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CONSTRUCTIVE APPROXIMATION

A Proof of Freud's Conjecture for Exponential Weights

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Abstract. Let $W(x)$ be a function nonnegative in $\mathbf{R}$, positive on a set of positive measure, and such that all power moments of $W^2(x)$ are finite. Let $\{p_n(W^2, x)\}_0^\infty$ denote the sequence of orthonormal polynomials with respect to the weight $W^2(x)$, and let $\{A_n\}_1^\infty$ and $\{B_n\}_1^\infty$ denote the coefficients in the recurrence relation

$$xp_n(W^2, x) = A_{n+1}p_{n+1}(W^2, x) + B_np_n(W^2, x) + A_np_{n-1}(W^2, x).$$

When $W(x) = w(x)\exp(-Q(x))$, $x \in (-\infty, \infty)$, where $w(x)$ is a "generalized Jacobi factor," and $Q(x)$ satisfies various restrictions, we show that

$$\lim_{n\to\infty} A_n/a_n = \tfrac{1}{2} \quad \text{and} \quad \lim_{n\to\infty} B_n/a_n = 0,$$

where, for n large enough, a_n is the positive root of the equation

$$n = (2/\pi)\int_0^1 a_nxQ'(a_nx)(1-x^2)^{-1/2}\,dx.$$

In the special case, $Q(x) = |x|^\alpha$, $\alpha > 0$, this proves a conjecture due to G. Freud. We also consider various noneven weights, and establish certain infinite-finite range inequalities for weighted polynomials in $L_p(\mathbf{R})$.

1. Introduction

Let $w(x)$ be nonnegative in $\mathbf{R}$, positive on a set of positive measure, and such that all power moments of w,

$$\int_{-\infty}^{\infty} x^j w(x)\,dx, \qquad j = 0, 1, 2, \ldots,$$

are finite. Then we shall call w a *weight function*. Associated with w is the sequence of orthonormal polynomials $\{p_n(w, x)\}_0^\infty$, where

$$p_n(w, x) = \gamma_n x^n + \cdots$$

has degree n with $\gamma_n = \gamma_n(w) > 0$, and

$$\int_{-\infty}^{\infty} p_n(w, x)p_m(w, x)w(x)\,dx = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

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The orthonormal polynomials satisfy the recurrence relation [5]

$$xp_n(w,x) = A_{n+1}p_{n+1}(w,x) + B_n p_n(w,x) + A_n p_{n-1}(w,x),$$

where

$$A_n := A_n(w) := \gamma_{n-1}(w)/\gamma_n(w)$$

and

$$B_n := B_n(w) := \int_{-\infty}^{\infty} xp_n^2(w,x)w(x)\,dx, \qquad n = 1, 2, 3, \ldots.$$

When considering weights with unbounded support, we shall find it convenient (as did G. Freud) to formulate results for $w = W^2$, where W is a nonnegative function such that W^2 is a weight function. In 1974 Freud [6] made the following

Conjecture. *Let* $\alpha > 0$, $\rho > -1$, *and*

$$W_{\alpha,\rho(x)} := |x|^{\rho/2}\exp(-|x|^\alpha), \qquad x \in \mathbf{R}.$$

Then

$$\lim_{n\to\infty} n^{-1/\alpha} A_n(W_{\alpha,\rho}^2) = \beta_\alpha/2, \tag{1.1}$$

where

$$\beta_\alpha := \lambda_\alpha^{-1/\alpha} \tag{1.2}$$

and

$$\lambda_\alpha := \Gamma(\alpha)/(2^{\alpha-2}\{\Gamma(\alpha/2)\}^2). \tag{1.3}$$

Note that $B_n(W_{\alpha,\rho}^2) = 0$, as $W_{\alpha,\rho}$ is even. Freud proved the conjecture for $\alpha = 2$, 4, 6 and $\rho > -1$. Subsequently, Al. Magnus [13], [14] proved the conjecture for α a positive even integer and also the analogue of Freud's conjecture for noneven weights such as $\exp(-P(x))$, where $P(x)$ is a polynomial of positive even degree with positive leading coefficient. Máté, Nevai, and Zaslavsky [16] sharpened Magnus's result to an asymptotic expansion for A_n for the weight $W_{\alpha,0}^2$, α a positive even integer. Recently, Bauldry, Máté, and Nevai [2] extended the results of [16] to some of the weights considered by Magnus in [14]. Several applications of Freud's conjecture are discussed by Nevai [22], and related physical applications have been discussed in Bessis, Itzykson, and Zuber [3] and in Pettifor and Weaire [24].

It is the purpose of this paper to prove Freud's conjecture for $W_{\alpha,\rho}^2$ for all $\alpha > 0$ and $\rho > -1$. We shall also prove the analogue of Freud's conjecture for more general weights and establish certain infinite-finite range inequalities for weighted polynomials in $L_p(\mathbf{R})$. Some of our results were announced in [12].

The paper is organized as follows: Section 2 contains the statement of our main results. In Section 3 we prove infinite-finite range inequalities. Finally, in Section 4, we prove Freud's conjecture and its generalization.

2. Statement of Results

In order to state our results, we shall need to define two classes of weights:

Definition 2.1. Let $W(x) := \exp(-Q(x))$, where $Q(x)$ is even, continuous and $Q'(x)$ exists for $x > 0$, while $xQ'(x)$ remains bounded as $x \to 0+$. Further, assume that $Q'''(x)$ exists for x large enough and for some $C > 0$ and $\alpha > 0$,

$$Q'(x) > 0, \quad x \text{ large enough,} \tag{2.1}$$

$$x^2|Q'''(x)|/Q'(x) \le C, \quad x \text{ large enough,} \tag{2.2}$$

and

$$\lim_{x\to\infty} (1 + xQ''(x)/Q'(x)) = \alpha. \tag{2.3}$$

Then we shall call W a *very smooth Freud weight* of order α and write $W \in \mathrm{VSF}(\alpha)$.

Note that if $\alpha > 0$ and $\beta \in \mathbf{R}$ then

$$W(x) = \exp(-|x|^\alpha (\log(2+x^2))^\beta) \in \mathrm{VSF}(\alpha).$$

The class of weights $\mathrm{VSF}(\alpha)$ was introduced in Lubinsky and Saff [11], in solving a conjecture of Saff [27, p. 252] on weighted polynomial approximations. The restrictions on $Q(x)$ arise in construction of even entire functions with nonnegative Maclaurin series coefficients that behave like $W^{-1}(x)$ on $\mathbf{R}$ (see Lubinsky [9], [10]) and in ensuring that the zero distribution of certain extremal polynomials is the Ullman distribution of order α [11], [19], [26].

Associated with each $W \in \mathrm{VSF}(\alpha)$, we define $a_n := a_n(W)$ to be the positive root of the equation

$$n = (2/\pi) \int_0^1 a_n x Q'(a_n x)(1-x^2)^{-1/2}\, dx. \tag{2.4}$$

It is shown in [11, Lemma 3.2] that for n large enough, a_n is uniquely defined. The number a_n was introduced by Mhaskar and Saff [18], [19] in studying the L_∞ norms, and asymptotic behavior, of weighted polynomials. When Q is even and convex, and $W(x) = \exp(-Q(x))$, Mhaskar and Saff [18], [19] showed that

$$\|PW\|_{L_\infty(\mathbf{R})} = \|PW\|_{L_\infty[-a_n, a_n]} \tag{2.5}$$

for each $P \in \mathcal{P}_n$, where $\mathcal{P}_n$ denotes the class of polynomials of degree at most n. Further, they showed that a_n is asymptotically best possible in (2.5).

In the special case $W := W_{\alpha,0}$, the quantity $a_n(W)$ takes a particularly simple form [17]:

$$a_n(W_{\alpha,0}) = \beta_\alpha n^{1/\alpha}, \qquad n = 1, 2, 3, \ldots, \quad \alpha > 0, \tag{2.6}$$

where β_α is given by (1.2) and (1.3).

Definition 2.2. Let

$$w(x) := \prod_{j=1}^{N} |x - z_j|^{\Delta_j}, \qquad x \in \mathbf{R}. \tag{2.7}$$

where $N \geq 1$; $z_1, z_2, \ldots, z_N$ are distinct complex numbers, $\Delta_1, \Delta_2, \ldots, \Delta_N \in \mathbf{R}$, and, for each real z_j, the corresponding $\Delta_j > -\frac{1}{2}$. Then we shall call w a *generalized Jacobi factor.*

We use the term "factor" rather than "weight" to distinguish w from the generalized Jacobi weights, considered by Nevai [21], which vanish outside $[-1, 1]$.

One of our main results is:

Theorem 2.3. *Let $W \in \mathrm{VSF}(\alpha)$ for some $\alpha > 0$, and let $a_n = a_n(W)$ be the root of (2.4) for n large enough. Further, let w be a generalized Jacobi factor, and let $P(x)$ be a polynomial of degree less than α. Finally, let*

$$\hat{W}(x) := \Psi(x) w(x) W(x) \exp(P(x)), \qquad x \in \mathbf{R}, \tag{2.8}$$

where $\Psi(x)$ is nonnegative in $\mathbf{R}$, $\Psi(x) \in L_\infty(\mathbf{R})$, and

$$\lim_{|x| \to \infty} \Psi(x) = 1. \tag{2.9}$$

Then the recurrence coefficients $A_n(\hat{W}^2)$, $B_n(\hat{W}^2)$ satisfy

$$\lim_{n \to \infty} A_n(\hat{W}^2)/a_n = \tfrac{1}{2} \tag{2.10}$$

and

$$\lim_{n \to \infty} B_n(\hat{W}^2)/a_n = 0. \tag{2.11}$$

The special case $\Psi \equiv 1$, $P \equiv 0$, $w(x) = |x|^{\rho/2}$, and $W(x) = \exp(-|x|^\alpha)$, $\alpha > 0$, establishes Freud's conjecture for $W^2_{\alpha,\rho}$, for all $\alpha > 0$, $\rho > -1$. The above result also contains the results of Magnus [14] for weights of the form $\exp(-c_1 x^m + p(x))$, where $c_1 > 0$, m is a positive even integer, and $p(x)$ is a polynomial of degree less than m.

We remark that the limits (2.10) and (2.11) may be reformulated in terms of Freud's quantity q_n, which, for n large enough, is taken to be the positive root of the equation

$$q_n Q'(q_n) = n.$$

In view of Lemma 3.2 in [11], we may rewrite (2.10) and (2.11) in the form

$$\lim_{n \to \infty} A_n(\hat{W}^2)/q_n = \beta_\alpha \alpha^{1/\alpha}/2 \quad \text{and} \quad \lim_{n \to \infty} B_n(\hat{W}^2)/q_n = 0,$$

where β_α is given by (1.2) and (1.3). In the special case

$$Q(x) := |x|^\alpha (\log(2 + x^2))^\beta, \qquad x \in \mathbf{R},$$

both q_n and a_n grow like $n^{1/\alpha}(\log n)^{-\beta/\alpha}$ as $n \to \infty$.

The factors Ψ, w, and e^P play no role in the description (2.10) and (2.11) of the asymptotic behavior of $A_n(\hat{W}^2)$ and $B_n(\hat{W}^2)$, and so one expects that (2.10) and (2.11) are valid for far more general $\hat{W}$. In this direction, we can replace e^P by a more general factor, but at the expense of a more cumbersome formulation:

Theorem 2.4. *Let W, w, and Ψ be as in Theorem* 2.3 *and let $V(x)$ be a function nonnegative in $\mathbf{R}$ with the following properties: $V(x)$ is bounded above in each finite interval;*

$$\lim_{|x|\to\infty} (\log V(x))/Q(x) = 0; \tag{2.12}$$

Given any sequence $\{\varepsilon_n\}_1^\infty$ of real numbers satisfying

$$\lim_{n\to\infty} \varepsilon_n = 0, \tag{2.13}$$

there exists, for large enough n, a positive integer l_n and $S_n \in P_{l_n}$ such that

$$\lim_{n\to\infty} l_n/n = 0, \tag{2.14}$$

$$\lim_{n\to\infty} |V(a_n(1+\varepsilon_n)x)S_n(x)| = 1 \quad \text{a.e. in } [-1, 1], \tag{2.15}$$

and, for some $C > 0$ and all n large enough,

$$\| V(a_n(1+\varepsilon_n)x)S_n(x)\|_{L_\infty[-1,1]} \le C. \tag{2.16}$$

Finally, let

$$\hat{W}(x) := \Psi(x)w(x)W(x)V(x), \qquad x \in \mathbf{R}. \tag{2.17}$$

Then the conclusions (2.10) *and* (2.11) *of Theorem* 2.3 *remain valid.*

We note that Theorem 2.3 follows from Theorem 2.4 by setting $V(x) := \exp(P(x))$ (see Section 4). In both Theorem 2.3 and Theorem 2.4, we may choose Ψ to be the characteristic function of the complement of finitely many compact intervals, so that the support of $\hat{W}$ consists of several disjoint intervals. Moreover, while it is not obvious, it can be shown that in both Theorems 2.3 and 2.4, the conditions on $\Psi(x)$ can be weakened in such a way that the support of $\hat{W}$ consists of infinitely many disjoint intervals whose complement is unbounded.

The proofs of Theorems 2.3 and 2.4 involve three main elements. The first, and possibly the most difficult, is the construction of polynomial approximations for the reciprocal of the weight $W(x)$—this task was completed in Lubinsky and Saff [11]. The second element is a sufficient condition for (2.10) and (2.11) to hold. This condition was established by Knopfmacher, Lubinsky, and Nevai [7], using the method of proof of Máté, Nevai, and Totik [15] for Rahmanov's theorem [25].

The final element consists of inequalities that relate the L_p norm of a weighted polynomial over $\mathbf{R}$ to its L_p norm over a finite interval. While Mhaskar and Saff [19], [20] established inequalities of this type for general weights and situations, the more refined inequalities of the type we need here were established by Lubinsky [8].

Theorem 2.5. *Let $0<p\le\infty$. Let $W=\exp(-Q)\in \mathrm{VSF}(\alpha)$ for some $\alpha>0$, except that we do not assume that $Q'''(x)$ exists for any $x\in\mathbf{R}$, or that (2.2) is satisfied. Let $\hat{W}(x)$ be a nonnegative function such that $\hat{W}(x)\in L_p(\mathbf{R})$ and*

$$\lim_{|x|\to\infty}\{\log(1/\hat{W}(x))\}/Q(x)=1. \tag{2.18}$$

Let $a_n=a_n(W)$ for n large enough. Then there exist sequences $\{\varepsilon_n\}_1^\infty$ and $\{\delta_n\}_1^\infty$ of positive real numbers such that

$$\lim_{n\to\infty}\varepsilon_n=0, \tag{2.19}$$

$$\lim_{n\to\infty}\delta_n\exp(n/\log n)=0, \tag{2.20}$$

and, for n large enough and each $P\in\mathcal{P}_n$,

$$\|P\hat{W}\|_{L_p(\mathbf{R})}\le(1+\delta_n)\|P\hat{W}\|_{L_p[-a_n(1+\varepsilon_n),a_n(1+\varepsilon_n)]}. \tag{2.21}$$

In (2.20) the numbers $\{\log n\}_1^\infty$ may be replaced by $\{\theta_n\}_1^\infty$, where $\theta_n\to\infty$ as $n\to\infty$, arbitrarily slowly. In special cases, we can also give a rate for $\{\varepsilon_n\}_1^\infty$.

Theorem 2.6. *Let $W(x):=\exp(-Q(x))$, where $Q(x):=|x|^\alpha$, some $\alpha>0$, or $Q(x)$ is even, continuous in $\mathbf{R}$, convex in $(0,\infty)$, and $Q'(x)$ exists and is positive in $(0,\infty)$. Let $0<p<\infty$ and let $\{K_n\}_1^\infty$ be a sequence of positive numbers such that, for n large enough,*

$$K_n\ge\begin{cases}(2+\alpha^{-1})/p & \text{if } Q(x)=|x|^\alpha,\ \alpha<1,\\ 4/p & \text{if } Q(X) \text{ is convex.}\end{cases} \tag{2.22}$$

Further, for $n=2,3,4,\dots$, let

$$\varepsilon_n:=\begin{cases}(4\alpha^{-1}K_n(\log n)/n)^{2/3} & \text{if } Q(x)=|x|^\alpha,\ \alpha<1,\\ (4\sqrt{6}K_n(\log n)/n)^{2/3} & \text{if } Q(X) \text{ is convex.}\end{cases} \tag{2.23}$$

and assume that

$$\lim_{n\to\infty}\varepsilon_n=0. \tag{2.24}$$

Then there exists $C_1>0$ such that, for $n=2,3,4,\dots$ and each $P\in\mathcal{P}_n$,

$$\|PW\|_{L_p(\mathbf{R})}\le(1+C_1n^{-K_n})\|PW\|_{L_p[-a_n(1+\varepsilon_n),a_n(1+\varepsilon_n)]}. \tag{2.25}$$

We remark that if $\alpha=1$, one may choose $\varepsilon_n:=(K_n(\log\log n)/n)^{2/3}$ and if $\alpha<1$, one may choose $\varepsilon_n:=(K_n/n)^{2/3}$ in (2.23), but the right-hand side of (2.25) then becomes more complicated. The difference to our present proof involves application of sharper Nikolskii inequalities for $\exp(-|x|^\alpha)$, $\alpha\le1$ (see Nevai and Totik [23]).

It seems likely that the above result should remain valid when the convexity of $Q(x)$ is replaced by the weaker condition that $xQ'(x)$ increases to $+\infty$ as x increases to $+\infty$.

3. Proofs of Theorems 2.5 and 2.6

Throughout, C, C_1, $C_2, \ldots$ denote positive constants independent of n and x. The same symbol does not necessarily denote the same constant from line to line.

The proofs of Theorems 2.5 and 2.6 will be organized as follows: we first list some technical lemmas, which summarize some results from [11] and [17]. Then we prove Theorem 2.6 in the special case $W(x)=\exp(-|x|^\alpha)$ and use this latter result to prove Theorem 2.5. Finally, we use the Poisson kernel for the exterior of a line segment to prove Theorem 2.6 for $W(x)=\exp(-Q(x))$ when $Q(x)$ is even and convex.

In describing the asymptotic behavior of extremal and orthogonal polynomials associated with weights such as

$$W_\alpha(x):=W_{\alpha,0}(x):=\exp(-|x|^\alpha), \qquad x\in\mathbf{R}, \quad \alpha>0,$$

an important role is played by the *Ullman distribution*

$$v(\alpha;x):=\begin{cases}\dfrac{\alpha}{\pi}\displaystyle\int_{|x|}^{1} t^{\alpha-1}(t^2-x^2)^{-1/2}\,dt, & x\in[-1,1],\\ 0, & \text{otherwise.}\end{cases} \tag{3.1}$$

Associated with $v(\alpha;x)$ is the function

$$U_\alpha(z):=\int_{-1}^{1}\log|z-t|\,v(\alpha;t)\,dt-|z|^\alpha/\lambda_\alpha+\log 2+1/\alpha, \tag{3.2}$$

$z\in\mathbf{C}$, where λ_α is given by (1.3).

Lemma 3.1. *Let $\alpha>0$. Then:*

(i) *$U_\alpha(z)$ is even, continuous in $\mathbf{C}$, and $U_\alpha(x)=0$ for $x\in[-1,1]$.*

(ii) *As $\varepsilon\to 0+$,*

$$U'_\alpha(1+\varepsilon)=-\alpha\sqrt{2\varepsilon}+O(\varepsilon). \tag{3.3}$$

(iii) *As $\varepsilon\to 0+$,*

$$U_\alpha(1+\varepsilon)=-\alpha\frac{2\sqrt{2}}{3}\varepsilon^{3/2}+O(\varepsilon^2). \tag{3.4}$$

(iv) *For $n=1,2,3,\ldots$ and each $P\in\mathcal{P}_n$, there holds, for all $x\in\mathbf{R}$,*

$$|P(x)W_\alpha^n(\beta_\alpha x)|\le e^{nU_\alpha(x)}\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_\infty[-1,1]}. \tag{3.5}$$

(v) *Let $\delta>0$, $0<p\le\infty$. There exists $C>0$ such that, for n large enough and $P\in\mathcal{P}_n$,*

$$\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(|x|\ge 1+\delta)}\le e^{-Cn}\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(\mathbf{R})}. \tag{3.6}$$

Proof. (i) This follows from Proposition 2.3 and equation (2.9) in [17, pp. 207–208].

(ii) Now

$$U'_\alpha(x) = \int_{-1}^{1} v(\alpha; t)/(x-t)\, dt - \alpha x^{\alpha-1}/\lambda_\alpha, \qquad x \in (1, \infty). \tag{3.7}$$

We see using the Monotone Convergence Theorem that, as $x \to 1+$,

$$U'_\alpha(x) \to \int_{-1}^{1} \frac{v(\alpha; t)}{1-t}\, dt - \frac{\alpha}{\lambda_\alpha} = 0,$$

by (A.6) in Lubinsky and Saff [11]. Further, writing $x = 1+\varepsilon$, and using (A.6) in [11],

$$\begin{aligned} U'_\alpha(1+\varepsilon) &= \int_{-1}^{1} \frac{v(\alpha; t)}{1+\varepsilon-t}\, dt - \frac{\alpha(1+\varepsilon)^{\alpha-1}}{\lambda_\alpha} \\ &= \int_{-1}^{1} \frac{v(\alpha; t)}{1+\varepsilon-t}\, dt - \frac{\alpha}{\lambda_\alpha} + O(\varepsilon) \\ &= \int_{-1}^{1} v(\alpha; t)\left[\frac{1}{1+\varepsilon-t} - \frac{1}{1-t}\right] dt + O(\varepsilon) \\ &= -\varepsilon \int_{-1}^{1} \frac{v(\alpha; t)}{(1+\varepsilon-t)(1-t)}\, dt + O(\varepsilon). \end{aligned}$$

It follows that

$$U'_\alpha(1+\varepsilon) = -\varepsilon \int_{1/2}^{1} \frac{v(\alpha; t)}{(1+\varepsilon-t)(1-t)}\, dt + O(\varepsilon), \tag{3.8}$$

as $\varepsilon \to 0+$. Now, by (A.4) in [11], uniformly for $\frac{1}{2} \le x \le 1$,

$$v(\alpha; x) = \frac{\alpha}{\pi}\sqrt{1-x^2} + O((1-x^2)^{3/2}). \tag{3.9}$$

Hence, uniformly for $0 \le \varepsilon u \le \frac{1}{2}$,

$$\begin{aligned} v(\alpha; 1-\varepsilon u) &= \frac{\alpha}{\pi}\sqrt{2\varepsilon u - (\varepsilon u)^2} + O((2\varepsilon u - (\varepsilon u)^2)^{3/2}) \\ &= \frac{\alpha\sqrt{2\varepsilon u}}{\pi} + O((\varepsilon u)^{3/2}). \end{aligned}$$

Letting $t = 1 - \varepsilon u$ in the integral in (3.8), we obtain

$$\begin{aligned} U'_\alpha(1+\varepsilon) &= -\int_{0}^{1/2\varepsilon} \frac{\alpha\pi^{-1}\sqrt{2\varepsilon u} + O((\varepsilon u)^{3/2})}{(u+1)u}\, du + O(\varepsilon) \\ &= -\alpha\pi^{-1}\sqrt{2\varepsilon} \int_{0}^{1/2\varepsilon} \frac{du}{u^{1/2}(u+1)} + O\left(\varepsilon^{3/2} \int_{0}^{1/2\varepsilon} u^{-1/2}\, du\right) + O(\varepsilon) \\ &= -\alpha\pi^{-1}\sqrt{2\varepsilon}\left(\int_{0}^{\infty} \frac{du}{u^{1/2}(u+1)} + O(\varepsilon^{1/2})\right) + O(\varepsilon) \\ &= -\alpha\sqrt{2\varepsilon} + O(\varepsilon), \end{aligned}$$

using a standard integral (see Dwight [4, p. 213, no. 856.02]).

(iii) This follows by integrating (ii) and using $U_\alpha(1)=0$.
(iv) This is (3.20) in [11].
(v) This is an immediate consequence of (3.18) in [11]. ■

We next prove a crude Nikolskii inequality:

Lemma 3.2. *Let $W(x)=\exp(-Q(x))$, where $Q(x)$ is bounded in each finite interval, and, for some $C>0$, the function $Q(x)$ is increasing in (C,∞) and decreasing in $(-\infty,-C)$, while*

$$\lim_{|x|\to\infty} Q(x)/\log|x| = \infty.$$

Let $0<p<\infty$. Then there exists C_1 depending on W and p only, such that, for $n=1$, 2, 3, ... and $P\in\mathscr{P}_n$,

$$\|PW\|_{L_\infty(\mathbf{R})} \le C_1 n^{2/p} \|PW\|_{L_p(\mathbf{R})}. \tag{3.10}$$

Proof. Let $P\in\mathscr{P}_n$, and let ξ be such that

$$|PW|(\xi) \ge (\tfrac{1}{2})\|PW\|_{L_\infty(\mathbf{R})}.$$

Suppose first that $\xi\ge C+1$. Then, by standard Nikolskii inequalities [21, pp. 106–114],

$$\begin{aligned}\|PW\|_{L_p(\mathbf{R})} &\ge \|PW\|_{L_p[\xi-1,\xi]}\\ &\ge W(\xi)\|P\|_{L_p[\xi-1,\xi]}\\ &\ge C_2 n^{-2/p} W(\xi)\|P\|_{L_\infty[\xi-1,\xi]}\\ &\ge C_2 n^{-2/p}|PW|(\xi) \ge (C_2/2)n^{-2/p}\|PW\|_{L_\infty(\mathbf{R})}.\end{aligned}$$

If $\xi\le -C-1$, the result follows similarly. If $|\xi|\le C+1$, the result again follows in a similar way, since $W^{\pm 1}(x)$ is bounded in $[-C-1, C+1]$. ■

Proof of Theorem 2.6 in the case $Q(x)=|x|^\alpha$, $\alpha>0$. Making a substitution in (2.25), we see that it suffices to prove that, for some $C_1>0$ and all $P\in\mathscr{P}_n$,

$$\|P(x)W_\alpha(a_n x)\|_{L_p(\mathbf{R})} \le (1+C_1 n^{-K_n})\|P(x)W_\alpha(a_n x)\|_{L_p[-1-\varepsilon_n, 1+\varepsilon_n]}, \tag{3.11}$$

where $W_\alpha(x) := W_{\alpha,0}(x) := \exp(-|x|^\alpha)$, $x\in\mathbf{R}$. Now, by (2.6),

$$a_n(W_\alpha) = \beta_\alpha n^{1/\alpha},$$

so that

$$W_\alpha(a_n x) = W_\alpha^n(\beta_\alpha x), \qquad x\in\mathbf{R}.$$

Hence (3.11) is equivalent to

$$\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(\mathbf{R})} \le (1+C_1 n^{-K_n})\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p[-1-\varepsilon_n, 1+\varepsilon_n]}. \tag{3.12}$$

It is not difficult to see that it then suffices to prove, for some $C_2>0$ and all $P\in\mathscr{P}_n$, $n=2,3,4,\ldots$,

$$\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(|x|\ge 1+\varepsilon_n)} \le C_2 n^{-K_n}\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(\mathbf{R})}, \tag{3.13}$$

where $\{\varepsilon_n\}_1^\infty$ is as in (2.23). Let $\delta > 0$ be so small that

$$U_\alpha(1+\varepsilon) \le -(\alpha/2)\varepsilon^{3/2}, \qquad \varepsilon \in [0, \delta],$$

such a choice of δ being possible by (3.4). Then, for $\varepsilon \in [0, \delta]$, Lemma 3.1(iv) yields

$$\begin{aligned}
&\|P(x)\, W_\alpha^n(\beta_\alpha x)\|_{L_p(1+\varepsilon \le |x| \le 1+\delta)} \\
&\quad \le \exp(-n(\alpha/2)\varepsilon^{3/2}) \|P(x)\, W_\alpha^n(\beta_\alpha x)\|_{L_\infty(\mathbf{R})} (2(\delta-\varepsilon))^{1/p} \\
&\quad \le C_3(2(\delta-\varepsilon))^{1/p} \exp(-n(\alpha/2)\varepsilon^{3/2}) n^{2/p} n^{1/(\alpha p)} \|P(x)\, W_\alpha^n(\beta_\alpha x)\|_{L_p(\mathbf{R})},
\end{aligned}$$

by Lemma 3.2 and a substitution. Note that C_3 is independent of n, ε, δ, and P. Choosing δ small enough, we obtain, for $P \in \mathcal{P}_n$, $\varepsilon \in [0, \delta]$,

$$\begin{aligned}
&\|P(x)\, W_\alpha^n(\beta_\alpha x)\|_{L_p(1+\varepsilon \le |x| \le 1+\delta)} \\
&\quad \le \exp(-n(\alpha/2)\varepsilon^{3/2} + (2+\alpha^{-1})p^{-1}\log n) \|P(x)\, W_\alpha^n(\beta_\alpha x)\|_{L_p(\mathbf{R})}.
\end{aligned} \tag{3.14}$$

Combining Lemma 3.1(v) and (3.14), we see that there exists $\delta_0 \le \delta$ such that, for $\varepsilon \in [0, \delta_0]$ and $P \in \mathcal{P}_n$, n large enough,

$$\begin{aligned}
&\|P(x)\, W_\alpha^n(\beta_\alpha x)\|_{L_p(|x| \ge 1+\varepsilon)} \\
&\quad \le 2\exp(-n(\alpha/2)\varepsilon^{3/2} + (2+\alpha^{-1})p^{-1}\log n) \|P(x)\, W_\alpha^n(\beta_\alpha x)\|_{L_p(\mathbf{R})}.
\end{aligned} \tag{3.15}$$

Next, note that if $\varepsilon = \varepsilon_n$ is given by (2.23) and $\varepsilon_n \le \delta_0$, then

$$\begin{aligned}
-n(\alpha/2)\varepsilon_n^{3/2} + (2+\alpha^{-1})p^{-1}\log n &\le (\log n)\{-2K_n + (2+\alpha^{-1})p^{-1}\} \\
&\le -K_n \log n,
\end{aligned}$$

by (2.22), and by considering separately the cases $\alpha < 1$ and $\alpha \ge 1$. Thus, by (3.15), for n large enough and $P \in \mathcal{P}_n$,

$$\|P(x)\, W_\alpha^n(\beta_\alpha x)\|_{L_p(|x| \ge 1+\varepsilon_n)} \le 2n^{-K_n} \|P(x)\, W_\alpha^n(\beta_\alpha x)\|_{L_p(\mathbf{R})},$$

establishing (3.13) and hence Theorem 2.6 in this case. ■

In the proof of Theorem 2.5, we shall need two lemmas that are essentially drawn from [11].

Lemma 3.3. *Let α, p, W, and $\hat{W}$ be as in Theorem 2.5. Let $0 < \varepsilon < \alpha$ and $a_n = a_n(W)$ for n large enough. Then:*

(i) *For $|x|$ large enough,*

$$|x|^{\alpha-\varepsilon-1} \le |Q'(x)| \le |x|^{\alpha+\varepsilon-1}. \tag{3.16}$$

(ii) *For $|x|$ large enough,*

$$|x|^{\alpha-\varepsilon} \le |Q(x)| \le |x|^{\alpha+\varepsilon}. \tag{3.17}$$

(iii) *For n large enough,*

$$n^{1/(\alpha+\varepsilon)} \le a_n \le n^{1/(\alpha-\varepsilon)}. \tag{3.18}$$

(iv) *For each $K > 0$,*

$$\lim_{n\to\infty} Q(Ka_n)/n = K^\alpha/\lambda_\alpha. \tag{3.19}$$

(v) *Uniformly in any compact subset of $\mathbf{R}\backslash\{0\}$,*

$$\lim_{n\to\infty} \hat{W}(a_n x)^{1/n} = W_\alpha(\beta_\alpha x). \tag{3.20}$$

Proof. (i) By (2.3), for x large enough,

$$(\alpha-\varepsilon/2-1)/x \le Q''(x)/Q'(x) \le (\alpha+\varepsilon/2-1)/x$$

and integrating, we obtain, for x large enough,

$$(\alpha-\varepsilon-1)\log x \le \log Q'(x) \le (\alpha+\varepsilon-1)\log x,$$

which yields (3.16).

(ii) By Lemma 3.1(iii) in [11],

$$\lim_{x\to\infty} xQ'(x)/Q(x)=\alpha, \tag{3.21}$$

and integrating as before yields (3.17).

(iii) By Lemma 3.2(ii) in [11],

$$\lim_{n\to\infty} a_nQ'(a_n)/n=\alpha/\lambda_\alpha. \tag{3.22}$$

Combined with (3.16), this yields (3.18).

(iv) By Lemma 3.1(iv) in [11], for each $K>0$,

$$\lim_{n\to\infty} Q(Ka_n)/Q(a_n)=K^\alpha. \tag{3.23}$$

Further, by (3.21),

$$\lim_{n\to\infty} Q(a_n)/(a_nQ'(a_n))=\alpha^{-1}.$$

Multiplying this limit by those in (3.22) and (3.23) yields (3.19).

(v) Let

$$J(x):=\hat{W}(x)/W(x), \qquad x\in\mathbf{R}. \tag{3.24}$$

Then $J(x)\in L_p[-a,a]$ for each $a>0$, and, by (2.18),

$$\lim_{|x|\to\infty} Q(x)^{-1}\log J(x)=0. \tag{3.25}$$

Since $Q(x)$ is even, and increasing for x large enough, we then see that, if $0<A<B<\infty$,

$$\sup\{J(a_nx)^{\pm 1/n}\colon |x|\in[A,B]\}\le \exp\left(\{Q(Ba_n)/n\}\left\{\sup_{|u|\ge Aa_n}|Q(u)^{-1}\log J(u)|\right\}\right)$$

$$\to 1 \qquad \text{as} \quad n\to\infty,$$

by (3.19) and (3.25). Hence, also, uniformly for $A\le|x|\le B$,

$$\lim_{n\to\infty}\hat{W}(a_nx)^{1/n}=\lim_{n\to\infty}W(a_nx)^{1/n}=W_\alpha(\beta_\alpha x),$$

where the last equality follows from Lemma 3.2(iii) in [11]. ■

Lemma 3.4. *Let α, p, W, and $\hat{W}$ be as in Theorem 2.5. Let $a_n=a_n(W)$ for n large enough.*

(i) *Let $\delta > 0$. There exists $C > 0$ such that, for n large enough and $P \in \mathcal{P}_n$,*

$$\|P(x)\hat{W}(a_n x)\|_{L_p(|x| \geq 1+\delta)} \leq e^{-Cn}\|P(x)\hat{W}(a_n x)\|_{L_p(\mathbf{R})}. \tag{3.26}$$

(ii) *There exists a sequence $\{\rho_n\}_1^\infty$ of positive numbers such that*

$$\lim_{n\to\infty} \rho_n^{1/n} = 1 \tag{3.27}$$

and, for $n = 1, 2, 3, \ldots$ and all $P \in \mathcal{P}_n$,

$$\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(\mathbf{R})} \leq \rho_n\|P(x)\hat{W}(a_n x)\|_{L_p(\mathbf{R})}. \tag{3.28}$$

Proof. (i) Let $J(x)$ be given by (3.24). Since (3.25) holds, Lemma 3.5 in [11] yields (3.26).

(ii) This follows from (3.43) and (3.45) in Lemma 3.6 in [11]. ■

Proof of Theorem 2.5. In view of (3.26), it suffices to show that, for some $\delta > 0$, there exist $\{\varepsilon_n\}_1^\infty$ and $\{\delta_n\}_1^\infty$ satisfying (2.19) and (2.20) such that

$$\|P(x)\hat{W}(a_n x)\|_{L_p(1+\varepsilon_n < |x| \leq 1+\delta)} \leq \delta_n\|P(x)\hat{W}(a_n x)\|_{L_p(\mathbf{R})}; \tag{3.29}$$

a substitution and some elementary manipulations then yield (2.21). By Lemma 3.3(v), there exists $\{\rho_n\}_1^\infty$ satisfying (3.27) such that

$$\hat{W}(a_n x) \leq \rho_n W_\alpha^n(\beta_\alpha x), \qquad 1 \leq |x| \leq 1+\delta, \quad n = 1, 2, 3, \ldots .$$

Then, for $0 \leq \varepsilon \leq \delta_0$, (3.15) yields, for $P \in \mathcal{P}_n$, n large enough,

$$\begin{aligned}\|P(x)\hat{W}(a_n x)\|_{L_p(1+\varepsilon < |x| \leq 1+\delta)} &\leq 2\rho_n \exp(-n(\alpha/2)\varepsilon^{3/2} \\ &\quad + (2+\alpha^{-1})p^{-1}\log n)\|P(x)W_\alpha^n(\beta_\alpha x)\|_{L_p(\mathbf{R})} \\ &\leq 2\rho_n\hat{\rho}_n \exp(-n(\alpha/2)\varepsilon^{3/2} + (2+\alpha^{-1})p^{-1}\log n) \\ &\quad \|P(x)\hat{W}(a_n x)\|_{L_p(\mathbf{R})},\end{aligned}$$

where, by Lemma 3.4(ii), $\{\hat{\rho}_n\}_1^\infty$ is independent of ε and P and

$$\lim_{n\to\infty} \hat{\rho}_n^{1/n} = 1.$$

Thus we may write

$$2\rho_n\hat{\rho}_n \exp((2+\alpha^{-1})p^{-1}\log n) = \exp(n\eta_n), \qquad n = 1, 2, 3, \ldots,$$

where

$$\lim_{n\to\infty} \eta_n = 0.$$

Then, for $0 \leq \varepsilon \leq \delta_0$, n large enough, and $P \in \mathcal{P}_n$,

$$\|P(x)\hat{W}(a_n x)\|_{L_p(1+\varepsilon \leq |x| \leq 1+\delta)} \leq \exp(n\{\eta_n - (\alpha/2)\varepsilon^{3/2}\})\|P(x)\hat{W}(a_n x)\|_{L_p(\mathbf{R})}.$$

Choose $\varepsilon := \varepsilon_n$ such that

$$(\alpha/2)\varepsilon_n^{3/2} \geq 4 \max\{\eta_n, (\log n)^{-1}\}, \qquad n = 2, 3, 4, \ldots,$$

and such that (2.19) holds. It is then easily seen that

$$\delta_n := \exp(n\{\eta_n - (\alpha/2)\varepsilon_n^{3/2}\}), \qquad n = 2, 3, 4, \ldots,$$

satisfies (2.20). Then (3.29) and the theorem follow. ■

In proving Theorem 2.6 for $W = e^{-Q}$, where Q is even and convex, we shall make use of the Poisson kernel for the exterior of a segment, as did Rahmanov [26]. In fact, the function $U_\alpha(x)$ may be derived using the Poisson kernel for the exterior of $[-1, 1]$. Let $Q(x)$ be continuous in $[-R, R]$ for some $R > 0$ and for $z \in \mathbf{C}\backslash[-R, R]$, let

$$(3.30) \qquad u(z, R, Q) := \pi^{-1} \int_{-R}^{R} \operatorname{Re}\left\{\frac{\sqrt{z^2 - R^2}}{z - t}\right\} \frac{Q(t)}{\sqrt{R^2 - t^2}}\, dt,$$

where the branch of the square root is chosen so that $\sqrt{z^2 - R^2}$ behaves like z as $z \to \infty$.

The function $u(z, R, Q)$ is harmonic in $\bar{\mathbf{C}}\backslash[-R, R]$, and has boundary values $Q(x)$ in $[-R, R]$. More precisely, as $Q(x)$ is continuous in $[-R, R]$,

$$u(z, R, Q) \to Q(x) \quad \text{as} \quad z \to x \in [-R, R].$$

These properties may be derived from well-known properties of the Poisson kernel for the unit disc [1] with the aid of a conformal map of $\mathbf{C}\backslash[-1, 1]$ onto $\{z : |z| > 1\}$, and then an inversion $z \to 1/z$.

Lemma 3.5. *Let $R > 0$, $Q(x)$ be continuous in $\mathbf{R}$, $W(x) := \exp(-Q(x))$, and let*

$$(3.31) \qquad H_n(x, R) := u(x, R, Q) - Q(x) + n \log|\Phi(x/R)|,$$

$|x| > R$, where, for $z \in \mathbf{C}\backslash[-1, 1]$,

$$(3.32) \qquad \Phi(z) := z + \sqrt{z^2 - 1}.$$

Then, for $n = 1, 2, 3, \ldots$, $P \in \mathcal{P}_n$, and $|x| > R$,

$$(3.33) \qquad |P(x)W(x)| \leq \|PW\|_{L_\infty[-R,R]} \exp(H_n(x, R)).$$

Proof. Now $f(z) := \log|P(z)\Phi(z/R)^{-n}| - u(z, R, Q)$ is subharmonic in $\bar{\mathbf{C}}\backslash[-R, R]$ and as $z \to x \in [-R, R]$,

$$f(z) \to \log|P(x)W(x)|.$$

By the maximum principle for subharmonic functions,

$$f(z) \leq \log\|PW\|_{L_\infty[-R,R]}, \qquad z \in \bar{\mathbf{C}}\backslash[-R, R].$$

Exponentiating this inequality, and using (3.31), we obtain (3.33). ■

We next need some properties of $H_n(x, R)$:

Lemma 3.6. *Let $Q(x)$ be even, continuous in $\mathbf{R}$, convex in $(0, \infty)$ and assume that $Q'(x)$ exists and is positive in $(0, \infty)$. Let $W = e^{-Q}$. Let $H_n'(x, R)$ denote the* *derivative of* $H_n(x, R)$ *with respect to* x *for fixed* $R>0$. *Then, for* $s\in(1,\infty)$,

$$H_n'(Rs, R)R(s^2-1)^{1/2}\le n-\frac{2s}{\pi}\int_0^1\frac{Q(Rs)-Q(Rt)}{s-t}\frac{s}{s+t}\frac{dt}{\sqrt{1-t^2}}. \tag{3.34}$$

Further, if $a_n=a_n(W)$ *for* n *large enough,*

$$H_n'(a_ns, a_n)\le -\frac{n}{a_n}\sqrt{\frac{s-1}{s+1}},\qquad s\in(1,\infty), \tag{3.35}$$

and if $0<\varepsilon<1$,

$$H_n(a_ns, a_n)\le -n\sqrt{\varepsilon/3}(s-1-\varepsilon),\qquad s\in(1+\varepsilon,\infty). \tag{3.36}$$

Proof. Using the evenness of $Q(x)$, we see from (3.30) that, for $x>R$,

$$u(x, R, Q)=\frac{2}{\pi}(x^2-R^2)^{1/2}\int_0^1\frac{Q(Rt)}{\sqrt{1-t^2}}\frac{x}{(x-Rt)}\frac{dt}{(x+Rt)}.$$

Further, since $u(x, R, 1)\equiv 1$, (3.31) yields, for $x>R$,

$$H_n(x, R)=-\frac{2}{\pi}(x^2-R^2)^{1/2}\int_0^1\frac{Q(x)-Q(Rt)}{x-Rt}\frac{x}{x+Rt}\frac{dt}{\sqrt{1-t^2}} \tag{3.37}$$
$$+n\log\Phi\left(\frac{x}{R}\right).$$

Since $\Phi'(x)/\Phi(x)=(x^2-1)^{-1/2}$, $x>1$, we obtain

$$H_n'(x, R)=-\frac{2}{\pi}\frac{x}{\sqrt{x^2-R^2}}\int_0^1\frac{Q(x)-Q(Rt)}{x-Rt}\frac{x}{x+Rt}\frac{dt}{\sqrt{1-t^2}} \tag{3.38}$$
$$-\frac{2}{\pi}\sqrt{x^2-R^2}\int_0^1\frac{d}{dx}\left\{\frac{Q(x)-Q(Rt)}{x-Rt}\frac{x}{x+Rt}\right\}\frac{dt}{\sqrt{1-t^2}}$$
$$+n/\sqrt{x^2-R^2}.$$

Since $Q(x)$ is convex, $(Q(x)-Q(Rt))/(x-Rt)$ increases as $x>Rt$ increases. Further, $x/(x+Rt)$ is also an increasing function for $x>0$. Hence, the second integral in the right-hand side of (3.38) is nonnegative and so, for $x>R$,

$$H_n'(x, R)\sqrt{x^2-R^2}\le n-\frac{2x}{\pi}\int_0^1\frac{Q(x)-Q(Rt)}{x-Rt}\frac{x}{x+Rt}\frac{dt}{\sqrt{1-t^2}}. \tag{3.39}$$

Letting $x=Rs$ then yields (3.34). Next, note that by definition (2.4) of a_n and an integration by parts,

$$n=\frac{2}{\pi}\int_0^1 a_ntQ'(a_nt)(1-t^2)^{-1/2}\,dt \tag{3.40}$$
$$=\frac{2}{\pi}\int_0^1\frac{Q(a_n)-Q(a_nt)}{(1-t^2)^{3/2}}\,dt.$$

Since $(Q(a_ns)-Q(a_nt))/(s-t)$ and $s/(s+t)$ are increasing with s, (3.34) shows that, for $s>1$,

$$H_n'(a_ns, a_n)a_n(s^2-1)^{1/2}\le n-\frac{2s}{\pi}\int_0^1\frac{Q(a_n)-Q(a_nt)}{1-t}\frac{1}{1+t}\frac{dt}{\sqrt{1-t^2}}$$
$$=n(1-s),$$

by (3.40). Then (3.35) follows on division by $a_n(s^2-1)^{1/2}$. Next, in view of the continuity of $Q(x)$, as $s \to 1^+$,

$$H_n(a_n s, a_n) \to Q(a_n) - Q(a_n) + n \log|\Phi(1)| = 0.$$

Then, from (3.35), for $s > 1+\varepsilon$,

$$\begin{aligned} H_n(a_n s, a_n) &= H_n(a_n(1+\varepsilon), a_n) + a_n \int_{1+\varepsilon}^{s} H_n'(a_n t, a_n)\, dt \\ &\le H_n(a_n, a_n) - n \int_{1+\varepsilon}^{s} \sqrt{\frac{t-1}{t+1}}\, dt \\ &\le 0 - n\sqrt{\frac{\varepsilon}{2+\varepsilon}}\,(s-1-\varepsilon), \end{aligned}$$

and, as $\varepsilon < 1$, (3.36) follows. ■

Proof of Theorem 2.6 for $Q(x)$ even and convex. Let $\varepsilon > 0$, $n \ge 1$, and $P \in \mathscr{P}_n$. By Lemma 3.5,

$$\begin{aligned} &\|PW\|_{L_p(|x| \ge a_n(1+2\varepsilon))} \\ &\quad \le \|PW\|_{L_\infty[-a_n, a_n]} \left(2 \int_{a_n(1+2\varepsilon)}^{\infty} \exp(pH_n(x, a_n))\, dx\right)^{1/p} \\ &\quad \le C_1 n^{2/p} \|PW\|_{L_p(\mathbf{R})} \left(2a_n \int_{1+2\varepsilon}^{\infty} \exp(-pn\sqrt{\varepsilon/3}\{s-1-\varepsilon\})\, ds\right)^{1/p}, \end{aligned}$$

by Lemma 3.2 and (3.36) in Lemma 3.6. Evaluating this last integral, we see that, for $\varepsilon > 0$, $n \ge 1$, and $P \in \mathscr{P}_n$,

$$\begin{aligned} &\|PW\|_{L_p(|x| \ge a_n(1+2\varepsilon))} \\ &\quad \le C_1 n^{2/p} \|PW\|_{L_p(\mathbf{R})} \{2\sqrt{3}\, a_n (pn\sqrt{\varepsilon})^{-1}\}^{1/p} \exp(-n\sqrt{\varepsilon/3}\,\varepsilon). \end{aligned} \tag{3.41}$$

Since $Q'(x)$ is nondecreasing, it is easily seen from (2.4) that, for some C_3,

$$a_n \le C_3 n, \quad n \text{ large enough.}$$

Further, it follows from (2.22) and (2.23) that, for some C_4,

$$\varepsilon_n \ge C_4 n^{-2/3}, \quad n \text{ large enough.}$$

Then taking $\varepsilon = \varepsilon_n/2$ in (3.41), we see that, for some C_5 independent of P and n,

$$\begin{aligned} \|PW\|_{L_p(|x| \ge a_n(1+\varepsilon_n))} &\le C_5 \|PW\|_{L_p(\mathbf{R})} \exp((\log n)(2+\tfrac{1}{3})/p - n\varepsilon_n^{3/2}/(2\sqrt{6})) \\ &= C_5 \|PW\|_{L_p(\mathbf{R})} \exp((\log n)\{(7/(3p)) - 2K_n\}) \\ &\le C_5 \|PW\|_{L_p(\mathbf{R})} \exp(-(\log n) K_n), \end{aligned}$$

by (2.22) and (2.23). Then (2.25) follows. ■

4. Proof of Theorems 2.3 and 2.4

We shall need the following result of Knopfmacher, Lubinsky, and Nevai [7].

Lemma 4.1. *Let $\hat{W}(x)$ be a nonnegative function such that $\hat{W}^2(x)$ is a weight function. Suppose there exists an increasing sequence of positive numbers $\{c_n\}_1^\infty$ and a decreasing sequence of positive numbers $\{\delta_n\}_1^\infty$ such that*

$$\lim_{n\to\infty} \delta_n = 0, \tag{4.1}$$

and such that, for $n = 1, 2, 3, \ldots$ and $P \in \mathcal{P}_n$,

$$\|P\hat{W}\|_{L_2(\mathbf{R})} \le (1+\delta_n)\|P\hat{W}\|_{L_2(-c_n,c_n)}. \tag{4.2}$$

Suppose, further, that there exist real polynomials $S_{n-2}(x)$ of degree at most $n-2$, $n = 2, 3, 4, \ldots$, such that, for $p = \frac{1}{2}$ and $p = 2$,

$$\lim_{n\to\infty} \pi^{-1} \int_{-1}^{1} |\hat{W}(c_n x) S_{n-2}(x)(1-x^2)^{1/4}|^p \, dx/\sqrt{1-x^2} = 1. \tag{4.3}$$

Then

$$\lim_{n\to\infty} A_n(\hat{W}^2)/c_n = \tfrac{1}{2} \tag{4.4}$$

and

$$\lim_{n\to\infty} B_n(\hat{W}^2)/c_{n+1} = 0. \tag{4.5}$$

The proof of this result uses the method of proof of Máté, Nevai, and Totik [15] of Rahmanov's theorem [25].

Proof of Theorem 2.4. In view of Definition 2.2, (2.9), (2.12), (2.17), and Lemma 3.3(ii), we see that

$$\lim_{|x|\to\infty} (\log 1/\hat{W}(x))/Q(x) = 1.$$

Further, $\hat{W} \in L_2(\mathbf{R})$ by the conditions on w, Ψ, V, and W. Hence, Theorem 2.5 is applicable: if $a_n = a_n(W)$ for n large enough, there exist $\{\delta_n\}_1^\infty$ and $\{\varepsilon_n\}_1^\infty$ satisfying (2.19) and (2.20) such that if

$$c_n := a_n(1+\varepsilon_n), \quad n \text{ large enough},$$

then

$$\|P\hat{W}\|_{L_2(\mathbf{R})} \le (1+\delta_n)\|P\hat{W}\|_{L_2[-c_n,c_n]}$$

for n large enough and all $P \in \mathcal{P}_n$. Further, by Theorem 2.6 in Lubinsky and Saff [11], there exist polynomials $S_{n-2}(x)$ of degree at most $n-2$, $n=2, 3, 4, \ldots$, such that

$$\lim_{n\to\infty} \int_{-1}^{1} |1-|(1-x^2)^{1/4}\hat{W}(c_n x)S_{n-2}(x)|^p|\, dx/\sqrt{1-x^2}=0$$

for $p=2$ and $p=\frac{1}{2}$. Then, for $p=2$ and $p=\frac{1}{2}$,

$$\lim_{n\to\infty} \pi^{-1}\int_{-1}^{1} |(1-x^2)^{1/4}\hat{W}(c_n x)S_{n-2}(x)|^p\, dx/\sqrt{1-x^2}=\pi^{-1}\int_{-1}^{1} dx/\sqrt{1-x^2}=1.$$

Thus (4.2) and (4.3) hold. Since (see Lemma 3.2(iv) in [11])

$$\lim_{n\to\infty} a_{n+1}/a_n=1,$$

the result follows. ∎

Proof of Theorem 2.3. It suffices to show $V(x):=\exp(P(x))$ satisfies (2.12)-(2.16) in Theorem 2.4. Let m be the degree of P. Since $m<\alpha$, Lemma 3.3(ii) shows that (2.12) is satisfied. Choose $\varepsilon>0$ such that $m<\alpha-\varepsilon$. For n large enough, let l_n be the largest integer $\le n^{m/(\alpha-\varepsilon)}$. Then (2.14) is satisfied. Further, for n large enough,

$$(4.6)\qquad l_n^{1/m}/a_n \ge n^{1/(\alpha-\varepsilon)}/(2a_n) \to \infty \quad \text{as} \quad n\to\infty,$$

by Lemma 3.3(iii). Now let $\hat{S}_n(x)$ denote the l_nth partial sum of the Maclaurin series of $\exp(-P(x))$. It is easily seen that, for some $C_1>0$ independent of n,

$$(4.7)\qquad |\exp(-P(x))-\hat{S}_n(x)|\le 2^{-l_n}, \qquad |x|\le C_1 l_n^{1/m},$$

n large enough. Note too that, uniformly for $|x|\le a_n$,

$$(4.8)\qquad \exp(P(x))=\exp(O(a_n^m))=\exp(o(l_n)),$$

by (4.6). Combining (4.6)-(4.8), we see that, uniformly for $|x|\le 2a_n$ and some $0<\eta<1$,

$$|1-\hat{S}_n(x)\exp(P(x))|\le \eta^{l_n},$$

n large enough. Given $\{\varepsilon_n\}_1^\infty$ satisfying (2.13), it then follows that $S_n(x):=\hat{S}_n(a_n(1+\varepsilon_n)x)$, for n large enough, is a sequence of polynomials satisfying (2.15) and (2.16). ∎

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Note Added in Proof. Since this paper was written, strong or Szegö type asymptotics have been obtained for $\gamma_n(w)$ and for $P_n(w, z)$ in the plane, for a class of weights essentially larger than VSF(α). This will appear in a Springer Lecture Note entitled "Strong Asymptotics for Extremal Errors and Extremal Polynomials Associated with Weights on $(-\infty, \infty)$."

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