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CONSTRUCTIVE APPROXIMATION

Uniform and Mean Approximation by Certain Weighted Polynomials, with Applications

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Abstract. Let $W(x):=\exp(-Q(x))$, where, for example, $Q(x)$ is even and convex on $\mathbf{R}$, and $Q(x)/\log x \to \infty$ as $x \to \infty$. A result of Mhaskar and Saff asserts that if $a_n = a_n(W)$ is the positive root of the equation

$$n=(2/\pi)\int_0^1 a_n x Q'(a_n x)/\sqrt{1-x^2}\,dx,$$

then, given any polynomial $P_n(x)$ of degree at most n, the sup norm of $P_n(x)W(a_n x)$ over $\mathbf{R}$ is attained on $[-1, 1]$. In addition, any sequence of weighted polynomials $\{P_n(x)W(a_n x)\}_1^\infty$ that is uniformly bounded on $\mathbf{R}$ will converge to 0, for $|x|>1$.

In this paper we show that under certain conditions on W, a function $g(x)$ continuous in $\mathbf{R}$ can be approximated in the uniform norm by such a sequence $\{P_n(x)W(a_n x)\}_1^\infty$ if and only if $g(x)=0$ for $|x|\geq 1$. We also prove an L_p analogue for $0<p<\infty$. Our results confirm a conjecture of Saff for $W(x)=\exp(-|x|^\alpha)$, when $\alpha>1$. Further applications of our results are upper bounds for Christoffel functions, and asymptotic behavior of the largest zeros of orthogonal polynomials. A final application is an approximation theorem that will be used in a forthcoming proof of Freud's conjecture for $|x|^\rho \exp(-|x|^\alpha)$, $\alpha>0$, $\rho>-1$.

1. Introduction

In recent years, much attention has been given to approximation on $\mathbf{R}$ by weighted polynomials $P_n(x)W(x)$, where $P_n(x)$ is of degree at most n, and, for example, $W(x):=W_\alpha(x):=\exp(-|x|^\alpha)$, $\alpha>0$. See Ditzian and Totik [4], Levin and Lubinsky [13], Mhaskar [26], Nevai [36], and Saff [41] for references, results, and reviews from different perspectives.

One of the important ideas in such approximation is the reduction of an L_p norm over an infinite interval to an L_p norm over a finite interval. For example, if $\mathcal{P}_n$ denotes the class of polynomials of degree at most n, there exists (under

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mild conditions on W) a number a_n such that for each $P_n \in \mathcal{P}_n$,

$$\| P_n W \|_{L_\infty(\mathbf{R})} = \| P_n W \|_{L_\infty[-a_n, a_n]}. \tag{1.1}$$

While related identities and inequalities have been considered by Freud [6] and Nevai [32], the sharp form of (1.1), namely, the "best" choice of a_n, has only been found recently. See Mhaskar and Saff [27, 29] for the L_∞ version and [31] for the L_p version. Methods similar to those of Mhaskar and Saff were used by Rahmanov [39], although Rahmanov did not investigate identities such as (1.1). Subsequently, Lubinsky [19] used the potential theoretic methods of Mhaskar, Rahmanov, and Saff to investigate L_p analogues of (1.1).

One feature of (1.1) is the growing nested sequence of intervals $[-a_n, a_n]$, $n = 1, 2, 3, \ldots$. In order to work with functions on a fixed finite interval, it is convenient to contract $[-a_n, a_n]$ to $[-1, 1]$ and to consider weighted polynomials of the form $p_n(x)W(a_n x)$, $p_n \in \mathcal{P}_n$, $n = 1, 2, 3, \ldots$, whose norm "lives" on $[-1, 1]$. This contraction is important in describing the asymptotic behavior of orthogonal and extremal polynomials associated with weights on $\mathbf{R}$ and typically leads to the Ullman distribution—see Mhaskar and Saff [27], [29], Nevai and Dehesa [37], Rahmanov [39], and Ullman [45], [46].

A second aspect of (1.1) is the following: under mild conditions on W, any sequence of weighted polynomials $\{p_n(x)W(a_n x)\}_1^\infty$, $p_n \in \mathcal{P}_n$, uniformly bounded on $\mathbf{R}$, will tend to zero for $|x| > 1$. But what can be said about the behavior of such a sequence in $[-1, 1]$? In this paper we consider this question for certain weights $W(x) = \exp(-Q(x))$. We prove that if $g(x)$ is a function continuous on $\mathbf{R}$, there exists $p_n \in \mathcal{P}_n$, $n = 1, 2, 3, \ldots$, such that

$$\lim_{n \to \infty} \| g(x) - p_n(x) W(a_n x) \|_{L_\infty(\mathbf{R})} = 0, \tag{1.2}$$

if and only if $g(x) = 0$, $|x| \geq 1$. Here, for suitable W, and n large enough, a_n is the number in (1.1) that may be defined to be the positive root of the equation

$$n = (2/\pi) \int_0^1 a_n x Q'(a_n x)/\sqrt{1 - x^2}\, dx. \tag{1.3}$$

We also prove an L_p analogue of (1.2), for $0 < p < \infty$. While the L_p theorem $(0 < p < \infty)$ may be applied to $W_\alpha(x)$ for any $\alpha > 0$ without restriction on $g(x)$, the L_∞ theorem may be applied to $W_\alpha(x)$ for any $\alpha > 1$ without restriction on $g(x)$, and for $0 < \alpha \leq 1$ if $g(0) = 0$. For $\alpha > 1$, the L_∞ theorem proves a conjecture of Saff [41] that had been proved for $\alpha = 2$ by Mhaskar and Saff [30].

A second theme in this paper is "mean relative approximation." Under certain conditions on W and on a function $f(x)$, positive almost everywhere in $[-1, 1]$, we show that there exists $P_n \in \mathcal{P}_n$, $n = 1, 2, 3, \ldots$, such that

$$f(x) P_n(x) W(a_n x) \to 1, \qquad n \to \infty, \quad x \in [-1, 1],$$

in a mean sense. This type of result is an important step (see [12]) in the proof of Freud's conjecture for weights on the form $|x|^\rho \exp(-|x|^\alpha)$, $\alpha > 0$, $\rho > -1$, which appears in [21].

A final application of our results is the estimation of some quantities in the theory of orthogonal polynomials. We obtain the asymptotic behavior of the largest zero x_{1n} of the nth orthogonal polynomial for $W^2(x)$, for weights including $\exp(-|x|^\alpha(\log(2+x^2))^\beta)$, $\alpha>0$, $\beta\in\mathbf{R}$, thereby extending previous results of Rahmanov [39]. We also obtain upper bounds for the Christoffel functions

$$\lambda_n(W^2,x) := \inf_{P\in\mathcal{P}_{n-1}} \int_{-\infty}^{\infty} (P(u)W(u))^2\,du/P^2(x),$$

in the interval $|x|\le(1-\varepsilon)x_{1n}$, for any $\varepsilon>0$. Except for the weights $\exp(-x^{2m})$, $m=1,2,3$, and $\exp(-x^4+S(x))$, $S\in\mathcal{P}_3$ [1], [34], [43], such upper bounds were previously known to be true only in $|x|\le cx_{1n}$, for some unspecified positive number c. These upper bounds lead to extensions of the range of validity of several results of Bonan [2], Freud [9], Lubinsky, Máté, and Nevai [20], and Nevai [35].

The paper is organized as follows: Section 2 contains the statements of our main approximation results. Section 3 contains notation and background. In Section 4 we prove a theorem concerning approximation by $P_n(x)/H_Q(a_nx)$, $P_n\in\mathcal{P}_n$, where $H_Q(x)$ is a certain entire function studied in Lubinsky [17], [18]. In Section 5 we use the result of Section 4 to prove the results of Section 2. In Section 6 we state and prove the upper bounds for Christoffel functions. Finally, the Appendix contains the detailed calculations required for investigating the positivity of a function associated with the Ullman distribution.

2. Statement of Approximation Results

For $n=1,2,3,\ldots$, let $\mathcal{P}_n$ denote the class of polynomials of degree at most n. Further, let

$$W(x) := \exp(-Q(x)),$$

where $Q(x)$ is even and continuous in $\mathbf{R}$, and $Q'(x)$ exists for $x>0$. Throughout this paper, whenever it is uniquely defined, we let $a_n=a_n(W)$ denote the positive root of the equation

$$n=(2/\pi)\int_0^1 a_nxQ'(a_nx)/\sqrt{1-x^2}\,dx, \tag{2.1}$$

$n=1,2,3,\ldots$. When $Q(x)$ is convex or $Q(x)=|x|^\alpha$, $\alpha>0$, results of Mhaskar and Saff [27, p. 210], [29, p. 77] imply that, for $n=1,2,3,\ldots$, and each $P_n\in\mathcal{P}_n$,

$$\|P_nW\|_{L_\infty(\mathbf{R})}=\|P_nW\|_{L_\infty[-a_n,a_n]}. \tag{2.2}$$

In the special case $W(x)=W_\alpha(x)$, where

$$W_\alpha(x) := \exp(-|x|^\alpha), \qquad \alpha>0, \tag{2.3}$$

$a_n(W)$ takes a particularly simple form:

$$a_n(W_\alpha)=\beta_\alpha n^{1/\alpha}, \qquad n=1,2,3,\ldots, \tag{2.4}$$

where

$$\beta_\alpha := \lambda_\alpha^{-1/\alpha}, \tag{2.5}$$

and

$$\lambda_\alpha := \Gamma(\alpha)/\{2^{\alpha-2}\Gamma(\alpha/2)^2\}. \tag{2.6}$$

Because of the homogeneity of $|x|^\alpha$,

$$W_\alpha(n^{1/\alpha}x) = W_\alpha^n(x), \qquad x \in \mathbf{R}.$$

Thus in this case, (2.2) may be rewritten in the form

$$\| P_n W_\alpha^n \|_{L_\infty(\mathbf{R})} = \| P_n W_\alpha^n \|_{L_\infty[-\beta_\alpha, \beta_\alpha]}, \tag{2.7}$$

$P_n \in \mathcal{P}_n$, $n = 1, 2, 3, \ldots$.

This form of (2.2) is significant for several reasons. When investigating extremal problems associated with general weights $W(x)$ on $\mathbf{R}$, having possibly disconnected support, it is more appropriate to consider weighted polynomials of the form $P_n(x)W(x)^n$, rather than $P_n(x)W(x)$ or $P_n(x)W(a_n x)$. The nth power of $W(x)$ also plays a natural role in questions concerning asymptotic behavior of extremal and orthogonal polynomials—see Goncar and Rahmanov [11], Mhaskar and Saff [29], [31], and Saff [41].

One consequence of the results in [27] is the following: let $\alpha > 0$ and let $\{P_n(x)W_\alpha^n(x)\}_1^\infty$ be a sequence of weighted polynomials that is uniformly bounded in $\mathbf{R}$. Then

$$\lim_{n\to\infty} P_n(x)W_\alpha^n(x) = 0, \qquad |x| > \beta_\alpha.$$

Thus, if we are to approximate a function $g(x)$ continuous on $\mathbf{R}$ by such a sequence in the uniform norm, then, necessarily, $g(x) = 0$ for $|x| \geq \beta_\alpha$. Saff [41, p. 252] conjectured that this should be the only restriction on $g(x)$ for such an approximation to be possible. The conjecture was generalized, and proved for $\alpha = 2$, by Mhaskar and Saff [30]. Here we prove the conjecture for all $\alpha < 1$ and, with a minor modification, for $\alpha > 0$:

Theorem 2.1. *Let $\alpha > 0$. Let $\{k_n\}_1^\infty$ be a sequence of nonnegative integers such that*

$$\lim_{n\to\infty} k_n/n = 0. \tag{2.8}$$

Let $g(x)$ be continuous in $\mathbf{R}$. If $0 < \alpha \leq 1$, we assume in addition that $g(0) = 0$. Then there exists $P_n \in \mathcal{P}_{n-k_n}$, $n = 1, 2, 3, \ldots$, such that

$$\lim_{n\to\infty} \| g(x) - P_n(x)W_\alpha^n(x) \|_{L_\infty(\mathbf{R})} = 0 \tag{2.9}$$

if and only if $g(x) = 0$ for $|x| \geq \beta_\alpha$.

The following is the L_p analogue:

Theorem 2.2. *Let $0 < p < \infty$ and $\alpha > 0$. Let $\{k_n\}_1^\infty$ be a sequence of nonnegative* *integers satisfying* (2.8). *Let* $g(x) \in L_p(\mathbf{R})$. *Then there exists* $P_n \in \mathcal{P}_{n-k_n}$, $n = 1, 2, 3, \ldots$, *such that*

(2.10) $$\lim_{n\to\infty} \| g(x) - P_n(x) W_\alpha^n(x) \|_{L_p(\mathbf{R})} = 0$$

if and only if $g(x) = 0$ *for almost all* $|x| \geq \beta_\alpha$.

Both Theorems 2.1 and 2.2 follow from more general results involving approximation by weighted polynomials of the form $P_n(x) W(a_n x)$. In order to state them, we first define a suitable class of weights.

Definition 2.3. Let $W(x) := \exp(-Q(x))$, where $Q(x)$ is even, continuous and $Q'(x)$ exists for $x > 0$, while $xQ'(x)$ remains bounded as $x \to 0+$. Further, assume that $Q'''(x)$ exists for x large enough, and for some $C > 0$ and $\alpha > 0$,

(2.11) $$Q'(x) > 0, \quad x \text{ large enough},$$

(2.12) $$x^2 |Q'''(x)| / Q'(x) \leq C, \quad x \text{ large enough},$$

and

(2.13) $$\lim_{x\to\infty} (1 + xQ''(x)/Q'(x)) = \alpha.$$

Then we shall call W a *very smooth Freud weight* of order α and write $W \in \mathrm{VSF}(\alpha)$.

We remark that if $\alpha > 0$ and $\beta \in \mathbf{R}$, then $W(x) = \exp(-|x|^\alpha (\log(2 + x^2))^\beta) \in \mathrm{VSF}(\alpha)$. The conditions on $Q(x)$ above arise in the construction of even entire functions with nonnegative Maclaurin series coefficients and behaving like $W^{-1}(x)$ on $\mathbf{R}$, see Lubinsky [17], [18]. They can be somewhat weakened, but we retain the above formulation for the sake of simplicity.

Our two main results follow:

Theorem 2.4. *Let* $W \in \mathrm{VSF}(\alpha)$ *for some* $\alpha > 0$. *Let* $a_n = a_n(W)$ *be the root of* (2.1) *for n large enough. Let* $g(x)$ *be continuous in* $\mathbf{R}$. *If*

(2.14) $$\limsup_{n\to\infty} a_n / n > 0,$$

we assume in addition that $g(0) = 0$. *Let* $\{k_n\}_1^\infty$ *be a sequence of nonnegative integers satisfying* (2.8). *Then there exists* $P_n \in \mathcal{P}_{n-k_n}$, $n = 1, 2, 3, \ldots$, *such that*

(2.15) $$\lim_{n\to\infty} \| g(x) - P_n(x) W(a_n x) \|_{L_\infty(\mathbf{R})} = 0$$

if and only if $g(x) = 0$ *for* $|x| \geq 1$.

We remark that the conditions on $W(x)$ ensure the existence of a_n for n large enough—see Lemma 3.2. Further, if $\alpha > 1$, then

(2.16) $$\lim_{n\to\infty} a_n / n = 0,$$

while, if $\alpha < 1$, then

$$\lim_{n\to\infty} a_n/n = \infty. \tag{2.17}$$

If $\alpha = 1$, then any of the relations (2.14), (2.16), or (2.17) may be valid—see the comments after Lemma 3.2.

Theorem 2.5. *Let $0 < p < \infty$. Let $W \in \mathrm{VSF}(\alpha)$ for some $\alpha > 0$. Let $a_n = a_n(W)$ be the root of (2.1) for n large enough. Let $\{k_n\}_1^\infty$ be a sequence of nonnegative integers satisfying (2.8). If $g \in L_p(\mathbf{R})$, there exists $P_n \in \mathcal{P}_{n-k_n}$, $n = 1, 2, 3, \ldots$, such that*

$$\lim_{n\to\infty} \| g(x) - P_n(x) W(a_n x) \|_{L_p(\mathbf{R})} = 0 \tag{2.18}$$

if and only if $g(x) = 0$ for almost all $|x| \geq 1$.

We remark that in Theorems 2.4 and 2.5, one may replace $W(a_n x)$ by $W(a_n(1+\varepsilon_n)x)$, where $\{\varepsilon_n\}_1^\infty$ is any sequence of real numbers satisfying

$$\lim_{n\to\infty} \varepsilon_n = 0.$$

This may be seen by substituting $x = (1+\varepsilon_n)u$ in (2.15) and (2.16). Further, if we replace $W(a_n x)$ in (2.14) by $1/H_Q(a_n x)$, where $H_Q(x)$ is a certain entire function that behaves like $W^{-1}(x)$ in $\mathbf{R}$, then the restriction involving $g(0) = 0$ is no longer necessary—see Theorem 4.1.

One application of Theorem 2.4 is in obtaining upper bounds for Christoffel functions—see Section 6. As a prerequisite, we investigate asymptotic behavior of the largest zeros of orthogonal polynomials—see Section 3. A final application is to the mean relative approximation of functions, in a sense made precise in the following result. The rather cumbersome formulation will be required in a forthcoming proof [21] of Freud's conjecture for $|x|^\rho W_\alpha(x)$, $\alpha > 0$, $\rho > -1$.

Theorem 2.6. *Let $W \in \mathrm{VSF}(\alpha)$ for some $\alpha > 0$. Let $a_n = a_n(W)$ be the root of (2.1) for n large enough. Let $\{k_n\}_1^\infty$ be a sequence of nonnegative integers satisfying (2.8), and let $\{\varepsilon_n\}_1^\infty$ be a sequence of real numbers satisfying*

$$\lim_{n\to\infty} \varepsilon_n = 0. \tag{2.19}$$

Let $\Psi(u)$ be a function nonnegative in $\mathbf{R}$, such that $\Psi(u) \in L_\infty(\mathbf{R})$ and

$$\lim_{|u|\to\infty} \Psi(u) = 1. \tag{2.20}$$

Further, let $V(u)$ be a function nonnegative in $\mathbf{R}$, with the following property: for $n = 1, 2, 3, \ldots$, there exists a positive integer l_n and $S_n \in \mathcal{P}_{l_n}$ such that

$$\lim_{n\to\infty} l_n/n = 0, \tag{2.21}$$

$$\lim_{n\to\infty} | V(a_n(1+\varepsilon_n)u) S_n(u) | = 1, \quad \textit{almost everywhere in } [-1, 1], \tag{2.22}$$

and for some $C > 0$,

$$\| V(a_n(1+\varepsilon_n)u)S_n(u)\|_{L_\infty[-1,1]} \le C, \qquad n = 1, 2, 3, \dots. \tag{2.23}$$

Next, let $N \ge 1$; $z_1, z_2, \dots, z_N \in \mathbf{C}$; $\Delta_1, \Delta_2, \dots, \Delta_N \in \mathbf{R}$;

$$\Delta := \sum_{j=1}^{N} \Delta_j, \tag{2.24}$$

and

$$w(x) := \prod_{j=1}^{N} |x - z_j|^{\Delta_j}, \qquad x \in \mathbf{R}. \tag{2.25}$$

Let

$$\hat{W}(x) := W(x)V(x)\Psi(x)w(x), \qquad x \in \mathbf{R}. \tag{2.26}$$

Finally, let $r_1, r_2 > 0$, *and if, for some* $1 \le k \le N$, $z_k \in \mathbf{R}$, *assume that*

$$r_j \Delta_k > -1, \qquad j = 1, 2. \tag{2.27}$$

Then if $f(x)$ *is continuous and positive almost everywhere in* $[-1, 1]$, *there exists* $P_n \in \mathcal{P}_{n-k_n}$, $n = 1, 2, 3, \dots$, *such that, for* $j = 1, 2$,

$$\lim_{n\to\infty} \int_{-1}^{1} \left|1 - |f(x)\hat{W}(a_n(1+\varepsilon_n)x)P_n(x)|^{r_j}\right| (1-x^2)^{-1/2}\, dx = 0. \tag{2.28}$$

Note that if $V(u)$ is taken to be a positive constant, the conditions (2.21)-(2.23) are trivially satisfied.

3. Notation and Background

Throughout $C, C_1, C_2, \dots$ denote positive constants independent of n and x. The same symbol does not necessarily denote the same constant from line to line. Given two sequences $\{c_n\}_1^\infty$ and $\{d_n\}_1^\infty$ of real numbers, we write

$$c_n \asymp d_n$$

if there exist $C_1, C_2 > 0$ such that

$$C_1 \le c_n / d_n \le C_2, \quad n \text{ large enough}.$$

Similarly, we may define

$$f_n(x) \asymp g_n(x)$$

uniformly for x in a specified range.

We first establish some elementary properties of $W \in \mathrm{VSF}(\alpha)$. Note that the only results in this section whose proofs require the existence of $Q'''(x)$, or (2.12) to be satisfied, are Lemmas 3.10 and 3.11.

Lemma 3.1. *Let $W \in \mathrm{VSF}(\alpha)$ for some $\alpha > 0$. Then:*

(i) *$xQ'(x)$ is strictly increasing for x large enough.*
(ii) *Uniformly for r in any compact subinterval of* $(0, \infty)$,

(3.1) $$\lim_{x\to\infty} Q'(rx)/Q'(x) = r^{\alpha-1}.$$

(iii)

(3.2) $$\lim_{x\to\infty} xQ'(x)/Q(x) = \alpha.$$

(iv) *Uniformly for r in any compact subinterval of* $(0, \infty)$,

(3.3) $$\lim_{x\to\infty} Q(rx)/Q(x) = r^{\alpha}.$$

Proof. (i) For large enough x, (2.11) and (2.13) imply that

$$\begin{aligned}(d/dx)(xQ'(x)) &= Q'(x)\{1 + xQ''(x)/Q'(x)\}\\ &\geq Q'(x)\{\alpha/2\} > 0.\end{aligned}$$

(ii) Let $0 < \varepsilon < \alpha$. By (2.13) there exists $A > 0$ such that, for $x \geq A$,

$$\alpha - 1 - \varepsilon \leq xQ''(x)/Q'(x) \leq \alpha - 1 + \varepsilon.$$

Let $r \geq 1$. Then, if $x \geq A$,

$$\begin{aligned}Q'(rx)/Q'(x) &= \exp\left(\int_x^{rx} Q''(u)/Q'(u)\,du\right)\\ &\geq \exp\left(\int_x^{rx} (\alpha - 1 - \varepsilon)/u\,du\right) = r^{\alpha-1-\varepsilon}.\end{aligned}$$

Similarly, for $x \geq A$ and $r \geq 1$,

$$Q'(rx)/Q'(x) \leq r^{\alpha-1+\varepsilon}.$$

If $0 < r < 1$, we similarly obtain

(3.4) $$r^{\alpha-1+\varepsilon} \leq Q'(rx)/Q'(x) \leq r^{\alpha-1-\varepsilon}, \qquad rx \geq A.$$

Then (3.1) follows, since $r^{\pm\varepsilon} \to 1$ as $\varepsilon \to 0$, uniformly for r in any compact subinterval of $(0, \infty)$.

(iii) First note from (3.1) that

(3.5) $$\lim_{x\to\infty} rxQ'(rx)/(xQ'(x)) = r^{\alpha}$$

uniformly for r in any compact subinterval of $(0, \infty)$. The monotonicity of $xQ'(x)$ and (3.5) imply that

(3.6) $$\lim_{x\to\infty} xQ'(x) = \infty.$$

Let $A>0$ be so large that $uQ'(u)$ is positive and increasing for $u\geq A$ and (3.4) holds for some $0<\varepsilon<\alpha$. Let $0<\eta<1$. Then, for $\eta x>A$,

$$Q(x)=Q(A)+\left\{\int_A^{\eta x}+\int_{\eta x}^{x}\right\}Q'(u)\,du,$$

so that

$$Q(x)/(xQ'(x))=Q(A)/(xQ'(x))+\int_{A/x}^{\eta} Q'(sx)/Q'(x)\,ds + \int_{\eta}^{1} Q'(sx)/Q'(x)\,ds. \tag{3.7}$$

In view of (3.6), the first term on the right-hand side of (3.7) tends to 0 as $x\to\infty$. Next, by (3.4),

$$\int_{A/x}^{\eta} Q'(sx)/Q'(x)\,dx \leq \int_0^{\eta} s^{\alpha-1-\varepsilon}\,ds=\eta^{\alpha-\varepsilon}/(\alpha-\varepsilon).$$

Finally, by (ii),

$$\lim_{x\to\infty}\int_{\eta}^{1} Q'(sx)/Q'(x)\,ds=\int_{\eta}^{1} s^{\alpha-1}\,ds=(1-\eta^{\alpha})/\alpha.$$

Since $\eta>0$ is arbitrary, we deduce that

$$\lim_{x\to\infty} Q(x)/(xQ'(x))=1/\alpha.$$

(iv) Using the expression

$$Q(rx)/Q(x)=\exp\left(\int_x^{rx} Q'(u)/Q(u)\,du\right),$$

and using (3.2), we obtain (3.3). ∎

Let $W\in \mathrm{VSF}(\alpha)$. For n large enough, we let q_n denote the positive root of the equation

$$n=q_nQ'(q_n). \tag{3.8}$$

In view of (3.6) and Lemma 3.1(i), this is uniquely defined for n large enough. The quantity q_n was considered in detail by Freud [8], [9]. Throughout, we denote by $a_n=a_n(W)$, the positive root of the equation

$$n=(2/\pi)\int_0^1 a_nxQ'(a_nx)/\sqrt{1-x^2}\,dx, \tag{3.9}$$

whenever it is uniquely defined. Further, we shall abbreviate $a_n(W)$ as a_n, except when confusion may arise.

Lemma 3.2. *Let $W\in \mathrm{VSF}(\alpha)$ for some $\alpha>0$. Then:*

(i) *For n large enough, a_n is uniquely defined.*

(ii) *If* λ_α *is given by* (2.6), *then*

$$\lim_{n\to\infty} n/(a_nQ'(a_n)) = \lambda_\alpha/\alpha \tag{3.10}$$

and

$$\lim_{n\to\infty} q_n/a_n = (\lambda_\alpha/\alpha)^{1/\alpha}. \tag{3.11}$$

(iii) *Uniformly in compact subsets of* $\mathbf{R}$,

$$\lim_{n\to\infty} W(a_nx)^{1/n} = W_\alpha(\beta_\alpha x). \tag{3.12}$$

(iv) *Let* $\{k_n\}_1^\infty$ *be any sequence of integers with*

$$\lim_{n\to\infty} k_n/n = 0. \tag{3.13}$$

Then

$$\lim_{n\to\infty} a_n/a_{n+k_n} = 1. \tag{3.14}$$

Proof. (i) Let

$$I(a) := \int_0^1 axQ'(ax)/\sqrt{1-x^2}\,dx.$$

From (3.6) and Lemma 3.1(i), it is easily seen that

$$\lim_{a\to\infty} I(a) = \infty.$$

It then suffices to show that $I(a)$ is increasing for a large enough, and to this end we express $I(a)$ in the form

$$I(a) = \int_0^A uQ'(u)/\sqrt{a^2-u^2}\,du + \int_{A/a}^1 axQ'(ax)/\sqrt{1-x^2}\,dx,$$

where $A > 0$ is so large that $uQ'(u)$ is positive and increasing for $u \geq A$ and

$$1 + uQ''(u)/Q'(u) > \alpha/2, \qquad u \geq A.$$

Then we see that

$$\begin{aligned} I'(a) &= -\int_0^A uQ'(u)a(a^2-u^2)^{-3/2}\,du + AQ'(A)(1-(A/a)^2)^{-1/2}Aa^{-2} \\ &\quad + \int_{A/a}^1 xQ'(ax)\{1+axQ''(ax)/Q'(ax)\}/\sqrt{1-x^2}\,dx \\ &\geq -C_1a^{-2} + (\alpha/2)\int_{A/a}^1 xQ'(ax)/\sqrt{1-x^2}\,dx \\ &\geq -C_1a^{-2} + (\alpha/(2a))AQ'(A)\int_{A/a}^1 1/\sqrt{1-x^2}\,dx \\ &\geq C_2a^{-1} \end{aligned}$$

for a large enough and some $C_2 > 0$.

(ii) Now,

$$\frac{n}{a_nQ'(a_n)}=\frac{2}{\pi}\int_0^1 \frac{a_nxQ'(a_nx)}{a_nQ'(a_n)}\frac{dx}{\sqrt{1-x^2}}.$$

Since (3.6) holds and $uQ'(u)$ is increasing for u large enough, while $uQ'(u)$ remains bounded as $u\to 0$, it is easily seen that there exists $C_3>0$ such that

$$a_nxQ'(a_nx)/(a_nQ'(a_n))\le C_3, \qquad x\in(0,1),$$

and n large enough. Then (3.5) and Lebesgue's dominated convergence theorem show that

$$\lim_{n\to\infty}\frac{n}{a_nQ'(a_n)}=\frac{2}{\pi}\int_0^1\frac{x^\alpha}{\sqrt{1-x^2}}\,dx.$$

In the special case $Q(x)=|x|^\alpha$, this limit relation is still valid, and the left-hand side is $(\alpha\beta_\alpha^\alpha)^{-1}=\lambda_\alpha/\alpha$, see (2.4) and (2.5). Thus (3.10) is valid. Next, by (3.8), we may rewrite (3.10) in the form

$$\lim_{n\to\infty} q_nQ'(q_n)/\{a_nQ'(a_n)\}=\lambda_\alpha/\alpha.$$

Using (3.5) and the monotonicity of $xQ'(x)$, (3.11) follows.

(iii) Now, by Lemma 3.1(iii) and (iv) and by (3.10), we have

$$\begin{aligned}\lim_{n\to\infty} Q(a_nx)/n&=\lim_{n\to\infty}\left(\frac{Q(a_nx)}{Q(a_n)}\right)\left(\frac{Q(a_n)}{a_nQ'(a_n)}\right)\left(\frac{a_nQ'(a_n)}{n}\right)\\&=|x|^\alpha\alpha^{-1}(\alpha/\lambda_\alpha)=(\beta_\alpha|x|)^\alpha,\end{aligned}$$

uniformly in any compact subset of $(0,\infty)$. Thus (3.12) holds uniformly in any compact set not containing 0. It remains to show that (3.12) holds uniformly in $[0,1]$. Let $\varepsilon>0$. Then there exists n_0 such that, for $n\ge n_0$,

$$|Q(a_nx)/n-(\beta_\alpha|x|)^\alpha|\le 3(\beta_\alpha\varepsilon)^\alpha, \qquad \varepsilon\le|x|\le 1.$$

Further, since $Q(x)$ is continuous in $\mathbf{R}$ and increasing to ∞ as $|x|\to\infty$ (by (3.3)), for n large enough and $x\in[0,\varepsilon]$,

$$|Q(a_nx)/n-(\beta_\alpha|x|)^\alpha|\le Q(a_n\varepsilon)/n+(\beta_\alpha\varepsilon)^\alpha\to 2(\beta_\alpha\varepsilon)^\alpha \quad \text{as} \quad n\to\infty.$$

Thus, for some $n_1=n_1(\varepsilon)$, $n\ge n_1$, and $x\in[0,1]$,

$$|Q(a_nx)/n-(\beta_\alpha|x|)^\alpha|\le 3(\beta_\alpha\varepsilon)^\alpha.$$

Then (3.12) follows uniformly in $[0,1]$.

(iv) It follows from (3.10) that

$$\lim_{n\to\infty} a_{n+k_n}Q'(a_{n+k_n})/\{a_nQ'(a_n)\}=\lim_{n\to\infty}(n+k_n)/n=1.$$

Then (3.14) follows easily from (3.5). ∎

Note that if $\alpha>1$, (3.1) shows that $Q'(x)\to\infty$ as $x\to\infty$. Then (3.8) and (3.11) imply that

$$\lim_{n\to\infty} n/q_n=\lim_{n\to\infty} n/a_n=\infty.$$

Similarly, if $\alpha<1$, $Q'(x)\to 0$ as $x\to\infty$, and so

$$\lim_{n\to\infty} n/q_n=\lim_{n\to\infty} n/a_n=0.$$

If $\alpha=1$, consideration of $Q(x)=|x|(\log(2+x^2))^\beta$, with $\beta<0$, $\beta=0$, or $\beta>0$, shows that any of (2.14), (2.16), or (2.17) may hold.

Next, we need a lemma that reduces the L_∞ or L_p norm of a weighted polynomial over an infinite interval to that over a finite interval. If $W=e^{-Q}$, where Q is even and convex, recent results of Mhaskar and Saff [29], [31] are almost immediately applicable. However, when Q is nonconvex, we have to use extra arguments, which explains the lengthy procedure below.

Throughout, given $\alpha>0$, $v(\alpha;t)$ denotes the Ullman distribution, defined by

$$v(\alpha;x):=\frac{\alpha}{\pi}\int_{|x|}^{1} t^{\alpha-1}/\sqrt{t^2-x^2}\,dt,\qquad |x|<1. \tag{3.15}$$

Associated with $v(\alpha;t)$, there is the function

$$U_\alpha(z):=\int_{-1}^{1}\log|z-t|\,v(\alpha;t)\,dt-|z|^\alpha/\lambda_\alpha+\log 2+1/\alpha, \tag{3.16}$$

which plays an important role in decribing the asymptotic behavior of extremal polynomials associated with weights on $\mathbf{R}$. We first state the inequalities for W_α:

Lemma 3.3. *Let α, $\delta>0$ and $0<p$, $r\le\infty$.*

(i) *There exist $C_1, C_2>0$ such that, for $n=1,2,3,\ldots$ and $P\in\mathscr{P}_n$,*

$$\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_r(\mathbf{R})}\le C_1 n^{C_2}\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(\mathbf{R})}. \tag{3.17}$$

(ii) *Given $\delta>0$, there exists $C=C(\delta)$ and $n_0=n_0(\delta)$ such that, for $n\ge n_0(\delta)$ and $P\in\mathscr{P}_n$,*

$$\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(|x|\ge 1+\delta)}\le e^{-Cn}\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p[-1-\delta,1+\delta]}, \tag{3.18}$$

and

$$\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(\mathbf{R})}\le (1+e^{-Cn})\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p[-1-\delta,1+\delta]}. \tag{3.19}$$

Proof. (i) This follows from Theorem 3.1 in [27, p. 213], with a substitution in the expressions defining the L_p and L_r norms.

(ii) By Theorem 2.2 in [27, p. 208], with $a=(n/\lambda_\alpha)^{1/\alpha}$ and a suitable substitution, and by (2.9) in [27, p. 207], we see that

$$|P(x)W_\alpha^n(\beta_\alpha x)|\le\|P(u)W_\alpha^n(\beta_\alpha u)\|_{L_\infty[-1,1]}\exp(nU_\alpha(x)) \tag{3.20}$$

for $P\in\mathscr{P}_n$ and $|x|>1$. Next, by Theorem 2.6 in [27, p. 209], $U_\alpha(x)<0$ for $|x|>1$. Further, it is clear from (3.16) that $U_\alpha(x)\to-\infty$ as $|x|\to\infty$. Thus, if

$$C:=C(\delta):=\min\{-U_\alpha(x): |x|\ge 1+\delta\},$$

we have, for $P\in\mathscr{P}_n$ and $|x|\ge 1+\delta$,

$$|P(x)W_\alpha^n(\beta_\alpha x)|\le e^{-Cn}\|P(u)W_\alpha^n(\beta_\alpha u)\|_{L_\infty[-1,1]}. \tag{3.21}$$

Thus (3.18) and (3.19) are valid for $p=\infty$. Next, let $p<\infty$. It follows from Theorem A in [15, p. 264] or results in [27, pp. 230-231] (see also [31]) that there exists n_0, $K>0$ and $0<\eta<1$ such that, for $n\geq n_0$ and $P\in\mathcal{P}_n$,

$$\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(|x|\geq K)}\leq\eta^n\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(\mathbf{R})}. \tag{3.22}$$

Next, from (3.21),

$$\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(1+\delta\leq|x|\leq K)}\leq e^{-Cn}\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_\infty[-1,1]}(2K)^{1/p}$$
$$\leq C_3n^{C_2}e^{-Cn}\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(\mathbf{R})},$$

by (i) of this lemma. Together with (3.22), this yields

$$\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(|x|\geq 1+\delta)}\leq e^{-C_4n}\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(\mathbf{R})}. \tag{3.23}$$

Then (3.18) and (3.19) follow. ■

Lemma 3.4. *Let $I:=[a,b]$ be a compact interval. Then, if $d(x,I):=\min\{|x-a|,|x-b|\}$ denotes the distance from $x\in\mathbf{R}\backslash I$ to I, we have, for $P\in\mathcal{P}_n$ and $x\notin I$,*

$$|P(x)|\leq(1+8\{d(x,I)/(b-a)+(d(x,I)/(b-a))^{1/2}\})^n\|P\|_{L_\infty(I)}. \tag{3.24}$$

Proof. If $I=[-1,1]$, the Walsh-Bernstein lemma [47] asserts that, for $P\in\mathcal{P}_n$ and $x\notin I$,

$$|P(x)|\leq|x+\sqrt{x^2-1}|^n\|P\|_{L_\infty(I)}.$$

Here the branch of the square root is chosen so that $|x+\sqrt{x^2-1}|>1$ for $|x|>1$. For general intervals $I:=[a,b]$, we see from the above that

$$|P(x)|\leq|u+\sqrt{u^2-1}|^n\|P\|_{L_\infty(I)},\qquad x\notin I,$$

where

$$u:=-1+2(x-a)/(b-a)$$

maps I onto $[-1,1]$. It is easily seen that

$$d(u,[-1,1])=(2/(b-a))d(x,I),$$

and

$$|u+\sqrt{u^2-1}|\leq 1+4d(u,[-1,1])+4d(u,[-1,1])^{1/2}.$$

Then (3.24) follows. ■

We next generalize parts of Lemma 3.3. Note that the conditions on W below are equivalent to $\hat{W}$ being nonnegative, L_p integrable over each finite interval, and satisfying

$$\lim_{|x|\to\infty}\log\hat{W}(x)/Q(x)=-1.$$

Lemma 3.5. *Let $W \in \mathrm{VSF}(\alpha)$ for some $\alpha > 0$, and let $\delta > 0$ and $0 < p \leq \infty$. Let $J(x)$ be a nonnegative function in $\mathbf{R}$, whose restriction to each finite interval I belongs to $L_p(I)$ and assume also that*

$$\lim_{|x|\to\infty} (\log J(x))/Q(x) = 0. \tag{3.25}$$

Let $a_n = a_n(W)$ for n large enough, and

$$\hat{W}(x) := W(x)J(x), \qquad x \in \mathbf{R}. \tag{3.26}$$

Let $\{\varepsilon_n\}_1^\infty \subset (0, \infty)$ satisfy

$$\lim_{n\to\infty} \varepsilon_n = 0. \tag{3.27}$$

Then there exist n_0, $C > 0$ and $\{\rho_n\}_1^\infty \subset (0, \infty)$ with the following properties:

$$\lim_{n\to\infty} \rho_n^{1/n} = 1. \tag{3.28}$$

For $n \geq n_0$ and $P \in \mathcal{P}_n$,

$$\| P(x)\hat{W}(a_n x) \|_{L_p(|x| \leq \varepsilon_n)} \leq \rho_n \| P(x)\hat{W}(a_n x) \|_{L_p(\varepsilon_n \leq |x| \leq 1)}, \tag{3.29}$$

$$\| P(x)\hat{W}(a_n x) \|_{L_p(|x| \geq 1+\delta)} \leq e^{-Cn} \| P(x)\hat{W}(a_n x) \|_{L_p(\varepsilon_n \leq |x| \leq 1+\delta)}, \tag{3.30}$$

and

$$\| P(x)\hat{W}(a_n x) \|_{L_p(\mathbf{R})} \leq \rho_n \| P(x)\hat{W}(a_n x) \|_{L_p(\varepsilon_n \leq |x| < 1+\delta)}. \tag{3.31}$$

Proof. For the purposes of this proof, we say that $\{\rho_n\}_1^\infty$ is of class $\mathcal{A}$ if $\{\rho_n\}_1^\infty \subset (0, \infty)$ and (3.28) holds. Note first that (3.31) follows easily from (3.29) and (3.30), possibly with initially a different $\{\rho_n\}_1^\infty$ in (3.31) to that in (3.29). To prove (3.29) we first note, from Lemmas 3.1 and 3.2, that, for any fixed positive integer l,

$$Q(la_n) \asymp Q(a_{ln}) \asymp n \asymp Q(q_{ln}) \asymp Q(q_n). \tag{3.32}$$

In view of (3.25), we may choose B so large that

$$|\log J(x)|/Q(x) \leq 1, \qquad |x| \geq B.$$

It then also follows from (3.25) and (3.32) that, for any $K > 0$,

$$\lim_{n\to\infty} (\sup\{J^{\pm 1}(x)\colon B \leq |x| \leq Ka_n\})^{1/n} = 1. \tag{3.33}$$

We shall assume, without loss of generality, that $\{\varepsilon_n\}_1^\infty$ approaches 0 so slowly that

$$\varepsilon_n a_n \geq B, \quad n \text{ large enough}. \tag{3.34}$$

Next, for some n_0, $n \geq n_0$ and $P \in \mathcal{P}_n$,

$$\| P(x)\hat{W}(a_n x) \|_{L_p(|x| \leq \varepsilon_n)} \leq \| P \|_{L_\infty(|x| \leq \varepsilon_n)} \| \hat{W} \|_{L_p(\mathbf{R})}. \tag{3.35}$$

Now, from (3.27) and Lemmas 3.1 and 3.2, we see that

$$\lim_{n\to\infty} Q(a_n\varepsilon_n)/n = \lim_{n\to\infty} Q(a_n\varepsilon_n^{1/2})/n = 0. \tag{3.36}$$

Applying Lemma 3.4 to $I := I_n := [\varepsilon_n, \varepsilon_n + \varepsilon_n^{1/2}]$ and then Lemma 6.3.11 in [33, p. 111], we see that

$$\begin{aligned}\|P\|_{L_\infty(|x|\le\varepsilon_n)} &\le (1+8\{(2\varepsilon_n^{1/2})+(2\varepsilon_n^{1/2})^{1/2}\})^n \|P\|_{L_\infty(I_n)}\\ &\le (1+16\varepsilon_n^{1/4})^n \sup\{\hat{W}^{-1}(x): \varepsilon_n a_n \le |x| \le (\varepsilon_n+\varepsilon_n^{1/2})a_n\}\\ &\quad \times C'n^{2/p}\|P(x)\hat{W}(a_n x)\|_{L_p(\varepsilon_n\le|x|\le1)}\\ &=: \rho_n'\|P(x)\hat{W}(a_n x)\|_{L_p(\varepsilon_n\le|x|\le1)}.\end{aligned}$$

Here, by (3.33), (3.34), and (3.36), we see that $\{\rho_n'\}_1^\infty$ is of class $\mathscr{A}$. Combining this last inequality with (3.35), we obtain (3.29).

To prove (3.30) we first note from Lemma 3.4 that, for some n_1, $n \ge n_1$, $P \in \mathscr{P}_n$, and $|x| \ge 2$,

$$\begin{aligned}|P(x)| &\le (1+16|x|)^n \|P\|_{L_\infty(\varepsilon_n\le|x|\le1)}\\ &\le (20|x|)^n \|P\|_{L_p(\varepsilon_n\le|x|\le1)},\end{aligned}$$

by standard Nikolskii inequalities [33, p. 114 f]. From (3.32) to (3.34), we see there exists n_1 and $C > 1$ such that

$$\hat{W}^{-1}(a_n x) \le C^n, \qquad \varepsilon_n \le |x| \le 1, \qquad n \ge n_1.$$

Hence, given $K > 2$,

$$\begin{aligned}(3.37)\quad &\|P(u)\hat{W}(a_n u)\|_{L_p(|u|\ge K)}\\ &\le (20C)^n \|u^n\hat{W}(a_n u)\|_{L_p(|u|\ge K)} \|P(x)\hat{W}(a_n x)\|_{L_p(\varepsilon_n\le|x|\le1)}.\end{aligned}$$

Here, in view of (3.25), for n large enough,

$$\begin{aligned}(3.38)\quad \|u^n\hat{W}(a_n u)\|_{L_p(|u|\ge K)} &\le \|u^n W^{1/2}(a_n u)\|_{L_p(|u|\ge K)}\\ &\le \|u^{4n}W(a_n u)\|_{L_\infty(\mathbf{R})}^{1/2} \|u^{-n}\|_{L_p(|u|\ge K)}\\ &= a_n^{-2n}(q_{4n}^{4n}W(q_{4n}))^{1/2}\{K^{1-np}/(1-np)\}^{1/p}.\end{aligned}$$

Since, from Lemma 3.1 and 3.2,

$$a_n \asymp q_n \asymp q_{4n},$$

we can choose K so large that the right-hand side of (3.38) is bounded above by $(40C)^{-n}$. Thus, from (3.37),

$$(3.39)\qquad \|P(u)\hat{W}(a_n u)\|_{L_p(|u|\ge K)} \le 2^{-n}\|P(u)\hat{W}(a_n u)\|_{L_p(\varepsilon_n\le|u|\le1)}.$$

Next, by (3.25) and Lemma 3.2(iii),

$$\lim_{n\to\infty} \hat{W}(a_n u)^{1/n} = W_\alpha(\beta_\alpha u) \qquad \text{uniformly for} \quad 1+\delta \le |u| \le K.$$

Hence, for some $\{\rho_n'\}_1^\infty$ of class $\mathscr{A}$, we have

$$\begin{aligned}(3.40)\quad \|P(u)\hat{W}(a_n u)\|_{L_p(1+\delta\le|u|\le K)} &\le \rho_n'\|P(u)W_\alpha^n(\beta_\alpha u)\|_{L_p(1+\delta\le|u|\le K)}\\ &\le \rho_n' e^{-Cn}\|P(u)W_\alpha^n(\beta_\alpha u)\|_{L_p(|u|\le1+\delta)},\end{aligned}$$

by (3.18) in Lemma 3.3. Again, by Lemma 3.2(iii), there exists $\{\rho_n''\}_1^\infty$ of class $\mathscr{A}$ such that

$$\|P(u)W_\alpha^n(\beta_\alpha u)\|_{L_p(|u|\le 1+\delta)} \le \rho_n'' \|P(u)W(a_n u)\|_{L_p(|u|\le 1+\delta)} \le \rho_n''' \|P(u)W(a_n u)\|_{L_p(\varepsilon_n\le|u|\le 1+\delta)} \tag{3.41}$$

for some $\{\rho_n'''\}_1^\infty$ of class $\mathscr{A}$, by (3.29) applied to W, rather than to $\hat{W}$. In view of (3.33) and (3.34), we have, for some $\{\hat{\rho}_n\}_1^\infty$ of class $\mathscr{A}$,

$$\|P(u)W(a_n u)\|_{L_p(\varepsilon_n\le|u|\le 1+\delta)} \le \hat{\rho}_n \|P(u)\hat{W}(a_n u)\|_{L_p(\varepsilon_n\le|u|\le 1+\delta)}. \tag{3.42}$$

Combining (3.40) to (3.42), we have, for some $C'>0$,

$$\|P(u)\hat{W}(a_n u)\|_{L_p(1+\delta\le|u|\le K)} \le e^{-C'n} \|P(u)\hat{W}(a_n u)\|_{L_p(\varepsilon_n\le|u|\le 1+\delta)}$$

for $n\ge n_1$ and $P\in\mathscr{P}_n$. Together with (3.39), this yields (3.30). ■

For future use, we record the following lemma, whose proof is essentially contained in that of Lemma 3.5.

Lemma 3.6. *Assume that p, W, J and $\hat{W}$ are as in Lemma 3.5. Let $0<r\le\infty$. Then there exists n_0 and a sequence $\{\rho_n\}_1^\infty$ satisfying (3.28) such that, for $n\ge n_0$, $P\in\mathscr{P}_n$, and $P\not\equiv 0$,*

$$\rho_n^{-1} \le \|P(x)W(a_n x)\|_{L_p(\mathbf{R})} / \|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(\mathbf{R})} \le \rho_n, \tag{3.43}$$

$$\rho_n^{-1} \le \|P(x)W(a_n x)\|_{L_p(\mathbf{R})} / \|P(x)W(a_n x)\|_{L_r(\mathbf{R})} \le \rho_n, \tag{3.44}$$

and

$$\rho_n^{-1} \le \|P(x)\hat{W}(a_n x)\|_{L_p(\mathbf{R})} / \|P(x)W(a_n x)\|_{L_p(\mathbf{R})} \le \rho_n. \tag{3.45}$$

Proof. Since the L_p norms of $P(x)W(a_n x)$ and $P(x)W_\alpha^n(\beta_\alpha x)$ both "live" on $[-(1+\delta), 1+\delta]$, in the sense outlined in Lemmas 3.3 and 3.5, and since Lemma 3.2(iii) holds, (3.43) follows easily. Next, (3.44) is true for the special case $W=W_\alpha$, by Lemma 3.3. Then (3.43) implies (3.44) in the general case. To obtain (3.45), we note first that, for $\{\varepsilon_n\}_1^\infty$ satisfying (3.27) and (3.34),

$$\{\hat{W}(a_n x)/W(a_n x)\}^{1/n} = J(a_n x)^{1/n} \to 1 \quad \text{as} \quad n\to\infty,$$

uniformly for $\varepsilon_n\le|x|\le 1+\delta$. Since the L_p norms of both $P(x)\hat{W}(a_n x)$ and $P(x)W(a_n x)$ "live" on $\{x: \varepsilon_n\le|x|\le 1+\delta\}$, in the sense of (3.31), we then easily obtain (3.45). ■

We shall need various properties of certain L_p extremal polynomials associated with $\hat{W}$, primarily for $p=2$ and $p=\infty$. Accordingly, if p, W, $\hat{W}$, and J are as in Lemma 3.5 and $a_n=a_n(W)$ for n large enough, we set

$$E_{n,p}(\hat{W}) := \inf_{P\in\mathscr{P}_{n-1}} \|(x^n+P(x))\hat{W}(a_n x)\|_{L_p(\mathbf{R})}. \tag{3.46}$$

Further, we let $T_{n,p}(\hat{W}, x)$ denote any monic polynomial of degree n satisfying

$$\|T_{n,p}(\hat{W}, x)\hat{W}(a_n x)\|_{L_p(\mathbf{R})} = E_{n,p}(\hat{W}). \tag{3.47}$$

It is known that $T_{n,p}(\hat{W}, x)$ exists (if a_n does) and has all real zeros. The zero distribution of extremal polynomials closely related to those of $T_{n,p}(\hat{W}, x)$ has been studied by Mhaskar and Saff [27]-[29], Rahmanov [39], and Goncar and Rahmanov [11]. For n large enough, and $\mathscr{S} \subset \mathbf{R}$, we let

$$\nu_n(\mathscr{S}) := (\text{number of zeros of } T_{n,p}(\hat{W}, x) \text{ in } \mathscr{S})/n.$$

Using the results of [27], we prove:

Lemma 3.7. *Assume that p, W, J, and $\hat{W}$ are as in Lemma 3.5. Then*

(i)

$$\lim_{n\to\infty} E_{n,p}(\hat{W})^{1/n} = \lim_{n\to\infty} E_{n,\infty}(W_\alpha)^{1/n} = e^{-1/\alpha}/2. \tag{3.48}$$

(ii) *If $-Y_n$ and X_n denote the smallest and largest zeros of $T_{n,p}(\hat{W}, x)$, then*

$$\lim_{n\to\infty} Y_n = \lim_{n\to\infty} X_n = 1. \tag{3.49}$$

(iii) *For each function $f(x)$ bounded in $[-2, 2]$ and continuous, except possibly for finitely many discontinuities,*

$$\lim_{n\to\infty} \int_{-2}^{2} f(x)\, d\nu_n(x) = \int_{-1}^{1} f(x) v(\alpha; x)\, dx, \tag{3.50}$$

where $v(\alpha; x)$ is defined by (3.15).

(iv) *Uniformly in compact subsets of* $\mathbf{C}\backslash[-1, 1]$,

$$\lim_{n\to\infty} |T_{n,p}(\hat{W}, z)\, \hat{W}(a_n|z|)/E_{n,p}(\hat{W})|^{1/n} = \exp(U_\alpha(z)). \tag{3.51}$$

Proof. (i) If $P_n(x)$ is a monic polynomial of degree n, $n = 1, 2, 3, \ldots,$ it follows from Lemma 3.6 that

$$\lim_{n\to\infty} \{\| P_n(x)\, \hat{W}(a_n x) \|_{L_p(\mathbf{R})} / \| P_n(x)\, W_\alpha^n(\beta_\alpha x) \|_{L_\infty(\mathbf{R})}\}^{1/n} = 1.$$

Since

$$W_\alpha^n(\beta_\alpha x) = W_\alpha(a_n(W_\alpha)x), \qquad x \in \mathbf{R},$$

it follows that

$$\lim_{n\to\infty} E_{n,p}(\hat{W})^{1/n} = \lim_{n\to\infty} E_{n,\infty}(W_\alpha)^{1/n}.$$

This establishes the first equation in (3.48). The second follows from Theorem 2.9 in [27, p. 211] and a substitution.

(ii) and (iii) Assume that for some $\delta > 0$, and some infinite sequence of positive integers $\mathscr{N}$,

$$X_n \geq 1 + 3\delta, \qquad n \in \mathscr{N}.$$

Define, for $n \in \mathscr{N}$,

$$P_n(x) := \{(x - (1 + 2\delta))/(x - X_n)\}\, T_{n,p}(\hat{W}, x),$$

a monic polynomial of degree n. We see that the expression in { } has value in $(0, 1)$ for $x \in [-(1+\delta), 1+\delta]$, and increases to its maximum value at $x = -(1+\delta)$. Thus

$$\max_{|x| \le 1+\delta} |(x-(1+2\delta))/(x-X_n)| = (2+3\delta)/(1+\delta+X_n)$$

$$\le (2+3\delta)/(2+4\delta) =: \eta < 1$$

for $n \in \mathcal{N}$. Then, for some $C' > 0$, (3.30) in Lemma 3.5 implies that

$$\begin{aligned} \| P_n(x)\hat{W}(a_n x) \|_{L_p(\mathbf{R})} &\le (1+e^{-C'n}) \| P_n(x)\hat{W}(a_n x) \|_{L_p[-(1+\delta),1+\delta]} \\ &\le (1+e^{-C'n})\eta \| T_{n,p}(\hat{W}, x)\hat{W}(a_n x) \|_{L_p[-(1+\delta),1+\delta]} \\ &\le (1+e^{-C'n})\eta E_{n,p}(\hat{W}). \end{aligned}$$

For large enough n, this contradicts the definition of $E_{n,p}(\hat{W})$. Thus

(3.52) $$\limsup_{n\to\infty} X_n \le 1.$$

The corresponding lim inf will be established below. Next, by Lemma 3.6 and (i) of this lemma,

$$\lim_{n\to\infty} \| W_\alpha^n(\beta_\alpha x) T_{n,p}(\hat{W}, x) \|_{L_\infty(\mathbf{R})}^{1/n} = \lim_{n\to\infty} \| \hat{W}(a_n x) T_{n,p}(\hat{W}, x) \|_{L_p(\mathbf{R})}^{1/n} = e^{-1/\alpha}/2.$$

Then Theorem 3.6 in [27, p. 215] asserts that, for $[c, d] \subset [-2, 2]$,

(3.53) $$\lim_{n\to\infty} \int_c^d d\nu_n(t) = \int_c^d v(\alpha; t)\, dt.$$

Then (3.50) follows. Further, as $v(\alpha; t)$ is positive in $(-1, 1)$, it follows from (3.53) that 1 is a limit of zeros of $T_{n,p}(\hat{W}, x)$, so

$$\liminf_{n\to\infty} X_n \ge 1.$$

Together with (3.52), this establishes the second equation in (3.49). The first follows similarly.

(iv) This is an easy consequence of the definition (3.16) of $U_\alpha(z)$ and the limit relations (3.12), (3.33), (3.48), (3.49), and (3.50). ∎

We remark that if $p = 2$, (3.49) implies that the largest zero x_{1n} of the nth orthonormal polynomial associated with the weight $\hat{W}^2$ on $\mathbf{R}$ satisfies

(3.54) $$\lim_{n\to\infty} x_{1n}/a_n = 1.$$

In the special case $Q(x) = |x|^\alpha$, this result was established by Rahmanov [39]. Since Definition 2.3 allows more general growth of $Q(x)$ at ∞, such as

$$Q(x) = |x|^\alpha (\log(2+x^2))^\beta, \qquad \beta \in \mathbf{R},$$

(3.54) extends this result of [39]. For the weight $\exp(-x^m)$, m a positive even integer, sharper asymptotics have been obtained by Máté, Nevai, and Totik [24].

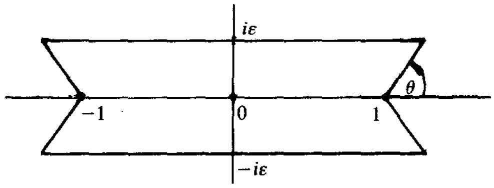

Fig. 3.1. $\mathscr{G}(\varepsilon;\theta)$.

We next define a region in which we can prove that $U_\alpha(z)$ is positive:

Definition 3.8 (Figure 3.1). Given $\varepsilon>0$ and $0<\theta<\pi$, we define

$$\begin{aligned}(3.55)\qquad \mathscr{G}(\varepsilon;\theta):=&\{x+iy: |x|\le 1, 0<|y|<\varepsilon\}\\ &\cup\{\pm(1+ye^{it}): t\in(\theta,\pi/2) \text{ and } 0<|y\sin t|<\varepsilon\}.\end{aligned}$$

Theorem 3.9. *Let $\alpha>0$ and $0<\delta<\pi/6$. There exists $\eta>0$ with the following property: let K be any compact subset of $\mathscr{G}(\eta;\pi/3+\delta)$. Then*

$$(3.56)\qquad \min\{U_\alpha(z): z\in K\}>0.$$

Proof. The detailed calculations required for this appear in the Appendix. ■

The final part of this section concerns an entire function that, along the real axis, behaves like the reciprocal of a weight. Let $W\in\mathrm{VSF}(\alpha)$ for some $\alpha>0$. Recall that q_n is defined by (3.8) for n large enough. For the finitely many remaining positive integers, we set $q_n=1$. Define

$$(3.57)\qquad G_Q(x):=1+\sum_{n=1}^{\infty}(x/q_n)^{2n}n^{-1/2}\exp(2Q(q_n)).$$

Lemma 3.10. *Let $W\in\mathrm{VSF}(\alpha)$, for some $\alpha>0$. Then*

$$(3.58)\qquad \lim_{|x|\to\infty} G_Q(x)W^2(x)=(\pi\alpha)^{1/2}.$$

Proof. The conditions on Q in Definition 2.3 ensure that W is a "smooth Freud weight" in the sense used in [18]. Then Theorem 5 in [18] shows that, as $|x|\to\infty$,

$$G_Q(x)=\{\pi T(x)\}^{1/2}\exp(2Q(x))\{1+O(Q(x)^{-1/2}(\log x)^{3/2})\},$$

where

$$T(x):=1+xQ''(x)/Q'(x).$$

By (2.13), $T(x)\to\alpha$ as $|x|\to\infty$. Then (3.58) follows. ■

We shall need an entire function growing like $W^{-1}(x)$, rather than $W^{-2}(x)$:

Lemma 3.11. *Let $W=e^{-Q}\in\mathrm{VSF}(\alpha)$ for some $\alpha>0$. Let*

$$(3.59)\qquad H_Q(x):=G_{Q/2}(x)(\pi\alpha)^{-1/2},$$

where $G_{Q/2}(x)$ *is the entire function associated with* $e^{-Q/2}$ *by* (3.57). *Let*

$$\phi(x) := \{H_Q(x)W(x)\}^{-1}, \qquad x \in \mathbf{R}. \tag{3.60}$$

Then

$$\lim_{|x|\to\infty} \phi(x) = 1, \tag{3.61}$$

and there exists $b > 0$ *such that*

$$b^{-1} \le \phi(x) \le b, \qquad x \in \mathbf{R}. \tag{3.62}$$

Proof. Note first that $e^{-Q/2} \in \mathrm{VSF}(\alpha)$ also. Then (3.58) and (3.59) yield (3.61). As both $H_Q(x)$ and $W(x)$ are positive and continuous in $\mathbf{R}$, (3.62) follows immediately. ■

Note that Lemmas 3.10 and 3.11 are the only results in this section whose proofs require the existence of $Q'''(x)$ and (2.12). Further, even if (2.12) does not hold and $Q'''(x)$ is not assumed to exist, (3.62) remains valid, since both $W(x)$ and $W^{1/2}(x)$ are still "Freud weights" in the sense of [18].

4. Uniform Approximation by $P_n(x)/H_Q(a_n x)$

The aim of this section is to prove

Theorem 4.1. *Let* $W := e^{-Q} \in \mathrm{VSF}(\alpha)$ *for some* $\alpha > 0$. *Let* $H_Q(x)$ *be the associated entire function defined by* (3.59). *Let* $h(x)$ *be continuous in* $\mathbf{R}$, *with* $h(x) = 0$ *for* $|x| > 1$. *Then there exists* $P_n \in \mathcal{P}_n$, $n = 1, 2, 3, \ldots$, *such that*

$$\lim_{n\to\infty} \| h(x) - P_n(x)/H_Q(a_n x) \|_{L_\infty(\mathbf{R})} = 0. \tag{4.1}$$

We remark that the proof of the above result does not require the existence of $Q'''(x)$ or (2.12) to hold, since it requires only (3.62) in Lemma 3.11. Further, the specific form of $H_Q(x)$ is not used, and Theorem 4.1 is true for any even entire function $H(x)$ with nonnegative Maclaurin series coefficients satisfying

$$H(x)W(x) \asymp 1, \qquad x \in \mathbf{R}.$$

Our strategy in approximating a function $h(x)$ is as follows: we form the Lagrange interpolation polynomial $L_n(x)$ of degree at most $n-1$, which interpolates to $h(x)H_Q(a_n x)$ at the zeros of $T_{n,\infty}(W, x)$, and then expect that $L_n(x)/H_Q(a_n x)$ is a good approximation to $h(x)$. This turns out to be correct. To show that the error in approximation approaches zero as n approaches ∞, we use the Hermite contour integral error formula for Lagrange interpolation. Because of the delicate nature of the error estimation, we need to choose the contour integral in the Hermite formula very carefully:

Definition 4.2. Let $W \in \mathrm{VSF}(\alpha)$, for some $\alpha > 0$. For n large enough, let ξ_n denote the largest positive number for which

$$|T_{n,\infty}(W, \xi_n) W(a_n \xi_n)| = E_{n,\infty}(W). \tag{4.2}$$

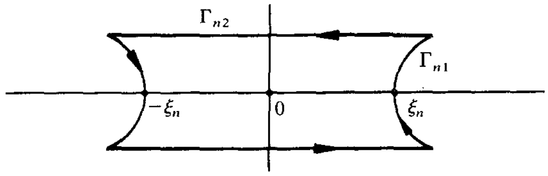

Fig. 4.1. Γ_n.

Let $0 < \varepsilon < 1$, and let

$$\Gamma_{n1} := \{(\xi_n^2 + iy)^{1/2} : 0 \le y \le \varepsilon\} \tag{4.3}$$

for n large enough, where the branch of the square root is the principal one. Let $-$ denote complex conjugation and Γ_{n2} denote the horizontal line segment joining Γ_{n1} and $(-\bar{\Gamma}_{n1})$, so that

$$\Gamma_{n2} := \{z : |\mathrm{Re}(z)| \le \mathrm{Re}(\xi_n^2 + i\varepsilon)^{1/2} \quad \text{and} \quad \mathrm{Im}(z) = \mathrm{Im}(\xi_n^2 + i\varepsilon)^{1/2}\}. \tag{4.4}$$

Finally, let

$$\Gamma_n := \Gamma_{n1} \cup \bar{\Gamma}_{n1} \cup (-\Gamma_{n1}) \cup (-\bar{\Gamma}_{n1}) \cup \Gamma_{n2} \cup \bar{\Gamma}_{n2}, \tag{4.5}$$

oriented in a positive sense (Figure 4.1).

Note that, as is well known, $T_{n,\infty}(x)\, W(a_n x)$ has $n+1$ points of equi-oscillation, and ξ_n above denotes the largest. Let $W \in \mathrm{VSF}(\alpha)$ for some $\alpha > 0$. Since ξ_n is to the right of the largest zero of $T_{n,\infty}(W, x)$, Lemma 3.7(ii) ensures that

$$\liminf_{n\to\infty} \xi_n \ge 1.$$

Further, (3.30) in Lemma 3.5 applied to $\hat{W} = W$ and $p = \infty$ ensures that given $\delta > 0$,

$$\xi_n < 1 + \delta, \quad n \text{ large enough.}$$

Thus

$$\lim_{n\to\infty} \xi_n = 1. \tag{4.6}$$

We remark that when $Q(x)$ is even and convex, results of Mhaskar and Saff [29] ensure, in addition to (4.6), that $\xi_n \le 1$ for n large enough.

Lemma 4.3. *Let $W \in \mathrm{VSF}(\alpha)$ for some $\alpha > 0$. Let $g(t)$ be analytic for $|t| \le 2$ and real valued in $(-2, 2)$. For n large enough, let*

$$h_n(x) := \begin{cases} g(x)(x^2 - \xi_n^2), & x \in (-\xi_n, \xi_n), \\ 0, & x \in \mathbf{R}\backslash(-\xi_n, \xi_n). \end{cases} \tag{4.7}$$

Further, let $L_n(x)$ denote the Lagrange interpolation polynomial of degree at most $n-1$ to $h_n(x) H_Q(a_n x)$, in the zeros of $T_{n,\infty}(W, x)$. Then, for n large enough and ε small enough, there exists C_1 (depending on ε but not on n or g) such that

$$\| h_n(x) - L_n(x)/H_Q(a_n x) \|_{L_\infty(\mathbf{R})} \tag{4.8}$$

$$\le C_1 \max\{|g(t)| : |t| \le 2\} \int_{\Gamma_{n1} \cup \Gamma_{n2}} E_{n,\infty}(W)/|T_{n,\infty}(W, t)\, W(a_n |t|)| \, |dt|.$$

Proof. Inside and on Γ_n, we continue $h_n(x)$ to the complex plane, using the first of the formulae in (4.7). The Hermite error formula for Lagrange interpolation states that, for $x \in (-\xi_n, \xi_n)$,

$$H_Q(a_n x)h_n(x) - L_n(x) = \frac{1}{2\pi i}\int_{\Gamma_n} \frac{h_n(t)H_Q(a_n t)}{t-x}\,\frac{T_{n,\infty}(W,x)}{T_{n,\infty}(W,t)}\,dt. \tag{4.9}$$

Further, the contour integral representation for $L_n(x)$ [47] shows that, for $|x| > \xi_n$,

$$-L_n(x) = \frac{1}{2\pi i}\int_{\Gamma_n} \frac{h_n(t)H_Q(a_n t)}{t-x}\,\frac{T_{n,\infty}(W,x)}{T_{n,\infty}(W,t)}\,dt.$$

Then, taking account of the definition (4.7) of $h_n(x)$, we see that (4.9) holds for all real $x \neq \pm\xi_n$. Next, as H_Q has nonnegative Maclaurin series coefficients, we see that, for $t \in \mathbf{C}$ and $x \in \mathbf{R}$,

$$\begin{aligned} |H_Q(a_n t)/H_Q(a_n x)| &\le H_Q(a_n|t|)/H_Q(a_n x) \\ &\le b^2 W(a_n x)/W(a_n|t|), \end{aligned} \tag{4.10}$$

by Lemma 3.11. Then, by (4.9), (4.10), and continuity of $h_n(x)$, $L_n(x)$, and $H_Q(x)$, we see that

$$\begin{aligned} &\|h_n(x) - L_n(x)/H_Q(a_n x)\|_{L_\infty(\mathbf{R})} \\ &\quad \le b^2(2\pi)^{-1}\max\{|g(t)(t^2-\xi_n^2)/(t-x)| : t \in \Gamma_n, x \in \mathbf{R}\} \\ &\quad \times \int_{\Gamma_n} E_{n,\infty}(W)/|T_{n,\infty}(W,t)W(a_n|t|)|\,|dt|. \end{aligned} \tag{4.11}$$

Recall that (4.6) is valid. Then, for n large enough, $x \in \mathbf{R}$, and $t \in \Gamma_{n1}$, say $t = (\xi_n^2 + iy)^{1/2}$,

$$\begin{aligned} |t^2 - \xi_n^2|/|t-x| &\le (|t| + |\xi_n|)\,|t-\xi_n|/|t-x| \\ &\le 4|(1+iy/\xi_n^2)^{1/2} - 1|/|\mathrm{Im}(1+iy/\xi_n^2)^{1/2}| \\ &\le 8 \end{aligned}$$

for $0 < y \le \varepsilon$, by the binomial expansion, provided only that ε is small enough. Similar estimates hold for $x \in \mathbf{R}$ and $t \in \bar{\Gamma}_{n1} \cup (-\Gamma_{n1}) \cup (-\bar{\Gamma}_{n1})$. Next, for $x \in \mathbf{R}$ and $t \in \Gamma_{n2} \cup \bar{\Gamma}_{n2}$,

$$|t^2 - \xi_n^2|/|t-x| \le 6/|\mathrm{Im}(\xi_n^2 + i\varepsilon)^{1/2}| \le 24/\varepsilon$$

for n large enough and ε small enough. Combined with the symmetry in the definition of Γ_n, and the fact that $T_{n,\infty}(W,t)W(a_n|t|)$ is even and real valued, these last two estimates and (4.11) yield (4.8). ∎

We turn next to estimation of the integral in the right-hand side of (4.8). Note that the integral is independent of g.

Lemma 4.4. *Assume the hypotheses and notation of Lemma 4.3. For ε small enough, there exists $0 < \hat{\delta} < 1$ and $C > 0$ such that, for n large enough,*

$$\int_{\Gamma_{n2}} E_{n,\infty}(W)/|T_{n,\infty}(W,t)W(a_n|t|)|\,|dt| \le C\hat{\delta}^n. \tag{4.12}$$

Proof. Let $0<\delta<\pi/6$, and let $\eta>0$ be the corresponding number in Theorem 3.9, so that $U_\alpha(z)$ is positive inside $\mathscr{G}(\eta;\ \pi/3+\delta)$. We show that if ε is small enough, Γ_{n2} is contained in $\mathscr{G}(\eta;\ \pi/3+\delta)$. First note that if $\mathscr{G}(\eta;\ \pi/3+\delta)$ contains the endpoints of a horizontal line segment, it also contains the segment—see Figure 3.1. Because of symmetry, it suffices to show that the right endpoint of Γ_{n2}, namely, $(\xi_n^2+i\varepsilon)^{1/2}$, is contained in $\mathscr{G}(\eta;\ \pi/3+\delta)$. Let

$$\begin{aligned} z_n &:= (\xi_n^2+i\varepsilon)^{1/2}-1 \\ &= (\xi_n-1)+i\varepsilon/(2\xi_n)+\varepsilon^2/(4\xi_n^3)+\cdots \end{aligned}$$

for n large enough. Then, for small enough, but fixed, ε and for $n\geq n_1(\varepsilon)$, we see that

$$\arg(z_n):=\arctan(\operatorname{Im}(z_n)/\operatorname{Re}(z_n))\geq \arctan(\varepsilon^{-1})>\pi/3+\delta$$

and

$$|z_n|\leq\varepsilon,$$

provided ε is small enough. Hence, for $n\geq n_1(\varepsilon)$ (compare Figure 3.1),

$$(\xi_n^2+i\varepsilon)^{1/2}=1+z_n\in\mathscr{G}(\eta;\ \pi/3+\delta).$$

The above arguments also show that there is a compact set K of $\mathscr{G}(\eta;\ \pi/3+\delta)$ that contains Γ_{n2} for n large enough. By Lemma 3.7(iv), we have, uniformly for $z\in\Gamma_{n2}\subset K$,

$$\limsup_{n\to\infty}\{E_{n,\infty}(W)/|T_{n,\infty}(W,z)W(a_n|z|)|\}^{1/n}\leq\exp(-\min_K U_\alpha(z))<\hat{\delta}<1,$$

by Theorem 3.9. Then (4.12) follows. ■

Lemma 4.5. *Assume the hypotheses and notation of Lemma* 4.3. *Given* $\delta>0$, *we have, for* ε *small enough and* n *large enough,*

$$I_{n1}:=\int_{\Gamma_{n1}} E_{n,\infty}(W)/|T_{n,\infty}(W,t)W(a_n|t|)|\,|dt|\leq\delta n^{-1/2}. \tag{4.13}$$

Proof. Let

$$s(y):=(\xi_n^2+iy)^{1/2},\qquad 0\leq y\leq\varepsilon.$$

Then, by definition of Γ_{n1} and ξ_n,

$$I_{n1}=\int_0^\varepsilon \frac{T_{n,\infty}(W,\xi_n)W(a_n\xi_n)\,dy}{|T_{n,\infty}(W,s(y))W(a_n|s(y)|)|\,|2s(y)|}. \tag{4.14}$$

Firstly,

$$|s(y)|=(\xi_n^4+y^2)^{1/4}\leq\xi_n+2y^2$$

for $0\leq y\leq\varepsilon$, ε small enough, and $n\geq n_0(\varepsilon)$. Now, as $Q(x)$ is increasing for large enough x, we have, for $0\leq y\leq\varepsilon$ and $n\geq n_0(\varepsilon)$,

$$Q(a_n|s(y)|)-Q(a_n\xi_n)\leq Q(a_n(\xi_n+2y^2))-Q(a_n\xi_n)=2y^2a_nQ'(a_nc)$$

for some c between ξ_n and ξ_n+2y^2, say $c\le 2$. In view of (3.1) and (3.10), we see that, for n large enough,

$$\begin{aligned} a_nQ'(a_nc) &\le 4a_nQ'(2a_n) \\ &\le 2^{\alpha+2}a_nQ'(a_n)\le An, \end{aligned}$$

where $A:=2^{\alpha+3}\alpha/\lambda_\alpha$ depends only on α. Thus

$$W(a_n\xi_n)/W(a_n|s(y)|)\le \exp(2y^2An) \tag{4.15}$$

uniformly for $0\le y\le \varepsilon$ and $n\ge n_0(\varepsilon)$. Next, if $\langle x\rangle$ denotes the greatest integer $\le x$, and $z_{1n}, z_{2n}, \ldots, z_{nn}$ denote the zeros of $T_{n,\infty}(W, x)$, and if $0<r<1$,

$$\begin{aligned} |T_{n,\infty}(W,\xi_n)/T_{n,\infty}(W,s(y))| &= \left\{\prod_{z_{jn}>0} |\xi_n^2-z_{jn}^2|/|\xi_n^2+iy-z_{jn}^2|\right\}\{|\xi_n|/|s(y)|\}^{n-2\langle n/2\rangle} \\ &\le \prod_{z_{jn}>0}(1+\{y/(\xi_n^2-z_{jn}^2)\}^2)^{-1/2} \\ &\le \exp\left(-\tfrac{1}{2}\sum_{0<z_{jn}<1-r}\log\{1+y^2/(1+r-z_{jn}^2)^2\}\right) \end{aligned}$$

for any $0<r<1$, and n so large that $\xi_n^2<1+r$. If $\varepsilon<\varepsilon_0(r)$, we see that, for $0\le y\le\varepsilon$ and $0\le s\le 1-r$,

$$\log\{1+y^2/(1+r-s^2)^2\}\ge(\tfrac{1}{2})y^2/(1+r-s^2)^2.$$

Hence, for $n\ge n_0(r)$, $\varepsilon<\varepsilon_0(r)$, and $0<y\le\varepsilon$,

$$|T_{n,\infty}(W,\xi_n)|/|T_{n,\infty}(W,s(y))|\le\exp\left(-(\tfrac{1}{4})y^2\sum_{0<z_{jn}<1-r}1/(1+r-z_{jn}^2)^2\right). \tag{4.16}$$

Here, by Lemma 3.7(iii),

$$\lim_{n\to\infty} n^{-1}\sum_{0<z_{jn}<1-r}1/(1+r-z_{jn}^2)^2=(\tfrac{1}{2})\int_{-1+r}^{1-r}v(\alpha;t)/(1+r-t^2)^2\,\mathrm{d}t.$$

As $r\to 0+$, Lebesgue's monotone convergence theorem shows that this last integral increases to

$$\int_{-1}^{1}v(\alpha;t)/(1-t^2)^2\,dt.$$

It is easily seen from (3.15) that this last integral diverges (see also (A.5)). Thus, given $\delta>0$, we can find $r>0$ and $n_1(r)$ such that

$$(\tfrac{1}{4})\sum_{0<z_{jn}<1-r}1/(1+r-z_{jn}^2)^2\ge n(2A+\delta^{-2})$$

for $n\ge n_1(r)$. Combining (4.14), (4.15), and (4.16), we obtain

$$I_{n1}\le\int_0^{\varepsilon}\exp(-\delta^{-2}ny^2)\,dy\le\delta n^{-1/2}\int_0^{\infty}\exp(-x^2)\,dx$$

for n large enough. ■

We summarize Lemmas 4.3-4.5 in the following:

Lemma 4.6. *Let $W\in\mathrm{VSF}(\alpha)$ for some $\alpha>0$. Let $g(t)$ be analytic for $|t|\le 2$ and real valued in $(-2,2)$. Let $L_n(x)$ be defined as in Lemma 4.3, for n large enough.*

Let

$$h(x) := \begin{cases} g(x)(x^2-1), & x \in (-1, 1), \\ 0, & |x| \geq 1. \end{cases}$$

Then there exist $C > 0$, $n_0 > 0$ *independent of n and g and* $\{\delta_n\}_1^\infty$ *independent of g such that* $\lim_{n\to\infty} \delta_n = 0$, *and, for* $n \geq n_0$,

$$\| h(x) - L_n(x)/H_Q(a_n x) \|_{L_\infty(\mathbf{R})} \leq C \max\{|g(t)|: |t| \leq 2\}(|1-\xi_n| + \delta_n n^{-1/2}).$$

Proof. With the notation of Lemma 4.3, we see, from Lemmas 4.3–4.5, that, for $n \geq n_1$ and some $\{\delta_n\}_1^\infty$ as above,

$$\| h_n(x) - L_n(x)/H_Q(a_n x) \|_{L_\infty(\mathbf{R})} \leq C_1 \max\{|g(t)|: |t| \leq 2\} \delta_n n^{-1/2}, \tag{4.17}$$

where both C_1 and n_1 are independent of g. Now, for $|x| \leq \min\{\xi_n, 1\}$,

$$\begin{aligned} |h(x) - h_n(x)| &= |g(x)| |1-\xi_n^2| \\ &\leq 3 \max\{|g(t)|: |t| \leq 2\} |1-\xi_n|, \end{aligned} \tag{4.18}$$

provided n is large enough. For $|x|$ between ξ_n and 1, we have

$$\begin{aligned} |h(x) - h_n(x)| &= \max\{|h(x)|, |h_n(x)|\} \\ &\leq \max\{|g(t)|: |t| \leq 2\} |1-\xi_n^2|. \end{aligned} \tag{4.19}$$

Since $h(x)$ and $h_n(x)$ vanish for $|x| \geq \max\{1, \xi_n\}$, we obtain the result from (4.17) to (4.19). ■

Finally, we complete the proof of Theorem 4.1.

Proof of Theorem 4.1. In view of Lemma 4.6 and (4.6), it suffices to show that, given $\varepsilon > 0$, we can find a polynomial $g(x)$ such that

$$\| h(x) - g(x)(x^2-1) \|_{L_\infty[-1,1]} < \varepsilon.$$

Let $S(x)$ be a polynomial such that

$$\| h(x) - S(x) \|_{L_\infty[-1,1]} < \varepsilon/3.$$

Let $P(x) := S(x) - S(-1)(1-x)/2 - S(1)(1+x)/2$. Then

$$\| h(x) - P(x) \|_{L_\infty[-1,1]} < \varepsilon/3 + |S(-1)| + |S(1)| < \varepsilon$$

as $h(\pm 1) = 0$. Further, $P(\pm 1) = 0$ so has the form $g(x)(x^2-1)$, where $g(x)$ is a polynomial. ■

For future use, we record the following lemma:

Lemma 4.7. *Let* $W \in \mathrm{VSF}(\alpha)$ *for some* $\alpha > 0$. *There exists a sequence* $\{\eta_n\}_1^\infty$ *of positive numbers such that*

$$\lim_{n\to\infty} \eta_n = 0, \tag{4.20}$$

and such that, for each function $h(x)$ continuous in $[-1, 1]$, there exists $P_n \in \mathscr{P}_{n-1}$, $n = 2, 3, 4, \dots$, such that

$$\lim_{n\to\infty} \| h(x) - P_n(x)/H_Q(a_n x) \|_{L_\infty(|x| \le \xi_n - \eta_n n^{-1/2})} = 0. \tag{4.21}$$

Proof. It suffices to consider the case where $h(t)$ is analytic for $|t| \le 2$ and real valued in $[-2, 2]$. Let $P_n(x)$ denote the Lagrange interpolation polynomial of degree at most $n-1$ to $h(x)H_Q(a_n x)$ in the zeros of $T_{n,\infty}(W, x)$. Then (4.9) is valid for $x \in (-\xi_n, \xi_n)$, provided we replace h_n by h. Proceeding as in the proof of Lemma 4.3, we obtain, for $|x| < \xi_n$ and with b as in Lemma 3.11,

$$\begin{aligned} &|h(x) - P_n(x)/H_Q(a_n x)| \\ &\quad\le b^2 \max\{|h(t)| : |t| \le 2\} \int_{\Gamma_n} E_{n,\infty}(W)/|(t-x)T_{n,\infty}(W, t)W(a_n|t|)||dt| \\ &\quad\le 4b^2 \max\{|h(t)| : |t| \le 2\} I_n(x), \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} I_n(x) := {}&(\xi_n - |x|)^{-1} \int_{\Gamma_{n1}} E_{n,\infty}(W)/|T_{n,\infty}(W, t)W(a_n|t|)||dt| \\ &+ |\mathrm{Im}(\xi_n^2 + i\varepsilon)^{1/2}|^{-1} \int_{\Gamma_{n2}} E_{n,\infty}(W)/|T_{n,\infty}(W, t)W(a_n|t|)||dt|. \end{aligned} \tag{4.23}$$

Note that $I_n(x)$ is independent of h. Now given $\delta > 0$, Lemmas 4.4 and 4.5 show that there exists $n_0 = n_0(\delta)$ and $\rho = \rho(\delta) \in (0, 1)$ such that, uniformly for $n \ge n_0(\delta)$ and $x \in (-\xi_n, \xi_n)$,

$$\begin{aligned} I_n(x) &\le (\xi_n - |x|)^{-1}\delta n^{-1/2} + \rho^n \\ &\le (\xi_n - |x|)^{-1}(\delta n^{-1/2} + 3\rho^n). \end{aligned}$$

Choosing $\delta = \delta_n > 0$ to tend to zero sufficiently slowly with n, we obtain

$$I_n(x) \le (\xi_n - |x|)^{-1} 2\delta_n n^{-1/2}.$$

If $|x| \le \xi_n - \eta_n n^{-1/2}$, n large enough, where $\{\eta_n\}_1^\infty$ satisfies

$$\delta_n = o(\eta_n), \qquad n \to \infty,$$

we obtain (4.21). ∎

It seems likely that $\xi_n - \eta_n n^{-1/2}$ in (4.21) can be replaced by $1 - C(\log n/n)^{2/3}$ for some $C > 0$.

5. Proof of the Results of Section 2

This section is organized as follows: we first prove Theorem 2.5 and deduce Theorem 2.2. We then prove Theorem 2.4 and deduce Theorem 2.1. Finally, we prove Theorem 2.6.

Proof of the "if" part of Theorem 2.5. We note that it is sufficient to consider the case where $g(x)$ is continuous in $\mathbf{R}$ and vanishes outside $[-1, 1]$. By Theorem 4.1, there exists $P_n \in \mathcal{P}_n$, $n = 1, 2, 3, \ldots$, such that

$$\lim_{n\to\infty} \| g(x) - P_n(x)/H_Q(a_n x) \|_{L_\infty(\mathbf{R})} = 0. \tag{5.1}$$

Then, if $\phi(x)$ is given by (3.60), we have

$$\lim_{n\to\infty} P_n(x) W(a_n x) = \lim_{n\to\infty} \{P_n(x)/H_Q(a_n x)\}\phi^{-1}(a_n x)$$
$$= g(x),$$

by (5.1) and (3.61), provided $x \neq 0$. Further, this even yields, for any $0 < \varepsilon < A < \infty$,

$$\lim_{n\to\infty} \| P_n(x) W(a_n x) - g(x) \|_{L_\infty(\varepsilon \le |x| \le A)} = 0. \tag{5.2}$$

Also, from (5.1) and (3.62), there exists $C > 0$ such that

$$\| P_n(x) W(a_n x) \|_{L_\infty(\mathbf{R})} \le C, \qquad n = 1, 2, 3, \ldots. \tag{5.3}$$

Hence, by Lebesgue's dominated convergence theorem, for any $A > 0$,

$$\lim_{n\to\infty} \| P_n(x) W(a_n x) - g(x) \|_{L_p[-A,A]} = 0.$$

By (3.30) in Lemma 3.5 and (5.3), we then have, if $A > 1$,

$$\lim_{n\to\infty} \| P_n(x) W(a_n x) \|_{L_p(|x| \ge A)} = 0.$$

Since $g(x) = 0$ for $|x| \ge 1$, we have

$$\lim_{n\to\infty} \| g(x) - P_n(x) W(a_n x) \|_{L_p(\mathbf{R})} = 0, \tag{5.4}$$

and so (2.18) is true in the special case $k_n = 0$, $n = 1, 2, 3, \ldots$. Now let $\{k_n\}_1^\infty$ be a sequence of nonnegative integers satisfying (2.8). We make the substitution $x = sa_n / a_{n-k_n}$ in the following consequence of (5.4):

$$\lim_{n\to\infty} \| g(x) - P_{n-k_n}(x) W(a_{n-k_n} x) \|_{L_p(\mathbf{R})} = 0.$$

In view of Lemma 3.2(iv), this substitution yields

$$\lim_{n\to\infty} \| g(sa_n / a_{n-k_n}) - P_n^*(s) W(a_n s) \|_{L_p(\mathbf{R})} = 0, \tag{5.5}$$

where, for $n = 1, 2, 3, \ldots$,

$$P_n^*(s) := P_{n-k_n}(x) \in \mathcal{P}_{n-k_n}.$$

Since $g(x)$ is continuous in $\mathbf{R}$ and vanishes outside $(-1, 1)$, Lemma 3.2(iv) shows that

$$\lim_{n\to\infty} \| g(s) - g(sa_n / a_{n-k_n}) \|_{L_p(\mathbf{R})} = 0. \tag{5.6}$$

Combining (5.5) and (5.6), we obtain the "if" part of Theorem 2.5, namely, (2.18). ■

Proof of the "only if" part of Theorem 2.5. Suppose $g \in L_p(\mathbf{R})$ and (2.18) holds. It is then an easy consequence of (2.18) and (3.30) in Lemma 3.5 that

$$\lim_{n\to\infty} \| P_n(u)\, W(a_n u) \|_{L_p(|u| \geq 1+\delta)} = 0$$

for each $\delta > 0$. Then (2.18) implies that

$$\| g \|_{L_p(|u| \geq 1+\delta)} = 0$$

for each $\delta > 0$. ■

Proof of Theorem 2.2. This follows from Theorem 2.5, since

$$W_\alpha(a_n(W_\alpha)u) = W_\alpha^n(\beta_\alpha u). \qquad ■$$

Proof of Theorem 2.4. We prove the "if" part. The "only if" part follows from Lemma 3.5, as above. Let $g(x)$ be continuous in $\mathbf{R}$ and $g(x) = 0$ for $|x| > 1$, and let $P_n \in \mathcal{P}_n$, $n = 1, 2, 3, \ldots$, satisfy (5.1). Then (5.2) and (5.3) also hold. In view of (3.30) in Lemma 3.5 (with $p = \infty$), (5.3) yields

$$\lim_{n\to\infty} \| P_n(x)\, W(a_n x) \|_{L_\infty(|x| \geq A)} = 0$$

for any $A > 1$. Combined with (5.2), this yields

$$\limsup_{n\to\infty} \| g(x) - P_n(x)\, W(a_n x) \|_{L_\infty(|x| \geq \varepsilon)} = 0 \tag{5.7}$$

for any $\varepsilon > 0$.

Let us now assume that $g(0) = 0$, and let $\eta > 0$. Then we can find $\varepsilon > 0$ such that

$$|g(x)| \leq \eta, \qquad |x| \leq \varepsilon.$$

Then (5.1) and Lemma 3.11 show that

$$\limsup_{n\to\infty} \| g(x) - P_n(x)\, W(a_n x) \|_{L_\infty[-\varepsilon,\varepsilon]} \leq \eta(1+b).$$

Since $\eta > 0$ is arbitrary, this last inequality and (5.7) show that

$$\lim_{n\to\infty} \| g(x) - P_n(x)\, W(a_n x) \|_{L_\infty(\mathbf{R})} = 0,$$

so that Theorem 2.4 is true in the special case when $g(0) = 0$ and all $k_n = 0$, $n = 1, 2, 3, \ldots$. As in the proof of Theorem 2.5, a substitution of the form $x = a_n s / a_{n-k_n}$ yields the conclusion of Theorem 2.4 when $g(0) = 0$ and $\{k_n\}_1^\infty$ satisfies (2.8).

Finally, we remove the restriction $g(0) = 0$, in the case when

$$\lim_{n\to\infty} a_n / n = 0. \tag{5.8}$$

Let $\{l_n\}_1^\infty$ be a sequence of positive integers with

$$\lim_{n\to\infty} a_n / l_n = 0 \quad \text{and} \quad \lim_{n\to\infty} l_n / n = 0. \tag{5.9}$$

Since $\phi(u)$ of Lemma 3.11 is uniformly continuous in $\mathbf{R}$, with modulus of continuity $w(\cdot)$, say, Jackson's theorem shows that there exists $S_n \in \mathcal{P}_{l_n}$, $n = 1, 2, 3, \ldots$, such that, for $|x| \leq 2a_n$,

$$|\phi(x) - S_n(x/a_n)| \leq Aw(a_n/l_n), \qquad n = 1, 2, 3, \ldots,$$

where A is an absolute constant. In view of (5.9), we obtain

$$\lim_{n\to\infty} |\phi(a_n u) - S_n(u)| = 0 \qquad \text{uniformly for} \quad |u| \leq 2. \tag{5.10}$$

Now, let $\hat{P}_n(u)$ be a polynomial of degree at most $n - k_n - l_n$, $n = 1, 2, 3, \ldots$, satisfying

$$\lim_{n\to\infty} \| g(u) - \hat{P}_n(u)/H_Q(a_n u) \|_{L_\infty(\mathbf{R})} = 0, \tag{5.11}$$

where $\{k_n\}_1^\infty$ satisfies (2.8). The existence of $\{\hat{P}_n\}_1^\infty$ follows from Theorem 4.1, and a substitution, as above.

Let

$$P_n(u) := \hat{P}_n(u) S_n(u) \in \mathcal{P}_{n-k_n}, \qquad n = 1, 2, 3, \ldots.$$

It follows from (3.61), (5.10), and (5.11) that

$$\lim_{n\to\infty} \| g(u) - P_n(u) W(a_n u) \|_{L_\infty[-2,2]} = 0.$$

Further, then, Lemma 3.5 shows that

$$\lim_{n\to\infty} \| P_n(u) W(a_n u) \|_{L_\infty(|u| \geq 2)} = 0.$$

These last two limit relations imply (2.15). ∎

Proof of Theorem 2.1. This follows directly from Theorem 2.4. ∎

Proof of Theorem 2.6. Let $\delta > 0$ and define

$$g_\delta(x) := \min\{1/\delta, 1/(f(x)|x|^\Delta)\}, \qquad |x| \leq 1 - \delta.$$

Further extend $g_\delta(x)$ continuously so that

$$g_\delta(x) = 0, \qquad |x| \geq 1,$$

and

$$g_\delta(x) \leq 1/(f(x)|x|^\Delta), \qquad 1 - \delta \leq |x| \leq 1.$$

Note that if

$$\mathscr{E} := \{x \in [-1, 1]: f(x)|x|^\Delta \leq \delta\} \cup (-1, -1 + \delta) \cup (1 - \delta, 1)$$

and

$$\mathscr{F} := [-1, 1] \backslash \mathscr{E}$$

we have

$$g_\delta(x) f(x) |x|^\Delta = 1, \qquad x \in \mathscr{F}, \tag{5.12}$$

and

$$0 \le g_\delta(x) f(x) |x|^\Delta \le 1, \qquad x \in [-1, 1]. \tag{5.13}$$

Recall that $\{k_n\}_1^\infty$ and $\{l_n\}_1^\infty$ satisfy respectively (2.8) and (2.21). By Theorem 4.1, we can find $\hat{P}_n \in \mathcal{P}_n$, $n = 1, 2, 3, \ldots$, such that

$$\lim_{n\to\infty} \| g_\delta(u) - \hat{P}_{n-k_n-l_n}(u)/H_Q(a_{n-k_n-l_n} u) \|_{L_\infty(\mathbf{R})} = 0.$$

Making the substitution $u = x a_n(1+\varepsilon_n)/a_{n-k_n-l_n}$, and taking account of (2.8), (2.19), (2.21), and the uniform continuity of $g_\delta(u)$, we see that

$$P_n^*(x) := \hat{P}_{n-k_n-l_n}(u) \in P_{n-k_n-l_n}, \qquad n = 1, 2, 3, \ldots,$$

satisfies

$$\lim_{n\to\infty} \| g_\delta(x) - P_n^*(x)/H_Q(a_n(1+\varepsilon_n)x) \|_{L_\infty(\mathbf{R})} = 0.$$

In view of (3.60), we may rewrite this as

$$\lim_{x\to\infty} \| g_\delta(x) - P_n^*(x) W(a_n(1+\varepsilon_n)x) \phi(a_n(1+\varepsilon_n)x) \|_{L_\infty(\mathbf{R})} = 0. \tag{5.14}$$

Let

$$P_n(x) := a_n^{-\Delta} P_n^*(x) S_n(x) \in \mathcal{P}_{n-k_n}, \qquad n = 1, 2, 3, \ldots.$$

Then, from (2.20), (2.22), (2.25), (2.26), (3.60), and (5.14), we see that

$$\begin{aligned} &\lim_{n\to\infty} | P_n(x) \hat{W}(a_n(1+\varepsilon_n)x) | \\ &\quad = \lim_{n\to\infty} \{a_n^{-\Delta} w(a_n(1+\varepsilon_n)x)\}\{| P_n^*(x) | \, W(a_n(1+\varepsilon_n)x)\} \\ &\qquad \times | V(a_n(1+\varepsilon_n)x) S_n(x) | \Psi(a_n(1+\varepsilon_n)x) \\ &\quad = |x|^\Delta g_\delta(x) \qquad \text{for almost all} \quad x \in [-1, 1]. \end{aligned} \tag{5.15}$$

We may assume that all z_j, $j = 1, 2, \ldots, N$, are distinct—if not, reorder them and add the relevant indices Δ_j. We may also assume $0 \in \{z_1, z_2, \ldots, z_N\}$—if not, simply append it to the set and define the corresponding Δ_j to be 0. Let

$$s := \min_{j \ne k} |z_j - z_k| \quad \text{and} \quad K := \max_j |z_j|,$$

and let

$$\mathcal{I}_{nj} := \{x : |(1+\varepsilon_n)x - z_j/a_n| \le s/(4a_n)\}, \qquad j = 1, 2, \ldots, N,$$

and

$$\mathcal{I}_n := \bigcup_{j=1}^N \mathcal{I}_{nj}, \qquad n = 1, 2, 3, \ldots.$$

Choose n_1 so large that

$$\tfrac{1}{2} \le 1 + \varepsilon_n \le 2, \qquad n \ge n_1.$$

For $x \notin \mathscr{I}_n$ and $|x| \geq 4K/a_n$, and $n \geq n_1$, we have

$$|x|/4 \leq |x|/2 - K/a_n \leq |(1+\varepsilon_n)x - z_j/a_n| \leq 2|x| + K/a_n \leq 3|x|,$$

while, for $x \notin \mathscr{I}_n$ and $|x| \leq 4K/a_n$, and $n \geq n_1$, we have

$$s|x|/(16K) \leq s/(4a_n) \leq |(1+\varepsilon_n)x - z_j/a_n| \leq 9K/a_n \leq 72K|x|/s.$$

The last inequality follows since $0 \in \{z_1, \ldots, z_N\}$.

Hence, for $n \geq n_1$ and some C_1, C_2 depending only on w,

$$C_1|x|^\Delta \leq a_n^{-\Delta} w(a_n(1+\varepsilon_n)x) \leq C_2|x|^\Delta, \tag{5.16}$$

$x \in [-1, 1]\backslash I_n$. Then, for some C_3 independent of n and δ, we obtain, from (5.14), (5.16), and (2.23), that, for $x \in [-1, 1]\backslash \mathscr{I}_n$,

$$\begin{aligned} f(x)\hat{W}(a_n(1+\varepsilon_n)x)|P_n(x)| &\leq C_3 f(x)(g_\delta(x)+1)|x|^\Delta \\ &\leq C_4(1+|x|^\Delta), \end{aligned}$$

by (5.13). Now let us assume first that

$$\Delta r_j > -1, \qquad j = 1, 2. \tag{5.17}$$

Combined with (5.15) and Lebesgue's dominated convergence theorem, this yields, for $j = 1, 2$,

$$\begin{aligned} \limsup_{n\to\infty} \int_{[-1,1]\backslash \mathscr{I}_n} &|1 - |f(x)\hat{W}(a_n(1+\varepsilon_n)x)P_n(x)|^{r_j}|(1-x^2)^{-1/2}\,dx \\ &= \int_{-1}^{1} |1 - |f(x)|x|^\Delta g_\delta(x)|^{r_j}|(1-x^2)^{-1/2}\,dx \\ &= \int_{\mathscr{E}} |1 - |f(x)|x|^\Delta g_\delta(x)|^{r_j}|(1-x^2)^{-1/2}\,dx \\ &\leq \int_{\mathscr{E}} (1-x^2)^{-1/2}\,dx, \end{aligned} \tag{5.18}$$

by (5.12) and (5.13). Next, if $x \in \mathscr{I}_{nl}$ for some $1 \leq l \leq N$, then, by definition of s, for $k \neq l$, $1 \leq k \leq N$, and $n \geq n_1$,

$$s/(2a_n) \leq |(1+\varepsilon_n)x - z_k/a_n| \leq 2|x| + K/a_n \leq 11K/a_n.$$

Hence, for $j = 1, 2$ and some C_5 independent of n,

$$\begin{aligned} \int_{\mathscr{I}_{nl}} |a_n^{-\Delta} w(a_n(1+\varepsilon_n)x)|^{r_j}\,dx &\leq C_5(1/a_n)^{r_j(\Delta-\Delta_l)} \int_0^{K/a_n} u^{r_j\Delta_l}\,du \\ &\leq C_6 a_n^{-(r_j\Delta+1)} \to 0 \qquad \text{as} \quad n\to\infty, \end{aligned}$$

by (5.17). Since, as above, for $x \in \mathcal{I}_{nl}$,

$$|f(x)\hat{W}(a_n(1+\varepsilon_n)x)P_n(x)| \le C_7 a_n^{-\Delta} w(a_n(1+\varepsilon_n)x),$$

and since $\operatorname{meas}(\mathcal{I}_{nl}) \to 0$ as $n \to \infty$, we have, for $j = 1, 2$; $l = 1, 2, \ldots, N$,

$$\lim_{n\to\infty} \int_{\mathcal{I}_{nl}} |1 - |f(x)\hat{W}(a_n(1+\varepsilon_n)x)P_n(x)|^{r_j}|(1-x^2)^{-1/2}\,dx = 0.$$

Thus, from (5.18), for $j = 1, 2$,

$$\limsup_{n\to\infty} \int_{-1}^{1} |1 - |f(x)\hat{W}(a_n(1+\varepsilon_n)x)P_n(x)|^{r_j}|(1-x^2)^{-1/2}\,dx \le \int_{\mathscr{E}} (1-x^2)^{-1/2}\,dx.$$

As $\operatorname{meas}(\mathscr{E}) \to 0$ as $\delta \to 0+$, the result follows.

Finally, suppose that (5.17) is not satisfied. Then let $z_{N+1} := i$ and choose Δ_{N+1} such that

$$\Delta^* := \sum_{j=1}^{N+1} \Delta_j$$

satisfies $\Delta^* r_j > -1$, $j = 1, 2$. For $x \in \mathbf{R}$, let

$$w^*(x) := \prod_{j=1}^{N+1} |x - z_j|^{\Delta_j}$$

and

$$W^*(x) := \exp(-Q^*(x)),$$

where

$$Q^*(x) := (\Delta_{N+1}/2)\log(1+x^2) + Q(x).$$

Then we see that $W^* \in \mathrm{VSF}(\alpha)$ also and $w^*W^* = wW$. Since (5.17) is satisfied for $\hat{W} = W^* V \Psi w^*$, the result follows. ■

6. Upper Bounds for Christoffel Functions

Let $W \in \mathrm{VSF}(\alpha)$ for some $\alpha > 0$, and define

$$\lambda_n(W^2; x) = \inf_{P \in \mathscr{P}_{n-1}} \int_{-\infty}^{\infty} (P(u)W(u))^2\,du / P^2(x), \tag{6.1}$$

the nth *Christoffel function* associated with $W^2(x)$. The Christoffel functions play an important role in weighted approximation and in the theory of orthogonal polynomials—see Nevai [36]. One of the many reasons for this is the identity [7], [36]

$$\lambda_n(W^2; x) = 1 \Big/ \sum_{j=0}^{n-1} \{p_j(W^2; x)\}^2, \tag{6.2}$$

where $p_j(W^2, x)$, $j=0,1,2,\ldots$, are the orthonormal polynomials associated with W^2, so that

$$\int_{-\infty}^{\infty} p_m(W^2; x)p_n(W^2; x)W^2(x)\,dx = \begin{cases} 1, & m=n, \\ 0, & m\neq n. \end{cases}$$

In order to place the new result we obtain below in perspective, we first briefly review existing estimates for Christoffel functions. For the sake of simplicity we restrict our discussion to $W_\alpha(x) := \exp(-|x|^\alpha)$, $\alpha>0$, and merely note that many of the quoted results hold in greater generality.

For $\alpha\geq 2$, Freud [9] (see Nevai [36]) established the lower bound

$$\lambda_n(W_\alpha^2; x) \geq Cn^{-1+1/\alpha}W_\alpha^2(x), \qquad x\in\mathbf{R}, \quad n\geq 1. \tag{6.3}$$

Subsequently, Levin and Lubinsky [13], [14] showed that (6.3) is also true for $1<\alpha<2$. For $\alpha=1$, the analogue of (6.3) (with $n^{-1+1/\alpha}$ replaced by $(\log n)^{-1}$) appears in Freud, Giroux, and Rahman [10] and, for $0<\alpha<1$, the analogue of (6.3) (with $n^{-1+1/\alpha}$ replaced by 1) appears in Lubinsky [16].

Concerning matching upper bounds, Freud [8] showed that, for $\alpha>1$, there exist $C_1, C_2>0$ such that

$$\lambda_n(W_\alpha^2; x) \leq C_1 n^{-1+1/\alpha}W_\alpha^2(x) \tag{6.4}$$

for

$$|x|\leq C_2 n^{1/\alpha}, \qquad n\geq 1. \tag{6.5}$$

The analogue for $\alpha=1$ appears in [10]. For $0<\alpha<1$, the upper bounds seem not to have been considered, though in this case there is a trivial constant upper bound to match the lower bound for x in any finite interval. In any event, upper bounds are not of such great interest for $0<\alpha<1$, see Nevai and Totik [38].

One unfortunate feature of the above upper bounds is the unspecified constant C_2 in (6.5). For $\alpha=2,4,6$ [34], [1], [43], much deeper results on asymptotic behavior of orthogonal polynomials show that (6.4) holds for

$$|x|\leq(1-\varepsilon)x_{1n}, \tag{6.6}$$

$n=1,2,3,\ldots$, where $\varepsilon>0$ is arbitrary and x_{1n} denotes the largest zero of $p_n(W_\alpha^2; x)$. One may use Lemmas 3.5 and 3.7 to show that one cannot replace $(1-\varepsilon)$ in (6.6) by $(1+\delta)$, for any $\delta>0$.

Even for $\alpha=8,10,12,\ldots$, (6.4) has not been proven true for the range (6.6). This is somewhat surprising, in view of A. Magnus's [22], [23] proof of Freud's conjecture for these values of α, and the subsequent sharpening of Magnus's results by Máté, Nevai, and Zaslavsky [25]. Evidently, Freud's conjecture is not related to upper bounds for Christoffel functions. The next result establishes (6.4) for the range (6.6) for any $\alpha\geq 1$.

Theorem 6.1. *Let $W\in \mathrm{VSF}(\alpha)$ for some $\alpha>0$. Let $\varepsilon>0$. Then there exists $C>0$ such that, for $n=1,2,3,\ldots$,*

$$\lambda_n(W^2; x)\leq C(a_n/n)W^2(x), \qquad |x|\leq(1-\varepsilon)x_{1n}, \tag{6.7}$$

where x_{1n} denotes the largest zero of $p_n(W^2; x)$.

In view of (3.11), one may replace a_n by q_n in (6.7). Further, (3.54) shows that one may replace x_{1n} by a_n in (6.7). Of course, the above result is of interest only if $\alpha \geq 1$.

Theorem 6.1 may be used to extend the range of several results in orthogonal polynomials. Again, for the sake of simplicity, we restrict ourselves to the weights $W_\alpha(x)$, $\alpha > 1$. For $n = 1, 2, 3, \ldots$, let us denote the zeros of $p_n(W_\alpha^2; x)$ by

$$-\infty < x_{nn} < x_{n-1,n} < \cdots < x_{1n} < \infty.$$

6.1. *Spacing of Zeros*

Using (6.7) and the method of proof for upper bounds [8, pp. 293–294], as well as the method of proof for lower bounds [9, p. 37], one may show that, given $0 < \varepsilon < 1$,

$$x_{jn} - x_{j+1,n} \asymp n^{-1+1/\alpha}$$

uniformly for $|x_{jn}|, |x_{j+1,n}| \leq (1-\varepsilon)x_{1n}$, $n = 2, 3, 4, \ldots$.

6.2. *Lower Bounds for Orthonormal Polynomials*

Let α be a positive even integer. Bonan [2] and Nevai [35] showed that the exist $C_1, C_2 > 0$ such that

$$p_{n-1}^2(W_\alpha^2; x_{jn})\, W_\alpha^2(x_{jn}) \geq C_1 n^{-1/\alpha} \tag{6.8}$$

uniformly for j and n such that

$$|x_{jn}| \leq C_2 n^{1/\alpha}. \tag{6.9}$$

In [35], Nevai suggested that (6.8) should be true in the larger range

$$|x_{jn}| \leq (1-\varepsilon)x_{1n} \tag{6.10}$$

for any $\varepsilon > 0$. A glance at the proof of either Bonan [2] or Nevai [35] indicates that the missing ingredient is the upper bound (6.7). Thus (6.8) is true in the range (6.10).

6.3. *Estimates of Quadrature Sums*

In [20, Corollary 9], Lubinsky, Máté, and Nevai estimated quadrature sums of the form

$$\sum_{|x_{jn}| \leq Cn^{1/\alpha}} \lambda_{jn} |P(x_{jn})|^p W_\alpha^{-r}(x_{jn}),$$

where $\lambda_{jn} := \lambda_n(W^2; x_{jn})$, p, $r > 0$, and $P(x)$ is a polynomial of degree at most a constant times n. The constant C was unspecified because of a lack of upper bounds for $\lambda_n(W_\alpha^2; x)$. Using (6.7), one may replace $Cn^{1/\alpha}$ in the range of summation by $(1-\varepsilon)x_{1n}$, where $\varepsilon > 0$ is arbitrary.

Proof of Theorem 6.1. Making the substitution $x = a_n u$ in (6.1), and using (3.30) in Lemma 3.5, we see that, for some n_1 and $n \geq n_1$, $x \in \mathbf{R}$,

$$\lambda_n(W^2; x) \leq 2a_n \inf_{P \in \mathcal{P}_{n-1}} \int_{-2}^{2} P^2(a_n u) W^2(a_n u)\, du / P^2(x) \tag{6.11}$$

$$= 2a_n \inf_{S \in \mathcal{P}_{n-1}} \int_{-2}^{2} S^2(u) W^2(a_n u)\, du / S^2(x/a_n).$$

Let $0 < \varepsilon < 1$. Suppose that we can find a sequence of polynomials $\{R_n\}_1^\infty$ with the following properties: there exist $C_1, C_2 > 0$ and $0 < \eta < 1$ such that, for n large enough,

$$R_n(x) \text{ has degree } p_n \leq n(1-\eta), \tag{6.12}$$

$$|R_n(x) W(a_n x)| \geq C_2, \qquad |x| \leq 1 - \varepsilon, \tag{6.13}$$

and

$$\| R_n(x) W(a_n x) \|_{L_\infty[-2,2]} \leq C_1. \tag{6.14}$$

Then, choosing $S := R_n P$ in (6.11) for some P of degree $\leq n - 1 - p_n$, we obtain, for n large enough and $|x| \leq (1-\varepsilon) a_n$,

$$\lambda_n(W^2; x) \leq 2a_n W^2(x)(C_1/C_2)^2 \inf_{P \in \mathcal{P}_{n-p_n-1}} \int_{-2}^{2} P^2(u)\, du / P^2(x/a_n)$$

$$\leq C_3(a_n/n) W^2(x)$$

for $|x| \leq (1-\varepsilon) a_n$, by standard estimates for the Christoffel function of the Legendre weight [33, p. 79] and since

$$n - p_n \geq \eta n/2, \quad n \text{ large enough}.$$

It remains to find polynomials $\{R_n\}$ satisfying (6.12)-(6.14). Let

$$h(x) := \begin{cases} 1 - x^2, & |x| \leq 1 \\ 0, & \text{otherwise}. \end{cases}$$

By Theorem 4.1, we can find polynomials $\hat{R}_n(x)$ *of* degree $\leq \langle n(1-\eta) \rangle$ (where $\langle x \rangle$ denotes the greatest integer $\leq x$), $n = 1, 2, 3, \ldots$, such that

$$\lim_{n \to \infty} \| h(x) - \hat{R}_n(x) / H_Q(a_{\langle n(1-\eta) \rangle} x) \|_{L_\infty(\mathbf{R})} = 0.$$

Making the substitutions $x = a_n u / a_{\langle n(1-\eta) \rangle}$ and $R_n(u) := \hat{R}_n(x)$, we obtain

$$\lim_{n \to \infty} \| h(a_n u / a_{\langle n(1-\eta) \rangle}) - R_n(u) / H_Q(a_n u) \|_{L_\infty(\mathbf{R})} = 0. \tag{6.15}$$

Now, by (3.10), we see that

$$\lim_{n \to \infty} a_{\langle n(1-\eta) \rangle} Q'(a_{\langle n(1-\eta) \rangle}) / \{a_n Q'(a_n)\} = (1 - \eta).$$

Hence, using (3.5), it follows that

$$\lim_{n\to\infty} a_{\langle n(1-\eta)\rangle}/a_n=(1-\eta)^{1/\alpha}.$$

Then, choosing η small enough, we have, for $n\geq n_2(\eta)$,

$$h(a_n u/a_{\langle n(1-\eta)\rangle})\geq C_4>0, \qquad |u|\leq 1-\varepsilon.$$

In view of Lemma 3.11, (6.15) then implies (6.13) for n large enough. Further, (6.14) follows immediately from (6.15). Finally, (6.12) is satisfied by choice of R_n. ■

Note that the proof of the above result does not require the existence of $Q'''(x)$ or (2.12) to be satisfied, since Theorem 4.1 and (3.62) do not require these conditions. Furthermore, the above method may also be applied to the L_p-Christoffel functions [16]

$$\lambda_{n,p}(W;x):=\inf_{P\in\mathscr{P}_{n-1}}\|PW\|_{L_p(\mathbf{R})}/|P(x)|, \qquad 0<p\leq\infty.$$

We may also estimate $\lambda_n(W^2w;x)$, where w is as in (2.25), by a modification of the above method.

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Appendix: Proof of Theorem 3.9

Recall that, if $\alpha>0$,

$$\text{(A.1)} \qquad v(\alpha;x):=\frac{\alpha}{\pi}\int_{|x|}^{1} t^{\alpha-1}/\sqrt{t^2-x^2}\,dt, \qquad |x|\leq 1,$$

and

$$\text{(A.2)} \qquad U_\alpha(z):=\int_{-1}^{1}\log|z-t|\,v(\alpha;t)\,dt-|z|^\alpha/\lambda_\alpha+\log 2+1/\alpha.$$

We shall split the proof of Theorem 3.9 into several steps: in Lemma A.1 we note some properties of $v(\alpha;x)$. In Lemma A.2 we establish the existence of an angular region at ±1, in which $U_\alpha(z)$ is positive-compare Figure 3.1. In Lemma A.3 we show that, for any $0<\delta<1$, there is a rectangular region above and below $[\delta,1]\cup[-1,-\delta]$, in which $U_\alpha(z)$ is positive. In Lemma A.4 we consider the interval $[-\delta,\delta]$ for $\alpha>1$, and in Lemma A.5 we establish the analogue of Lemma A.4 for $0<\alpha\leq1$. The latter requires special care.

Lemma A.1. *Let $\alpha > 0$. Then*

(A.3) $$v(\alpha; x) = \frac{\alpha}{\pi}\sqrt{1-x^2} - \frac{\alpha(\alpha-2)}{\pi}\int_{|x|}^{1} t^{\alpha-3}\sqrt{t^2-x^2}\,dt, \qquad |x|<1.$$

Given $\delta > 0$, we have, uniformly for $\delta \le |x| \le 1$,

(A.4) $$v(\alpha; x) = \frac{\alpha}{\pi}\sqrt{1-x^2} + O((1-x^2)^{3/2}).$$

There exists $C > 0$ such that, for $|x| \le 1$,

(A.5) $$v(\alpha; x) \ge C\sqrt{1-x^2}.$$

The following identity is valid:

(A.6) $$\int_{-1}^{1} v(\alpha; t)/(1-t)\,dt = \alpha/\lambda_\alpha.$$

Proof. Integrating (A.1) by parts, yields (A.3). Then (A.4) follows, as

$$\sqrt{t^2-x^2} = O((1-x^2)^{1/2}) \qquad \text{uniformly for} \quad |x| \le |t| \le 1.$$

For $|x|$ close enough to 1, (A.4) implies (A.5). For $|x| \le 1-\delta$, any $0<\delta<1$, (A.5) follows as $v(\alpha; x)$ is bounded by a positive number. To prove (A.6) we first note that, from equation (4.8) in [27, p. 218] and equation (4.36) in [27, p. 221],

(A.7) $$\int_{-1}^{1} \frac{v(\alpha; t)}{x-t}\,dt = \frac{\alpha x^{\alpha-1}}{\lambda_\alpha} - \alpha\int_{1}^{x} t^{\alpha-1}/\sqrt{x^2-t^2}\,dt$$

for $x > 1$. Since, for $t \in (-1, 1)$, $1/(x-t)$ increases as x decreases to 1, we may let $x \to 1+$ in (A.7) and use Lebesgue's monotone convergence theorem to deduce (A.6). ∎

Note that $U_\alpha(z)$ is continuous in $\mathbf{C}$, and vanishes in $[-1, 1]$, see Proposition 2.3 in [27, p. 208].

Lemma A.2. *Let $\alpha > 0$. Let $0 < \delta < \pi/6$. Then there exist $\varepsilon > 0$ and $C > 0$ such that*

(A.8) $$U_\alpha(1+ye^{i\theta}) \ge Cy^{3/2}, \qquad y \in [0, \varepsilon], \quad \theta \in [\pi/3+\delta, \pi/2].$$

Proof. Differentiating $U_\alpha(1+ye^{i\theta})$ partially with respect to y and letting $z := 1+ye^{i\theta}$, we see that

$$\frac{\partial U_\alpha}{\partial y}(z) = \int_{-1}^{1} \frac{(1-t)\cos\theta + y}{|z-t|^2}\, v(\alpha; t)\,dt - \frac{\alpha}{\lambda_\alpha}|z|^{\alpha-2}(\cos\theta + y).$$

Further, for some ε, $C_1 > 0$ and $y \in [0, \varepsilon]$, $\theta \in [0, \pi]$,

$$|z|^{\alpha-2} = |1+y\,e^{i\theta}|^{\alpha-2} \le 1 + C_1 y.$$

Using (A.6), we see that, for some C_2 and all $y \in [0, \varepsilon]$,

$$\frac{\partial U_\alpha}{\partial y}(z) \geq \int_{-1}^{1} \left\{ \frac{(1-t)\cos\theta + y}{|z-t|^2} - \frac{\cos\theta}{1-t} \right\} v(\alpha; t)\,dt - C_2 y$$
$$= \int_0^{2/y} \left\{ \frac{u\cos\theta + 1}{|u+e^{i\theta}|^2} - \frac{\cos\theta}{u} \right\} v(\alpha; 1-yu)\,du - C_2 y$$
(by the substitution $1-t = uy$)
$$= \int_0^{2/y} \{1 - 2\cos^2\theta - \cos\theta/u\} \frac{v(\alpha; 1-yu)}{|u+e^{i\theta}|^2}\,du - C_2 y.$$

Now, for $y \leq \frac{1}{4}$, $y^{-1/2} \leq u \leq 2/y$, and $\theta \in [\pi/3, \pi/2]$,

$$1 - 2\cos^2\theta - \cos\theta/u \geq \tfrac{1}{2} - y^{1/2} \geq 0.$$

Hence

$$\text{(A.9)} \qquad \frac{\partial U_\alpha}{\partial y}(z) \geq \int_0^{y^{-1/2}} \{1 - 2\cos^2\theta - \cos\theta/u\} \frac{v(\alpha; 1-yu)}{|u+e^{i\theta}|^2}\,du - C_2 y$$
$$=: I(\theta; y) - C_2 y.$$

Next, by (A.4), we have, uniformly for $0 \leq u \leq y^{-1/2}$ and $y \leq \frac{1}{4}$,

$$v(\alpha; 1-yu) = \frac{\alpha}{\pi}\sqrt{1-(1-yu)^2} + O(\{1-(1-yu)^2\}^{3/2})$$
$$= \frac{\alpha}{\pi}\sqrt{2yu - (uy)^2} + O((yu)^{3/2})$$
$$= \frac{\alpha}{\pi}\sqrt{2yu}(1 + O(y^{1/2})).$$

Then, uniformly for $\theta \in [\pi/3, \pi/2]$, as $y \to 0$,

$$I(\theta; y)/\sqrt{y} = \frac{\alpha\sqrt{2}}{\pi} \int_0^{y^{-1/2}} \{1 - 2\cos^2\theta - \cos\theta/u\} \frac{u^{1/2}}{|u+e^{i\theta}|^2}\,du\{1 + O(y^{1/2})\}$$
$$= \frac{\alpha\sqrt{2}}{\pi} \int_0^{\infty} \{1 - 2\cos^2\theta - \cos\theta/u\} \frac{u^{1/2}\,du}{|u+e^{i\theta}|^2}(1 + O(y^{1/2}))$$
$$+ O\left(\int_{y^{-1/2}}^{\infty} \{u^{-3/2} + u^{-5/2}\}\,du \right)(1 + O(y^{1/2})).$$

Now, letting $u = v^{-1}$, we see that

$$\int_0^\infty \frac{u^{1/2}}{|u+e^{i\theta}|^2}\,du = \int_0^\infty \frac{v^{-1/2}\,dv}{|v+e^{i\theta}|^2} =: A.$$

Hence, we see that, uniformly for $\theta \in [\pi/3, \pi/2]$,

$$\text{(A.10)} \qquad \frac{I(\theta; y)}{\sqrt{y}} = \frac{\alpha\sqrt{2}}{\pi}\{1 - 2\cos^2\theta - \cos\theta\}A(1 + O(y^{1/2})) + O(y^{1/4})$$

as $y \to 0$. Since $1-2\cos^2\theta - \cos\theta \geq C_3 > 0$ for $\theta \in [\pi/3+\delta, \pi/2]$, we obtain from (A.9) and (A.10) that, for $0<y\leq\varepsilon$, ε small enough, and $\theta\in[\pi/3, \pi/2]$,

$$y^{-1/2}\frac{\partial U_\alpha}{\partial y}(1+y\,e^{i\theta}) \geq C_4 > 0.$$

Multiplying by $y^{1/2}$, and integrating, we obtain (A.8), as $U_\alpha(1)=0$. ■

Lemma A.3. *Let $\alpha>0$. Let $0<\delta<1$. Then there exist $\varepsilon>0$ and $C_1>0$ such that*

(A.11) $$U_\alpha(x+iy) \geq C_1 y^{3/2}, \qquad \delta\leq|x|\leq 1, \quad 0<y<\varepsilon.$$

Proof. Since U_α is even and real valued in **R**, it suffices to prove (A.11) for $\delta\leq x\leq 1$ and $0<y\leq 1$. For $\delta\leq x\leq 1, 0<y\leq 1$, we see that, for some C_2 depending only on δ and α,

(A.12) $$\begin{aligned}\frac{\partial U_\alpha}{\partial y}(x+iy) &= \int_{-1}^{1}\frac{yv(\alpha;t)}{(x-t)^2+y^2}\,dt - \frac{\alpha}{\lambda_\alpha}\,y(x^2+y^2)^{\alpha/2-1}\\ &\geq \int_{x-\delta/2}^{x}\frac{yv(\alpha;t)}{(x-t)^2+y^2}\,dt - C_2 y\\ &= \int_0^{\delta/(2y)}\frac{v(\alpha;x-uy)}{u^2+1}\,du - C_2 y.\end{aligned}$$

Now, by (A.5),

$$\begin{aligned}v(\alpha;x-uy) &\geq C\sqrt{1-(x-uy)^2}\\ &= C\sqrt{1-x^2+2uyx-(uy)^2}\\ &\geq C\sqrt{2\delta uy-(uy)^2}\\ &\geq C_3(uy)^{1/2}\end{aligned}$$

for $0\leq u\leq\delta/(2y)$, with C_3 independent of x, u, and y. From (A.12) we obtain

(A.13) $$\begin{aligned}y^{-1/2}\frac{\partial U_\alpha}{\partial y}(x+iy) &\geq C_3\int_0^{\delta/(2y)}\frac{u^{1/2}\,du}{u^2+1} - C_2 y^{1/2}\\ &\geq C_4 > 0\end{aligned}$$

if $0<y\leq\varepsilon$ and ε is small enough. Here the upper bound on ε and C_4 are independent of x. Since $U_\alpha(x)=0$ for $\delta\leq x\leq 1$, we may multiply (A.13) by $y^{1/2}$ and integrate to obtain (A.11). ■

We next estimate $U_\alpha(x+iy)$ below, for x near 0 and $y>0$. The case $0<\alpha\leq 1$ is more difficult than the case $\alpha>1$. We first dispense with the latter:

Lemma A.4. *Let $\alpha>1$. There exist $\varepsilon>0$ and $C>0$ such that*

(A.14) $$U_\alpha(x+iy)\geq Cy^{3/2}, \qquad |x|\leq 1, \qquad 0<y<\varepsilon.$$

Proof. Let $a := \min\{1, \alpha-1\}$. Let $|x|\leq\frac{1}{2}$. We see from (A.12) that, for some C_1 independent of x and ε,

$$\begin{aligned}\frac{\partial U_\alpha}{\partial y}(x+iy) &\geq \int_{-1}^{1}\frac{yv(\alpha;t)}{(x-t)^2+y^2}\,dt - C_1 y^a\\ &\geq C_2\int_{x-1/4}^{x}\frac{y}{(x-t)^2+y^2}\,dt - C_1 y^a\end{aligned}$$

$$= C_2 \int_0^{1/(4y)} \frac{du}{u^2+1} - C_1 y^a,$$

where $C_2 := \min\{v(\alpha; t) : |t| \le \frac{3}{4}\} > 0$, and we made the substitution $x - t = uy$. Thus, as $a > 0$, there exist $\varepsilon > 0$ and $C_3 > 0$ such that

$$\frac{\partial U_\alpha}{\partial y}(x+iy) \ge C_3 > 0, \qquad |x| \le \tfrac{1}{2}, \quad 0 < y < \varepsilon.$$

Hence, integrating this inequality and using $U_\alpha(x) = 0$, we obtain

$$U_\alpha(x+iy) \ge C_3 y, \qquad |x| \le \tfrac{1}{2}, \quad 0 < y < \varepsilon.$$

Together with Lemma A.3, this implies (A.14). ■

Finally, we consider the harder case $\alpha \le 1$:

Lemma A.5. *Let $0 < \alpha \le 1$. There exist $\varepsilon > 0$ and $C > 0$ such that*

$$U_\alpha(x+iy) \ge Cy^2, \qquad |x| \le 1, \quad 0 < y < \varepsilon. \tag{A.15}$$

Proof. It suffices to consider $x \ge 0$. Since $\alpha < 2$, (A.3) shows that, for $|x| \le 1$,

$$\begin{aligned} v(\alpha; x) &\ge (\alpha(2-\alpha)/\pi) \int_{|x|}^1 t^{\alpha-3}(t^2 - x^2)^{1/2}\, dt \\ &= |x|^{\alpha-1} A(|x|), \end{aligned} \tag{A.16}$$

where

$$A(r) := (\alpha(2-\alpha)/\pi) \int_1^{1/r} s^{\alpha-3}(s^2-1)^{1/2}\, ds, \qquad r \in (0, 1). \tag{A.17}$$

Note that $A(r)$ decreases as r increases, and

$$A(0) := \lim_{r \to 0+} A(r) \begin{cases} <\infty, & \alpha < 1, \\ =\infty, & \alpha = 1. \end{cases} \tag{A.18}$$

Let $\Delta := (x^2 + y^2)^{1/2}$, and assume that $x, y \in (0, \frac{1}{2})$, so that $0 < \Delta \le 1/\sqrt{2}$. From (A.12),

$$\begin{aligned} \frac{\partial U_\alpha}{\partial y}(x+iy) &= \int_{-1}^1 \frac{yv(\alpha; t)}{(x-t)^2 + y^2}\, dt - \frac{\alpha y}{\lambda_\alpha} \Delta^{\alpha-2} \\ &\ge \int_{-1}^1 \frac{y|t|^{\alpha-1} A(|t|)}{\Delta^2 - 2xt + t^2}\, dt - \frac{\alpha y}{\lambda_\alpha} \Delta^{\alpha-2}, \end{aligned}$$

by (A.16). Letting $t = \Delta s$ in this last integral, we see that

$$\frac{\partial U_\alpha}{\partial y}(x+iy) \ge y\Delta^{\alpha-2} \left\{ \int_{-1/\Delta}^{1/\Delta} \frac{|s|^{\alpha-1} A(\Delta|s|)}{1 - 2xs/\Delta + s^2}\, ds - \frac{\alpha}{\lambda_\alpha} \right\}. \tag{A.19}$$

Now we see that

$$\begin{aligned} \int_{-1/\Delta}^{1/\Delta} \frac{|s|^{\alpha-1} A(\Delta|s|)}{1 - 2xs/\Delta + s^2}\, ds &= \int_0^{1/\Delta} s^{\alpha-1} A(\Delta s) \left\{ \frac{1}{1 - 2xs/\Delta + s^2} + \frac{1}{1 + 2xs/\Delta + s^2} \right\} ds \\ &= 2 \int_0^{1/\Delta} s^{\alpha-1} A(\Delta s)(1+s^2)/\{(1+s^2)^2 - 4x^2 s^2/\Delta^2\}\, ds \\ &\ge 2 \int_0^{1/\Delta} s^{\alpha-1} A(\Delta s)/(1+s^2)\, ds \end{aligned}$$

as $0 \le (1+s^2)^2 - 4x^2s^2/\Delta^2 \le (1+s^2)^2$. As Δ decreases to 0, this last integral increases, by Lebesgue's monotone convergence theorem, to $2A(0)B$, where

$$B := \int_0^\infty s^{\alpha-1}/(1+s^2)\, ds.$$

Let $\delta > 0$. It follows from (A.19) and these last considerations that there exists $\eta > 0$ depending only on α and δ such that, for $0 \le x \le \eta$ and $0 < y < \eta$,

$$\frac{\partial U_\alpha}{\partial y}(x+iy) \ge y\Delta^{\alpha-2}\{2A(0)B - \alpha/\lambda_\alpha - \delta\}. \tag{A.20}$$

We next show that $C_\alpha := 2A(0)B - \alpha/\lambda_\alpha$ is positive. Firstly, if $\alpha = 1$, (A.18) shows that $C_\alpha = \infty$. Now suppose $0 < \alpha < 1$. Making the substitution $s = 1/t$ in (A.17) and letting $r \to 0$, we see that

$$\begin{aligned} A(0) &= (\alpha(2-\alpha)/\pi) \int_0^1 t^{-\alpha}(1-t^2)^{1/2}\, dt \\ &= (\alpha(2-\alpha)/(2\pi))\Gamma(\tfrac{3}{2})\Gamma((1-\alpha)/2)/\Gamma((4-\alpha)/2), \end{aligned}$$

by a standard integral—see, for example, no. 855.41 in Dwight [5, p. 212]. Using the replication rule for the gamma function, we see that

$$A(0) = (\alpha/(2\sqrt{\pi}))\Gamma((1-\alpha)/2)/\Gamma(1-\alpha/2). \tag{A.21}$$

Next, by a standard integral—see, for example, no. 856.07 in [5, p. 213], we see that

$$B = \pi/(2\cos((\alpha-1)\pi/2)) = \pi/(2\sin(\alpha\pi/2)). \tag{A.22}$$

Then, combining (2.6) and (A.21) and (A.22),

$$C_\alpha = (\alpha\sqrt{\pi}/2)\Gamma((1-\alpha)/2)/\{\Gamma(1-\alpha/2)\sin(\alpha\pi/2)\} - \alpha 2^{\alpha-2}\Gamma(\alpha/2)^2/\Gamma(\alpha).$$

Next, using the identity [40, p. 21]

$$\Gamma(\alpha/2)\Gamma(1-\alpha/2) = \pi/\sin(\alpha\pi/2)$$

and Legendre's duplication formula [40, p. 24]

$$\sqrt{\pi}\Gamma(\alpha) = 2^{\alpha-1}\Gamma(\alpha/2)\Gamma(\alpha/2+\tfrac{1}{2}),$$

we see that

$$\begin{aligned} C_\alpha &= \alpha\Gamma(\alpha/2)\{\Gamma((1-\alpha)/2)\Gamma((1+\alpha)/2) - \pi\}/\{2\sqrt{\pi}\Gamma((\alpha+1)/2)\} \\ &= \alpha\Gamma(\alpha/2)\sqrt{\pi}\{1/\sin(\pi(\alpha+1)/2) - 1\}/\{2\Gamma((\alpha+1)/2)\}, \end{aligned}$$

by the rule

$$\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z) \qquad \text{with} \quad z = (\alpha+1)/2.$$

Hence $C_\alpha > 0$, and, choosing δ small enough in (A.20), we have

$$\frac{\partial U_\alpha}{\partial y}(x+iy) \ge yC_\alpha/2$$

for $0 \le x \le \eta$ and $0 < y < \eta$. Integrating, we obtain

$$U_\alpha(x+iy) \ge y^2C_\alpha/4, \qquad 0 \le x \le \eta, \quad 0 < y < \eta.$$

Together with Lemma A.3, this yields (A.15). ∎

Proof of Theorem 3.9. Lemmas A.2, A.4, and A.5 and the fact that $U_\alpha(z)=U_\alpha(\bar{z})=U_\alpha(-z)$ show that, for any $0<\delta<\pi/6$, there exists $\eta>0$ such that $U_\alpha(z)$ is positive in $\mathscr{G}(\eta;\ \pi/3+\delta)$. Since $U_\alpha(z)$ is continuous in **C**, the result follows. ∎

Note Added in Proof. Generalizations and improvements of the results of this paper, as well as applications to Szegö type asymptotics for extremal polynomials, appear in the authors' paper entitled "Strong Asymptotics for Extremal Errors and Extremal Polynomials Associated with Weights on $(-\infty,\infty)$."

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