$$\boldsymbol{P}_n(z)/Q_n(z) \to \boldsymbol{f}(z), \quad z \in D_\rho^- \tag{4.10}$$

and the rate of convergence of these approximants is governed by

$$\limsup_{n\to\infty} \| \boldsymbol{f} - \boldsymbol{P}_n/Q_n \|_K^{1/n} \leqslant \mu/\rho. \tag{4.11}$$

Additionally, if $Q_n(z)$ *is normalised to have leading coefficient unity,*

$$Q_n(z) \to Q(z), \quad z \in \mathbb{C}, \tag{4.12}$$

and the rate of convergence is governed by

$$\limsup_{n\to\infty} \| Q_n - Q \|_E^{1/n} \leqslant \max_{1 \leqslant i \leqslant k} |\zeta_i|/\rho. \tag{4.13}$$

Proof. We use the non-trivial polynomials $q_n(z)$ and

$$\boldsymbol{p}_n^{\mathrm{E}}(z) = (\boldsymbol{p}_n(z), \boldsymbol{p}_n^*(z))$$

which constitute $[n/2k]$ GIPPs to $\boldsymbol{f}^{\mathrm{E}}(z)$. Following Theorem 2.3, equations (2.18), (2.19), we assign non-negative integers σ_n, λ_n to the GIPPs and obtain the properties

$$\boldsymbol{p}_n(z) - \boldsymbol{f}(z) q_n(z) = \mathrm{O}(z^{n+\sigma_n+1}) \tag{4.14}$$

and

$$\boldsymbol{p}_n(z)\cdot\boldsymbol{p}_n^*(z)=\pi_{2n-2k}(z)q_n(z) \tag{4.15}$$

for some polynomial $\pi_{2n-2k}(z)$ of degree $2n-2k$ at most. From (4.14) and (4.15), we obtain

$$q_n(z)\{\pi_{2n-2k}(z)-\boldsymbol{p}_n(z)\cdot\boldsymbol{f}^*(z)-\boldsymbol{p}_n^*(z)\cdot\boldsymbol{f}(z)+q_n(z)\boldsymbol{f}(z)\cdot\boldsymbol{f}^*(z)\}$$
$$=\mathrm{O}(z^{2(n+\sigma_n+1)}). \tag{4.16}$$

By multiplying (4.16) by $Q(z)/q_n(z)$ and using Theorem 2.3, we find

$$Q(z)\pi_{2n-2k}(z)-\boldsymbol{p}_n(z)\cdot\boldsymbol{g}^*(z)-\boldsymbol{p}_n^*(z)\cdot\boldsymbol{g}(z)+\boldsymbol{g}(z)\cdot\boldsymbol{g}^*(z)q_n(z)Q(z)^{-1}$$
$$=\mathrm{O}(z^{2n+1}). \tag{4.17}$$

We can now use Hermite's formula to give precise form to the right-hand side of (4.17). We find that

$$Q(z)\pi_{2n-2k}(z)-\boldsymbol{p}_n(z)\cdot\boldsymbol{g}^*(z)-\boldsymbol{p}_n^*(z)\cdot\boldsymbol{g}(z)+\boldsymbol{g}(z)\cdot\boldsymbol{g}^*(z)q_n(z)Q(z)^{-1}$$
$$=A_n(z)+C_n(z), \tag{4.18}$$

where, for any $\rho'<\rho$ and $z\in D_{\rho'}$,

$$A_n(z):=-\frac{z^{2n+1}}{2\pi\mathrm{i}}\int_{|t|=\rho'}\{\boldsymbol{p}_n(t)\cdot\boldsymbol{g}^*(t)+\boldsymbol{p}_n^*(t)\cdot\boldsymbol{g}(t)\}\frac{\mathrm{d}t}{(t-z)t^{2n+1}} \tag{4.19}$$

and

$$C_n(z):=\frac{z^{2n+1}}{2\pi\mathrm{i}}\int_{|t|=\rho'}\boldsymbol{a}(t)\cdot\boldsymbol{a}^*(t)q_n(t)\frac{\mathrm{d}t}{(t-z)t^{2n+1}}. \tag{4.20}$$

From this representation of $C_n(z)$, we find that

$$C_n(z)=\theta(z/\rho)^{2n},\quad |z|<\rho. \tag{4.21}$$

To obtain a similar result for $A_n(z)$, we interpolate $q_n(z)$ in $\{\zeta_l,\ \zeta_l^*,\ l=1,2,\ldots,k\}$ by

$$q_n(z)=c_nQ(z)+\sum_{l=1}^{k}q_n(\zeta_l)B_l(z)+\sum_{l=1}^{k}q_n(\zeta_l^*)B_l^*(z), \tag{4.22}$$

where $\mathscr{L}:=\{B_l(z),\ B_l^*(z),\ l=1,2,\ldots,k\}$ forms a Lagrangian basis in which

$$\begin{aligned}B_l(\zeta_s)=B_l^*(\zeta_s^*)&=\delta_{ls},\\ B_l^*(\zeta_s)=B_l(\zeta_s^*)&=0,\end{aligned}\qquad l,\ s=1,2,\ldots,k.$$

We choose the normalisation for $q_n(z)$ in which $c_n\geqslant 0$ and

$$c_n+2\sum_{l=1}^{k}|q_n(\zeta_l)|=1, \tag{4.23}$$

so that $\{q_n(z)\}$ is uniformly bounded on any compact set for large n.

Using (4.22), we obtain

$$\boldsymbol{p}_n^*(t)=[q_n(t)\boldsymbol{f}^*(t)]_0^n$$
$$=c_n[Q(t)\boldsymbol{f}^*(t)]_0^n+\sum_{l=1}^{k}q_n(\zeta_l)[B_l(t)\boldsymbol{f}^*(t)]_0^n+\sum_{l=1}^{k}q_n(\zeta_l^*)[B_l^*(t)\boldsymbol{f}^*(t)]_0^n. \tag{4.24}$$

Substituting (4.24) into (4.19), we find for $|z|<\rho$

$$A_n(z)=\sum_{l=1}^{k} q_n(\zeta_l)\eta_l^{A}(z)+\sum_{l=1}^{k} q_n(\zeta_l^*)\eta_l^{A^*}(z)+\theta\left(\frac{z}{\rho}\right)^{2n}, \tag{4.25}$$

where

$$\eta_l^{A}(z):=-\frac{z^{2n+1}}{2\pi i}\int_{|t|=\rho'}\{\boldsymbol{g}(t)\cdot[B_l(t)\boldsymbol{f}^*(t)]_0^n+\boldsymbol{g}^*(t)[B_l(t)\boldsymbol{f}(t)]_0^n\}\frac{dt}{(t-z)t^{2n+1}}. \tag{4.26}$$

From (4.26), we can estimate

$$\eta_l^{A}(z)=\theta(z^2/\rho\zeta_l)^n. \tag{4.27}$$

Unless $q_n(\zeta_l^*)=0$, and this trivial case is easily included, (4.25) becomes

$$A_n(z)=\sum_{l=1}^{k} q_n(\zeta_l^*)\epsilon_l^{A}(z)+\theta\left(\frac{z}{\rho}\right)^{2n} \tag{4.28}$$

where

$$\epsilon_l^{A}(z):=\eta_l^{A^*}(z)+\eta_l^{A}(z)q_n(\zeta_l)/q_n(\zeta_l^*). \tag{4.29}$$

Again, we estimate

$$\epsilon_l^{A}(z)=\theta\left(\frac{z^2}{\rho\zeta_l}\right)^n,\quad l=1,2,\ldots,k. \tag{4.30}$$

Equations (4.21), (4.28) and (4.30) provide the bounds required for $A_n(z)$ and $C_n(z)$ in (4.18).
Next, we evaluate the left-hand side of (4.18) at ζ_j:

$$\begin{aligned}&\{Q\pi_{2n-2k}-\boldsymbol{p}_n\cdot\boldsymbol{g}^*-\boldsymbol{p}_n^*\cdot\boldsymbol{g}+\boldsymbol{a}\cdot\boldsymbol{a}^*q_n\}(\zeta_j)\\&\quad=\boldsymbol{g}(\zeta_j)\cdot\{\boldsymbol{f}^*q_n-\boldsymbol{p}_n^*\}(\zeta_j),\quad j=1,2,\ldots,k.\end{aligned} \tag{4.31}$$

In (4.31), we use the remainder formula again,

$$\{\boldsymbol{f}^*q_n-\boldsymbol{p}_n^*\}(\zeta_j)=\frac{\zeta_j^{n+1}}{2\pi i}\int_{\Gamma_j}\boldsymbol{f}^*(t)q_n(t)\frac{dt}{t^{n+1}(t-\zeta_j)} \tag{4.32}$$

where Γ_j is a simple contour enclosing 0, ζ_j but no singularities of $\boldsymbol{f}^*(z)$. Let $\boldsymbol{f}^*(z)$ have residue $\boldsymbol{\gamma}_l^*$ at its pole at ζ_l^*. From (4.7), we find that

$$\boldsymbol{\gamma}_j=\boldsymbol{g}(\zeta_j)/Q'(\zeta_j),\quad j=1,2,\ldots,k. \tag{4.33}$$

The contour Γ_j is 'expanded' to become the contour $|t|=\rho'$, with $\max_l|\zeta_l|<\rho'<\rho$, giving

$$\boldsymbol{g}(\zeta_j)\cdot\{\boldsymbol{f}^*q_n-\boldsymbol{p}_n^*\}(\zeta_j)=\epsilon_j^{D}-\zeta_j^{n+1}\boldsymbol{g}(\zeta_j)\cdot\sum_{l=1}^{k}\frac{\boldsymbol{\gamma}_l^*q_n(\zeta_l^*)}{\zeta_l^{*(n+1)}(\zeta_l^*-\zeta_j)} \tag{4.34}$$

where

$$\epsilon_j^{D}:=\frac{\zeta_j^{n+1}\boldsymbol{g}(\zeta_j)}{2\pi i}\int_{|t|=\rho'}\boldsymbol{f}^*(t)q_n(t)\frac{dt}{t^{n+1}(t-\zeta_j)}. \tag{4.35}$$

We have an easy error bound for (4.35):

$$\epsilon_j^{\mathrm{D}} := \theta\left(\frac{\zeta_j}{\rho}\right)^n. \tag{4.36}$$

Substitute (4.34) into (4.18) evaluated at ζ_j to find

$$\zeta_j^{n+1}\mathbf{g}(\zeta_j)\cdot\sum_{l=1}^{k}\frac{\gamma_l^* q_n(\zeta_l^*)}{\zeta_l^{*(n+1)}(\zeta_l^*-\zeta_j)}+\sum_{l=1}^{k} q_n(\zeta_l^*)\epsilon_l^{\mathrm{A}}(\zeta_j^*)=\theta\left(\frac{\zeta_j}{\rho}\right)^n \tag{4.37}$$

for $j=1, 2,\ldots, k$. Let

$$\tilde{q}_n(z) := q_n(z)z^{-n-1}, \tag{4.38}$$

and then (4.37) becomes

$$\sum_{l=1}^{k}\left[\frac{\mathbf{g}(\zeta_j)\cdot\gamma_l^*}{\zeta_l^*-\zeta_j}+\left(\frac{\zeta_l^*}{\zeta_j}\right)^{n+1}\epsilon_l^{\mathrm{A}}(\zeta_j^*)\right]\tilde{q}_n(\zeta_l^*)=\theta\left(\frac{1}{\rho}\right)^n, \quad j=1, 2,\ldots, k. \tag{4.39}$$

Using (4.33), this reduces to

$$\sum_{l=1}^{k}\left[\frac{\gamma_j\cdot\gamma_l^*}{\zeta_l^*-\zeta_j}+\epsilon_j^{(n)}\right]\tilde{q}_n(\zeta_l^*)=\theta\left(\frac{1}{\rho}\right)^n \tag{4.40}$$

where

$$\epsilon_j^{(n)} := Q'(\zeta_j)\left(\frac{\zeta_l^*}{\zeta_j}\right)^{n+1}\epsilon_l^{\mathrm{A}}(\zeta_j^*).$$

From (4.30), we obtain

$$\epsilon_j^{(n)} = \theta(\zeta_j/\rho)^n$$

and so $\epsilon_j^{(n)} \to 0$ as $n\to\infty$, $j=1, 2,\ldots, k$.

In the Appendix, the matrix M, whose elements are

$$M_{jl} := \frac{\gamma_j\cdot\gamma_l^*}{\zeta_l^*-\zeta_j}, \quad l, j=1, 2,\ldots, k,$$

is shown to be invertible. It follows from (4.40) that

$$\tilde{q}_n(\zeta_l^*) = \theta(1/\rho)^n.$$

and from (4.38) that

$$\limsup_{n\to\infty} |q_n(\zeta_l)|^{1/n} \leqslant |\zeta_l|/\rho, \quad l=1, 2,\ldots, k. \tag{4.41}$$

The proof now follows the lines of Theorem 3.1, and we find $c_n \to 1$. For n sufficiently large, let

$$Q_n(z) := q_n(z)/c_n$$

and then (4.12), (4.13) follow. With the representation equivalent to (3.46), equations (4.10) and (4.11) follow too and the proof is completed. □

5. Conjecture and conclusions

In Sections 3 and 4 we established two row convergence theorems, each appropriate to its own distinctive setting. There remains open, however, the general question of establishing a de Montessus de Ballore type theorem for any vector-valued meromorphic function of the form

$$\boldsymbol{f}(z)=\boldsymbol{g}(z)/Q(z),$$

where $Q(z)=\prod_{j=1}^{k}(z-z_j)$, $0<|z_j|<\rho$, $\boldsymbol{g}(z)$ is analytic in D_ρ: $|z|<\rho$ with $\boldsymbol{g}(z_j)\neq\boldsymbol{0}$ for all j. Without loss of generality, we may assume that Q is a real polynomial (cf. (2.1)–(2.4)). It seems plausible that for a suitable choice of fixed denominator degree, say 2ν, the GIPAs of type $[n/2\nu]$ will converge to f in $D_\rho\backslash\bigcup_{j=1}^{k}\{z_j\}$ as $n\to\infty$. Based on Theorems 3.1 and 4.1, the choice of 2ν seems likely to be fixed by the following criterion. Let m_j be the precise order of the pole of $\boldsymbol{f}$ at z_j and let λ_j be the order of the zero of $\boldsymbol{g}\cdot\boldsymbol{g}^*$ at z_j, with the convention that $\lambda_j=0$ if $(\boldsymbol{g}\cdot\boldsymbol{g}^*)(z_j)\neq 0$. (Note that $(\boldsymbol{g}\cdot\boldsymbol{g}^*)(z_j)=|\boldsymbol{g}(z_j)|^2\neq 0$ if z_j is real). We then take

$$2\nu := \sum_{j=1}^{k}\max\{(2m_j-\lambda_j),\,1\}.$$

Observe that the right-hand side of this expression is necessarily even because
(i) $\lambda_j=0$ for real poles,
(ii) non-real poles occur in conjugate pairs.

Note. After original submission of our manuscript, we were informed of the relevance to it of Madison Report MRC626 (1966) by P. Wynn. In this (unpublished) report, and in the context of the vector ϵ-algorithm, a result similar to our Corollary 2.2 is stated.

We are grateful to Prof. C. Brezinski for this information, and to Prof. P. Wynn for a copy of his report.

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Appendix

Theorem. *A set of distinct complex (or real) numbers* $\{\eta_i\}_{i=1}^{n}$ *is given, with the property that*

$$\operatorname{Re}\eta_i\neq 0,\quad i=1,2,\ldots\,. \tag{A.1}$$

A set $\{\boldsymbol{a}_i\}_{i=1}^{n}$ *of non-null complex (or real) vectors with* $\boldsymbol{a}_i\in\mathbb{C}^d$ *is also given. The elements of an* $n\times n$ *matrix* L *are defined by*

$$L_{ij}:=\frac{\boldsymbol{a}_i\cdot\boldsymbol{a}_j^*}{\eta_i+\eta_j^*},\quad i,\ j=1,2,\ldots,n, \tag{A.2}$$

where we assume that each L_{ij} *is finite.*

Then $\det L \neq 0$ *and*

$$\operatorname{sign}\{\det L\} = \operatorname{sign}\left\{ \prod_{i=1}^{n} \operatorname{Re} \eta_i \right\}. \tag{A.3}$$

Proof. The result is obvious for the case of $n = 1$. We will prove the result for the cases of $n = 2, 3$. The method used for the case of $n = 3$ exemplifies the method for larger values of n, and we feel that this style of proof is clearer for this theorem than the equivalent using multi-indices [1].

Case of $n = 2$. Let $\boldsymbol{a} := \boldsymbol{a}_1$, $\boldsymbol{b} := \boldsymbol{a}_2$ for short. By elementary algebra, we find that

$$P := (\eta_1 + \eta_1^*)(\eta_2 + \eta_2^*) \begin{vmatrix} \dfrac{|\boldsymbol{a}|^2}{\eta_1 + \eta_1^*} & \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}^*}{\eta_1 + \eta_2^*} \\ \dfrac{\boldsymbol{b} \cdot \boldsymbol{a}^*}{\eta_2 + \eta_1^*} & \dfrac{|\boldsymbol{b}|^2}{\eta_2 + \eta_2^*} \end{vmatrix}$$

$$= \begin{vmatrix} |\boldsymbol{a}|^2 & \boldsymbol{a} \cdot \boldsymbol{b}^* \\ \boldsymbol{b} \cdot \boldsymbol{a}^* & |\boldsymbol{b}|^2 \end{vmatrix} + |\boldsymbol{a} \cdot \boldsymbol{b}^*|^2 \left| \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2^*} \right|^2. \tag{A.4}$$

The determinant on the right-hand side of (A.4) is a Gram determinant. If $\boldsymbol{a} \neq \lambda \boldsymbol{b}$ for any λ, this Gram determinant is strictly positive. Otherwise, $\boldsymbol{a} = \lambda \boldsymbol{b}$ for some λ, and then the second term in the right-hand side of (A.4) is strictly positive, since $\{\eta_i\}$ are assumed to be distinct. Therefore $P > 0$.

Case of $n = 3$. This case is proved in four stages.

Stage 1. We first consider the case of $d = 1$, which is the one dimensional case, and let $a := \boldsymbol{a}_1$, $b := \boldsymbol{a}_2$, $c := \boldsymbol{a}_3$ for short. The formula for a Cauchy determinant [2] leads to the result

$$\begin{vmatrix} \dfrac{|a|^2}{\eta_1 + \eta_2^*} & \dfrac{ab^*}{\eta_1 + \eta_2^*} & \dfrac{ac^*}{\eta_1 + \eta_3^*} \\ \dfrac{ba^*}{\eta_2 + \eta_1^*} & \dfrac{|b|^2}{\eta_2 + \eta_2^*} & \dfrac{bc^*}{\eta_2 + \eta_3^*} \\ \dfrac{ca^*}{\eta_3 + \eta_1^*} & \dfrac{cb^*}{\eta_3 + \eta_2^*} & \dfrac{|c|^2}{\eta_3 + \eta_3^*} \end{vmatrix}$$

$$= \frac{|a|^2 |b|^2 |c|^2}{8(\operatorname{Re} \eta_1)(\operatorname{Re} \eta_2)(\operatorname{Re} \eta_3)} \, \frac{|\eta_1 - \eta_2|^2 |\eta_2 - \eta_3|^2 |\eta_3 - \eta_1|^2}{|\eta_1 + \eta_2^*|^2 |\eta_2 + \eta_3^*|^2 |\eta_3 + \eta_1^*|^2}. \tag{A.5}$$

Therefore the theorem holds for the particular case of $d = 1$, $n = 3$. To establish the result for $d > 1$, we make the inductive hypothesis that the theorem holds for the case of

$$\boldsymbol{a}_i \in \mathbb{C}^{(d-1)}, \quad i = 1, 2, 3.$$

Stage 2. For $\boldsymbol{a}_i \in \mathbb{C}^{(d)}$, let $\boldsymbol{a} := \boldsymbol{a}_1$, $\boldsymbol{b} := \boldsymbol{a}_2$, $\boldsymbol{c} := \boldsymbol{a}_3$ for short.
We define

$$\boldsymbol{a}^{\perp} := (0, a_2, a_3, \ldots, a_d)$$

so that

$$\boldsymbol{a} - \boldsymbol{a}^{\perp} = (a_1, 0, 0, \ldots, 0),$$

and similarly for $\boldsymbol{b}$ and $\boldsymbol{c}$. We assume, for the moment, that $\boldsymbol{a}^{\perp}$, $\boldsymbol{b}^{\perp}$ and $\boldsymbol{c}^{\perp}$ are non-null. We consider the determinant.

$$D^{(3)} := \begin{vmatrix} \frac{1}{\eta_1+\eta_1^*} & \frac{1}{\eta_1+\eta_2^*} & \frac{c_1^*}{\eta_1+\eta_3^*} & 0 & 0 \\ \frac{1}{\eta_2+\eta_1^*} & \frac{1}{\eta_2+\eta_2^*} & \frac{c_1^*}{\eta_2+\eta_3^*} & 0 & 0 \\ \frac{c_1}{\eta_3+\eta_1^*} & \frac{c_1}{\eta_3+\eta_2^*} & \frac{|\boldsymbol{c}|^2}{\eta_3+\eta_3^*} & \frac{\boldsymbol{c}^{\perp}\cdot\boldsymbol{a}^{\perp *}}{\eta_3+\eta_1^*} & \frac{\boldsymbol{c}^{\perp}\cdot\boldsymbol{b}^{\perp *}}{\eta_3+\eta_2^*} \\ 0 & 0 & \frac{\boldsymbol{a}^{\perp}\cdot\boldsymbol{c}^{\perp *}}{\eta_1+\eta_3^*} & \frac{|\boldsymbol{a}^{\perp}|^2}{\eta_1+\eta_1^*} & \frac{\boldsymbol{a}^{\perp}\cdot\boldsymbol{b}^{\perp *}}{\eta_1+\eta_2^*} \\ 0 & 0 & \frac{\boldsymbol{b}^{\perp}\cdot\boldsymbol{c}^{\perp *}}{\eta_2+\eta_3^*} & \frac{\boldsymbol{b}^{\perp}\cdot\boldsymbol{a}^{\perp *}}{\eta_2+\eta_1^*} & \frac{|\boldsymbol{b}^{\perp}|^2}{\eta_2+\eta_2^*} \end{vmatrix}, \tag{A.6}$$

which is a real quadratic form in the variable c_1. As such, we define $D_0^{(3)}$, $D_1^{(3)}$, $D_2^{(3)}$ implicitly by

$$D^{(3)} = |c_1|^2 D_2^{(3)} + c_1 D_1^{(3)*} + c_1^* D_1^{(3)} + D_0^{(3)}. \tag{A.7}$$

For example, the coefficient of $|c_1|^2$ in (A.6) is

$$D_2^{(3)} = \begin{vmatrix} \frac{1}{\eta_1+\eta_1^*} & \frac{1}{\eta_1+\eta_2^*} & \frac{1}{\eta_1+\eta_3^*} & 0 & 0 \\ \frac{1}{\eta_2+\eta_1^*} & \frac{1}{\eta_2+\eta_2^*} & \frac{1}{\eta_2+\eta_3^*} & 0 & 0 \\ \frac{1}{\eta_3+\eta_1^*} & \frac{1}{\eta_3+\eta_2^*} & \frac{1}{\eta_3+\eta_3^*} & 0 & 0 \\ 0 & 0 & 0 & \frac{|\boldsymbol{a}^{\perp}|^2}{\eta_1+\eta_1^*} & \frac{\boldsymbol{a}^{\perp}\cdot\boldsymbol{b}^{\perp *}}{\eta_1+\eta_2^*} \\ 0 & 0 & 0 & \frac{\boldsymbol{b}^{\perp}\cdot\boldsymbol{a}^{\perp *}}{\eta_2+\eta_1^*} & \frac{|\boldsymbol{b}^{\perp}|^2}{\eta_2+\eta_2^*} \end{vmatrix}. \tag{A.8}$$

Using (A.4), we find that $D_2^{(3)} \neq 0$ and

$$\text{sign}\{D_2^{(3)}\} = \text{sign}\{\text{Re } \eta_3\}. \tag{A.9}$$

The discriminant of the quadratic form (A.7) is

$$\Delta^{(3)} := D_1^{(3)} D_1^{(3)*} - D_2^{(3)} D_0^{(3)}. \tag{A.10}$$

Jacobi's identity is directly applicable to (A.10), and we find

$$\Delta^{(3)} = -D_{36,36}^{(4)} D^{(4)}, \tag{A.11}$$

where

$$D^{(4)} := \begin{vmatrix} \dfrac{1}{\eta_1+\eta_1^*} & \dfrac{1}{\eta_1+\eta_2^*} & \dfrac{1}{\eta_1+\eta_3^*} & 0 & 0 & 0 \\ \dfrac{1}{\eta_2+\eta_1^*} & \dfrac{1}{\eta_2+\eta_2^*} & \dfrac{1}{\eta_2+\eta_3^*} & 0 & 0 & 0 \\ \dfrac{1}{\eta_3+\eta_1^*} & \dfrac{1}{\eta_3+\eta_2^*} & \dfrac{1}{\eta_3+\eta_3^*} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{|\boldsymbol{a}^\perp|^2}{\eta_1+\eta_1^*} & \dfrac{\boldsymbol{a}^\perp\cdot\boldsymbol{b}^{\perp *}}{\eta_1+\eta_2^*} & \dfrac{\boldsymbol{a}^\perp\cdot\boldsymbol{c}^{\perp *}}{\eta_1+\eta_3^*} \\ 0 & 0 & 0 & \dfrac{\boldsymbol{b}^\perp\cdot\boldsymbol{a}^{\perp *}}{\eta_2+\eta_1^*} & \dfrac{|\boldsymbol{b}^\perp|^2}{\eta_2+\eta_2^*} & \dfrac{\boldsymbol{b}^\perp\cdot\boldsymbol{c}^{\perp *}}{\eta_2+\eta_3^*} \\ 0 & 0 & 0 & \dfrac{\boldsymbol{c}^\perp\cdot\boldsymbol{a}^{\perp *}}{\eta_3+\eta_1^*} & \dfrac{\boldsymbol{c}^\perp\cdot\boldsymbol{b}^{\perp *}}{\eta_3+\eta_3^*} & \dfrac{|\boldsymbol{c}^\perp|^2}{\eta_3+\eta_3^*} \end{vmatrix} \tag{A.12}$$

and $D^{(4)}_{36,36}$ is obtained by deleting rows 3, 6 and cols 3, 6 from the determinant in (A.12). In later equations, we continue to use this notation: for a generic determinant $D, D_{p,r}$ is defined to be the determinant D with row p and column r deleted, and $D_{pq,rs}$ is defined to be the determinant D with rows p, q and columns r, s deleted.

By using the inductive hypothesis and the result for Cauchy determinants (used in Stage 1), we find that $D^{(4)} > 0$, $D^{(4)}_{36,36} > 0$ and hence we obtain

$$\Delta^{(3)} < 0$$

in (A.11). This result, together with (A.7) and (A.9), shows that $D^{(3)} \neq 0$ and

$$\operatorname{sign}\{D^{(3)}\} = \operatorname{sign}\{\operatorname{Re}\,\eta_3\}. \tag{A.13}$$

We also need two corollaries of this result (A.13). By the same method, we find that $D^{(3)}_{2,2} \neq 0$, $D^{(3)}_{25,25} \neq 0$ and

$$\operatorname{sign}\{D^{(3)}_{5,5}\} = \operatorname{sign}\{(\operatorname{Re}\,\eta_2)(\operatorname{Re}\,\eta_3)\}, \tag{A.14}$$

$$\operatorname{sign}\{D^{(3)}_{25,25}\} = \operatorname{sign}\{\operatorname{Re}\,\eta_3\}. \tag{A.15}$$

Stage 3. Consider the determinant

$$D^{(2)} := \begin{vmatrix} \dfrac{1}{\eta_1+\eta_1^*} & \dfrac{b_1^*}{\eta_1+\eta_2^*} & \dfrac{c_1^*}{\eta_1+\eta_3^*} & 0 \\ \dfrac{b_1}{\eta_2+\eta_1^*} & \dfrac{|\boldsymbol{b}|^2}{\eta_2+\eta_2^*} & \dfrac{\boldsymbol{b}\cdot\boldsymbol{c}^*}{\eta_2+\eta_3^*} & \dfrac{\boldsymbol{b}^\perp\cdot\boldsymbol{a}^{\perp *}}{\eta_2+\eta_1^*} \\ \dfrac{c_1}{\eta_3+\eta_1^*} & \dfrac{\boldsymbol{c}\cdot\boldsymbol{b}^*}{\eta_3+\eta_2^*} & \dfrac{|\boldsymbol{c}|^2}{\eta_3+\eta_3^*} & \dfrac{\boldsymbol{c}^\perp\cdot\boldsymbol{a}^{\perp *}}{\eta_3+\eta_1^*} \\ 0 & \dfrac{\boldsymbol{a}^\perp\cdot\boldsymbol{b}^{\perp *}}{\eta_1+\eta_2^*} & \dfrac{\boldsymbol{a}^\perp\cdot\boldsymbol{c}^{\perp *}}{\eta_1+\eta_3^*} & \dfrac{|\boldsymbol{a}^\perp|^2}{\eta_1+\eta_1^*} \end{vmatrix} \tag{A.16}$$

which is a real quadratic form in b_1. We define $D_0^{(2)}$, $D_1^{(2)}$, $D_2^{(2)}$ implicitly by

$$D^{(2)} = |b_1|^2 D_2^{(2)} + b_1 D_1^{(2)*} + b_1^* D_1^{(2)} + D_0^{(2)}. \tag{A.17}$$

By inspection,

$$D_2^{(2)} = \begin{vmatrix} \frac{1}{\eta_1+\eta_1^*} & \frac{1}{\eta_1+\eta_2^*} & \frac{c_1^*}{\eta_1+\eta_3^*} & 0 \\ \frac{1}{\eta_2+\eta_1^*} & \frac{1}{\eta_2+\eta_2^*} & \frac{c_1^*}{\eta_2+\eta_3^*} & 0 \\ \frac{c_1}{\eta_3+\eta_1^*} & \frac{c_1}{\eta_3+\eta_2^*} & \frac{|\boldsymbol{c}|^2}{\eta_3+\eta_3^*} & \frac{\boldsymbol{c}^{\perp}\cdot\boldsymbol{a}^{\perp *}}{\eta_3+\eta_1} \\ 0 & 0 & \frac{\boldsymbol{a}^{\perp}\cdot\boldsymbol{c}^{\perp *}}{\eta_1+\eta_3^*} & \frac{|\boldsymbol{a}^{\perp}|^2}{\eta_1+\eta_1^*} \end{vmatrix}. \tag{A.18}$$

In fact, $D_2^{(2)} = D_{5,5}^{(3)}$. By (A.14) we obtain $D_2^{(2)} \neq 0$ and

$$\operatorname{sign}\{D_2^{(2)}\} = \operatorname{sign}\{(\operatorname{Re}\eta_2)(\operatorname{Re}\eta_3)\}. \tag{A.19}$$

The discriminant of the quadratic form (A.17) is

$$\Delta^{(2)} = D_1^{(2)} D_1^{(2)*} - D_0^{(2)} D_2^{(2)}. \tag{A.20}$$

Again Jacobi's identity applies directly to (A.20), and we find that

$$\Delta^{(2)} = -D^{(3)} D_{25,25}^{(3)}.$$

From (A.13) and (A.15), we find that $\Delta^{(2)} < 0$. From (A.17) and (A.19) we deduce that $D^{(2)} \neq 0$ and

$$\operatorname{sign}\{D^{(2)}\} = \operatorname{sign}\{(\operatorname{Re}\eta_2)(\operatorname{Re}\eta_3)\}. \tag{A.21}$$

As a corollary, we find by the same method that $D_{4,4}^{(2)} \neq 0$ and

$$\operatorname{sign}\{D_{4,4}^{(2)}\} = \operatorname{sign}\{(\operatorname{Re}\eta_1)(\operatorname{Re}\eta_2)(\operatorname{Re}\eta_3)\}. \tag{A.22}$$

Stage 4. Our aim is to analyse the determinant

$$D^{(1)} := \begin{vmatrix} \frac{|\boldsymbol{a}|^2}{\eta_1+\eta_1^*} & \frac{\boldsymbol{a}\cdot\boldsymbol{b}^*}{\eta_1+\eta_2^*} & \frac{\boldsymbol{a}\cdot\boldsymbol{c}^*}{\eta_1+\eta_3^*} \\ \frac{\boldsymbol{b}\cdot\boldsymbol{a}^*}{\eta_2+\eta_1^*} & \frac{|\boldsymbol{b}|^2}{\eta_2+\eta_2^*} & \frac{\boldsymbol{b}\cdot\boldsymbol{c}^*}{\eta_2+\eta_3^*} \\ \frac{\boldsymbol{c}\cdot\boldsymbol{a}^*}{\eta_3+\eta_1^*} & \frac{\boldsymbol{c}\cdot\boldsymbol{b}^*}{\eta_3+\eta_2^*} & \frac{|\boldsymbol{c}|^2}{\eta_3+\eta_3^*} \end{vmatrix}, \tag{A.23}$$

which is a real quadratic form in a_1. We define $D_0^{(1)}$, $D_1^{(1)}$ and $D_2^{(1)}$ implicitly by

$$D^{(1)} = |a_1|^2 D_2^{(1)} + a_1 D_1^{(1)*} + a_1^* D_1^{(1)} + D_0^{(1)}. \tag{A.24}$$

By inspection, $D_2^{(1)} = D_{4,4}^{(2)}$. Hence $D_2^{(1)} \neq 0$ and

$$\operatorname{sign}\{D_2^{(1)}\} = \operatorname{sign}\{(\operatorname{Re}\eta_1)(\operatorname{Re}\eta_2)(\operatorname{Re}\eta_3)\}. \tag{A.25}$$

The discriminant of the quadratic form (A.24) is

$$\Delta^{(1)} := D_1^{(1)} D_1^{(1)*} - D_0^{(1)} D_2^{(1)} = -D^{(2)} D_{14,14}^{(2)}. \tag{A.26}$$

Using Stage 1, we find that $D_{14,14}^{(2)} \neq 0$ and

$$\operatorname{sign}\{D_{14,14}^{(2)}\} = \operatorname{sign}\{(\operatorname{Re} \eta_2)(\operatorname{Re} \eta_3)\}.$$

Taking this result with (A.21), (A.26), we find that $\Delta^{(1)} < 0$. Hence $D^{(1)} \neq 0$ and

$$\operatorname{sign}\{D^{(1)}\} = \operatorname{sign}\{D_2^{(1)}\} = \operatorname{sign}\{(\operatorname{Re} \eta_1)(\operatorname{Re} \eta_2)(\operatorname{Re} \eta_3)\}.$$

If $\boldsymbol{a}^\perp = \boldsymbol{0}$, $\boldsymbol{b}^\perp \neq \boldsymbol{0}$ and $\boldsymbol{c}^\perp \neq \boldsymbol{0}$,

$$D^{(1)} = |a_1|^2 D_{4,4}^{(2)}.$$

The same method of analysis as that given in stages 2,3 remains applicable, and (A.22) remains valid. The special cases in which one or more of $\boldsymbol{a}^\perp$, $\boldsymbol{b}^\perp$ and $\boldsymbol{c}^\perp$ are null are treated similarly.

This completes the inductive proof of the theorem for the case of $n = 3$, and exemplifies the method for the cases of $n = 4, 5, 6, \ldots$. □

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