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DEGREE OF APPROXIMATION OF REAL FUNCTIONS BY RECIPROCALS OF REAL AND COMPLEX POLYNOMIALS*

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Abstract. Let $E^c_{on}(f; I)(E^r_{on}(f; I))$ denote the error in best uniform approximation of a real continuous function f on a closed interval I by reciprocals of polynomials of degree $\leqq n$ with complex (real) coefficients. We investigate the rate at which $E^c_{on}(f; I)$ (or $E^r_{on}(f; I)$ provided $f \geqq 0$) can decrease. For example, we prove a Jackson type theorem and also present a class of functions for which reciprocal polynomial approximation is significantly better than polynomial approximation.

Key words. uniform approximation, reciprocals of polynomials, Jackson theorem

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1. Introduction. For any real continuous function f on a closed interval I, let $E^r_{on}(f; I)$ and $E^c_{on}(f; I)$ denote the errors in best Chebyshev (uniform) approximation of f on I by reciprocals of polynomials of degree $\leqq n$ with real and complex coefficients respectively.

If f changes its sign on I, then obviously $E^r_{on}(f; I)$ does not approach zero as $n \to \infty$. On the other hand, a result of Walsh [8, Thm. IV] implies that any continuous function on I can be approximated arbitrarily close by reciprocals of *complex* polynomials; that is, $E^c_{on}(f; I) \to 0$. The aim of this paper is to investigate the rate at which $E^c_{on}(f; I)$ can decrease. For example, we prove a Jackson type theorem (Theorem 2.1) and also present a class of functions for which reciprocal polynomial approximation is significantly better than polynomial approximation (of the same degree).

Most of our results are formulated for the case $I = [-1, 1]$ but can be easily restated for an arbitrary finite interval. We also present some examples of approximation on the real line and on the unit disk.

The paper is organized as follows. In § 2 we state and discuss our main results. The proofs of these results are presented in §§ 3-6.

2. Main results. Our first result is the following Jackson type theorem.

THEOREM 2.1. *There exists an absolute constant M such that for any real $f \in C[-1, 1]$,*

$$E^c_{on}(f; [-1, 1]) \leqq M\omega(f; n^{-1}), \qquad n = 1, 2, 3, \cdots, \tag{2.1}$$

where $\omega(f; \delta)$ denotes the modulus of continuity of f on $[-1, 1]$.

Moreover, if f does not change its sign on $[-1, 1]$, then one can replace E^c_{on} by E^r_{on}:

$$E^r_{on}(f; [-1, 1]) \leqq M\omega(f; n^{-1}), \qquad n = 1, 2, 3, \cdots. \tag{2.2}$$

We remark that the estimate (2.1) follows from the estimate (2.2) and (via the usual Jackson theorem) from the following general result.

THEOREM 2.2. *For any real $f \in C[-1, 1]$,*

$$E^c_{0,3n}(f; [-1, 1]) \leqq 5(E^r_{on}(|f|; [-1, 1]) + E^r_{no}(f; [-1, 1])), \tag{2.3}$$

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where $E^r_{no}(f; [-1, 1])$ *stands for the error in best Chebyshev approximation of* f *by polynomials of degree* $\leqq n$.

From Theorem 2.1 we obtain as a special case that

$$E^r_{on}(|x|^\alpha; [-1, 1]) \leqq Mn^{-\alpha}, \qquad 0<\alpha\leqq 1. \tag{2.4}$$

For $0<\alpha<1$, this improves a result of Lungu [5]. For the case $\alpha=1$, the estimate (2.4) was proved by Newman and Reddy [6]. It turns out that the method of [6] can be modified to establish the estimate (2.4) for any $\alpha>0$. Since the matching lower bounds are also available (see Lungu [5]) we obtain the following result.

THEOREM 2.3. *For any* $\alpha>0$, *there exist positive constants* A_α, B_α *such that for any* $n=1, 2, 3, \cdots$, *the following hold*:

$$A_\alpha n^{-\alpha} \leqq E^c_{on}(|x|^\alpha; [-1, 1]) \leqq E^r_{on}(|x|^\alpha; [-1, 1]) \leqq B_\alpha n^{-\alpha}, \tag{2.5}$$

$$A_\alpha n^{-\alpha} \leqq E^c_{on}(|x|^\alpha \operatorname{sgn}(x); [-1, 1]) \leqq B_\alpha n^{-\alpha}, \tag{2.6}$$

$$A_\alpha n^{-2\alpha} \leqq E^r_{on}(x^\alpha; [0, 1]) \leqq B_\alpha n^{-2\alpha}. \tag{2.7}$$

Moreover, the constants A_α, B_α *may be written in the form* $A_\alpha = A^{-\alpha}$, $B_\alpha = C(B\alpha)^\alpha$, *where* A, B, C *are absolute constants* >1.

Note that the upper bound in (2.6) follows from that in (2.5) and (via Jackson's Theorem) from Theorem 2.2. The upper bound in (2.7) follows from that in (2.5) by the standard substitution $x \to x^2$.

The lower bounds in (2.5), (2.6) show that the estimate given in Theorem 2.1 is, in general, the least possible. Moreover, by considering the function $f(x)=x$, it is easy to see that no estimate of the kind $E^c_{on}(f; [-1, 1]) \leqq Mn^{-k}\omega(f^{(k)}; n^{-1})$ (the analogue of Jackson's Theorem for differentiable functions) can be obtained. To get estimates better than $O(n^{-1})$ one has to make some assumptions concerning the zeros of f. The simplest theorem of this kind is the following one.

THEOREM 2.4. *Let* $f(\not\equiv 0)$ *be real-valued and analytic on* $[-1, 1]$ *and assume* f *vanishes somewhere on* $[-1, 1]$. *Denote by* r *the smallest order of the zeros of* f *in* $(-1, 1)$ *and, by* s, *the smallest order of the zeros of* f *at* ± 1 (*either* r *or* s *may be zero but not both*). *Define* k *by*

$$k := \begin{cases} r & \text{if } s=0, \\ 2s & \text{if } r=0, \\ \min(r, 2s) & \text{if } r>0, \quad s>0. \end{cases}$$

Then there exist positive constants $A(f)$, $B(f)$ *such that*

$$A(f)n^{-k} \leqq E^c_{on}(f; [-1, 1]) \leqq B(f)n^{-k}, \qquad n=1, 2, 3, \cdots. \tag{2.8}$$

Moreover, the same estimates hold for $E^r_{on}(f; [-1, 1])$ *provided* f *does not change sign on* $[-1, 1]$.

Remark. The situation is more delicate if f is differentiable and does not change sign on $[-1, 1]$. It may be true that in this case one can obtain the estimate $E^r_{on}(f; [-1, 1]) \leqq Mn^{-1}\omega(f'; n^{-1})$ without any further assumptions on the structure of f. Even so, since $E^r_{on}(x^2; [-1, 1]) \neq 0$, no further refinement involving the modulus of continuity of higher derivatives is possible.

Our next result exhibits a class of functions that can be approximated by reciprocals of polynomials much better than by polynomials (of the same degree). The common feature of functions of this class is that they vanish on a set of intervals but not at isolated points. To demonstrate why such functions are "well approximable" by reciprocals of polynomials, consider the following example. Let

$$x_+ := \begin{cases} x, & x \geqq 0, \\ 0, & x < 0. \end{cases}$$

Then, from the well-known degree of polynomial approximation to $|x|$, we have $E_{no}(x_+; [-1, 1]) \geqq Cn^{-1}$. On the other hand, we can find a real polynomial $p_n(x)$ such that (see (2.7)) $|x - 1/p_n(x)| \leqq An^{-2}$ for $0 \leqq x \leqq 1$. In particular, $p_n(0) \geqq A^{-1}n^2$. It can be shown that $p_n(x)$ is monotonic on $(-\infty, 0)$ and consequently $p_n(x) \geqq A^{-1}n^2$ for $x < 0$. It follows that

$$|x_+ - 1/p_n(x)| = 1/p_n(x) < An^{-2} \quad \text{for } x < 0,$$

and we obtain

$$E^r_{on}(x_+; [-1, 1]) \leqq An^{-2}.$$

We now formulate the general result.

THEOREM 2.5. *Let $[a_j, b_j], j = 1, 2, \cdots, N$ ($N \geqq 1$), be mutually disjoint subintervals of $[-1, 1]$. For each j, let $f_j \in C^3[a_j, b_j]$ be real-valued and satisfy $f_j(a_j) = f_j(b_j) = 0$, $f_j'(a_j + 0) \neq 0$, $f_j'(b_j - 0) \neq 0$, and assume that $f_j \neq 0$ in (a_j, b_j). Define the function f on $[-1, 1]$ by*

$$f(x) := \begin{cases} f_j(x) & a_j \leqq x \leqq b_j, \quad j = 1, 2, \cdots, N, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$E^c_{on}(f; [-1, 1]) \leqq A(f)n^{-2}, \qquad n = 1, 2, 3, \cdots. \tag{2.9}$$

Further, if f does not change sign on $[-1, 1]$ then $E^r_{on}(f; [-1, 1]) \leqq A(f)n^{-2}$.

Notice that if $N \geqq 2$ or if $N = 1$ and $[a_1, b_1] \neq [-1, 1]$, then f is not differentiable somewhere in $(-1, 1)$ and consequently $E_{no}(f; [-1, 1]) \neq O(n^{-1-\varepsilon})$ for any $\varepsilon > 0$.

So far we have discussed the direct theorems. What about inverse results? To have any chance of proving that f is differentiable on $[-1, 1]$, we have to assume at least (in view of Theorem 2.5) that $E^c_{on}(f; [-1, 1]) = o(n^{-2})$. Under some additional assumptions on the behavior of f near its zeros we can then prove the differentiability of f. For example, if f is piecewise continuously differentiable on $[-1, 1]$ and satisfies $E^c_{on}(f; [-1, 1]) = o(n^{-2})$, then f is continuously differentiable on $[-1, 1]$. At present, the proper formulation of an inverse theorem for differentiable functions is not clear. We confine ourselves to the following Bernstein type result, which was essentially proved by J. L. Walsh [8].

THEOREM 2.6. *For any complex-valued function $f(\neq 0)$ on $[-1, 1]$, the following conditions are equivalent:*

(i) $\limsup_{n\to\infty} \{E^c_{on}(f; [-1, 1])\}^{1/n} < 1$.

(ii) *f is analytic on $[-1, 1]$ and does not vanish there.*

Our final result deals with approximation on the real axis.

THEOREM 2.7. *Let $K(x)$, $L(x)$ be real polynomials of degrees k and l respectively, with $k \leqq l - 1$. If $L(x) \neq 0$ for x real, then*

$$E^r_{on}(|K(x)|/L(x); \mathbb{R}) = O(n^{(k/l)-1}), \tag{2.10}$$

$$E^c_{on}(K(x)/L(x); \mathbb{R}) = O(n^{(k/l)-1}). \tag{2.11}$$

Furthermore, if $K(x)$ does not change sign for $x \in \mathbb{R}$ then

$$E^r_{on}(K(x)/L(x); \mathbb{R}) = O(n^{(2k/l)-2}). \tag{2.12}$$

For the special case $K(x)=x$, $L(x)=1+x^{2m}$, the estimates (2.10), (2.12) were proved by Newman and Reddy [6]. The lower bounds obtained in [6] show that the estimates of Theorem 2.7 are, in general, the best possible.

3. Jackson type theorems.

Proof of Theorem 2.1. To prove the estimate (2.1) it suffices to prove the corresponding estimate for the case of approximation of 2π-periodic functions on the interval $[-\pi, \pi]$ by reciprocals of trigonometric polynomials of degree n. In what follows we use the notation and the estimates that appear in the book of Lorentz [4, p. 55-56].

For any 2π-periodic function g, let

$$J_n(g; x) := \int_{-\pi}^{\pi} g(x+t) K_n(t)\, dt$$

be the Jackson operator. Since

$$\int_{-\pi}^{\pi} K_n(t)\, dt = 1, \quad \int_{-\pi}^{\pi} |t|^k K_n(t)\, dt = O(n^{-k}), \quad k=1,2,$$

we obtain that

$$\int_{-\pi}^{\pi} |g(x+t)-g(x)| K_n(t)\, dt \leqq c_1 \omega(g; n^{-1}) \tag{3.1}$$

and that

$$\int_{-\pi}^{\pi} |g(x+t)-g(x)|^2 K_n(t)\, dt \leqq c_2 [\omega(g; n^{-1})]^2, \tag{3.2}$$

where c_1, c_2 are absolute constants.

Consider now the function

$$f_\varepsilon(x) := f(x) + i\varepsilon, \tag{3.3}$$

where f is a given real 2π-periodic function and $\varepsilon > 0$ will be chosen later. Since f is real, $1/f_\varepsilon$ is continuous on $[-\pi, \pi]$. Furthermore,

$$\omega(f_\varepsilon; n^{-1}) = \omega(f; n^{-1}) \tag{3.4}$$

and

$$|1/f_\varepsilon(x)| \leqq 1/\varepsilon, \qquad -\pi \leqq x \leqq \pi. \tag{3.5}$$

Define the trigonometric polynomial p_n of degree $\leqq n$ by

$$p_n(x) := J_n(1/f_\varepsilon; x).$$

Then

$$|1/f_\varepsilon(x) - p_n(x)| \leqq \int_{-\pi}^{\pi} |1/f_\varepsilon(x) - 1/f_\varepsilon(x+t)| K_n(t)\, dt$$
$$= \int_{-\pi}^{\pi} \frac{|f_\varepsilon(x+t) - f_\varepsilon(x)|}{|f_\varepsilon(x) f_\varepsilon(x+t)|} K_n(t)\, dt.$$

Hence,

$$|1-f_\varepsilon(x)p_n(x)| \leqq \int_{-\pi}^{\pi} |f_\varepsilon(x+t)-f_\varepsilon(x)| \frac{1}{|f_\varepsilon(x+t)|} K_n(t)\, dt$$
$$\leqq \frac{1}{\varepsilon} c_1 \omega(f_\varepsilon; n^{-1}) = \frac{1}{\varepsilon} c_1 \omega(f; n^{-1})$$

by (3.1), (3.4) and (3.5).

The choice

$$\varepsilon = 2c_1 \omega(f; n^{-1}) \tag{3.6}$$

therefore yields

$$|f_\varepsilon(x)p_n(x)| \geqq \tfrac{1}{2}, \qquad -\pi \leqq x \leqq \pi. \tag{3.7}$$

In particular, $p_n \neq 0$ on $[-\pi, \pi]$. Now

$$|f_\varepsilon(x) - 1/p_n(x)| = |1/f_\varepsilon(x) - p_n(x)| \cdot |f_\varepsilon(x)/p_n(x)|$$
$$\leqq \int_{-\pi}^{\pi} \frac{|f_\varepsilon(x+t)-f_\varepsilon(x)|}{|f_\varepsilon(x)f_\varepsilon(x+t)|} \cdot \left|\frac{f_\varepsilon(x)}{p_n(x)}\right| \cdot K_n(t)\, dt$$
$$\leqq 2\int_{-\pi}^{\pi} |f_\varepsilon(x+t)-f_\varepsilon(x)| \cdot \left|\frac{f_\varepsilon(x)}{f_\varepsilon(x+t)}\right| K_n(t)\, dt \quad \text{(by (3.7))}$$
$$\leqq 2\int_{-\pi}^{\pi} |f_\varepsilon(x+t)-f_\varepsilon(x)| K_n(t)\, dt$$
$$+2\int_{-\pi}^{\pi} \frac{|f_\varepsilon(x+t)-f_\varepsilon(x)|^2}{|f_\varepsilon(x+t)|} K_n(t)\, dt$$
$$\leqq 2c_1\omega(f_\varepsilon; n^{-1}) + \frac{2}{\varepsilon}\int_{-\pi}^{\pi} |f_\varepsilon(x+t)-f_\varepsilon(x)|^2 K_n(t)\, dt$$

by (3.1) and (3.5). Thus, from (3.2), (3.4) and (3.6) we deduce that

$$|f_\varepsilon(x) - 1/p_n(x)| \leqq (2c_1 + c_2/c_1)\omega(f; n^{-1}).$$

This yields (see (3.3), (3.6)) the first part of Theorem 2.1.

For the second part, we suppose that $f \geqq 0$ on $[-\pi, \pi]$ and set

$$f_\varepsilon(x) := f(x) + \varepsilon. \tag{3.3$'$}$$

Then the polynomial $J_n(1/f_\varepsilon; x)$ will have real coefficients. The rest of the proof remains the same. □

Remark. Although it does not follow immediately from the above argument, Theorem 2.1 holds, more generally, for any *complex-valued* continuous function f on $[-1, 1]$. The proof of this fact will appear in [2]. Moreover, for a special class of functions f, our methods can be adapted to obtain a Jackson-type theorem for approximation by reciprocal polynomials on the unit disk $|z| \leqq 1$. For example, in [3] we prove that

$$E^c_{on}((z-1)^\alpha; |z| \leqq 1) \leqq Mn^{-\alpha}, \qquad 0 < \alpha \leqq 1.$$

We now proceed to the proof of Theorem 2.2, which uses an idea of Trefethen [7].

LEMMA 3.1. *Let p_n be a real polynomial of degree $\leqq n$. Then*

$$E^c_{0,3n}(p_n; I) \leqq 4E^r_{on}(|p_n|; I).$$

Proof. Let q_n be a real polynomial of degree $\leqq n$ satisfying

$$\max_{x\in I} ||p_n(x)|-1/q_n(x)| = E^r_{on}(|p_n|; I) =: \varepsilon.$$

Then

$$|p_n^2(x)-1/q_n^2(x)| \leqq \varepsilon(2|p_n(x)|+\varepsilon), \qquad x\in I.$$

Define the complex polynomial Q_{3n} of degree $\leqq 3n$ by

$$Q_{3n}(x) := (p_n(x)-i\varepsilon)q_n^2(x).$$

Then

$$\begin{aligned} |p_n(x)-1/Q_{3n}(x)| &\leqq \left| p_n(x)-\frac{p_n^2(x)}{p_n(x)-i\varepsilon}\right| \\ &\quad + \left|\frac{p_n^2(x)}{p_n(x)-i\varepsilon}-\frac{1}{(p_n(x)-i\varepsilon)q_n^2(x)}\right| \\ &\leqq \varepsilon\left|\frac{ip_n(x)}{p_n(x)-i\varepsilon}\right| + \varepsilon\frac{2|p_n(x)|+\varepsilon}{|p_n(x)-i\varepsilon|} \leqq 4\varepsilon, \end{aligned}$$

since p_n is real. □

Proof of Theorem 2.2. Let p_n be any real polynomial of degree $\leqq n$. With obvious simplification of notation we obtain the following:

$$\begin{aligned} E^c_{0,3n}(f) &\leqq \|f-p_n\| + E^c_{0,3n}(p_n) \\ &\leqq \|f-p_n\| + 4E^r_{on}(|p_n|) \quad \text{(by Lemma 3.1)} \\ &\leqq \|f-p_n\| + 4[E^r_{on}(|f|) + \| \, |f|-|p_n| \, \|] \\ &\leqq 5\|f-p_n\| + 4E^r_{on}(|f|). \end{aligned}$$

Hence, on choosing p_n such that $\|f-p_n\| = E^r_{no}(f)$, Theorem 2.2 follows. □

4. Approximation of powers of x. The lower bounds for $E^r_{on}(|x|^\alpha; [-1, 1])$ and for $E^r_{on}(x^\alpha; [0, 1])$ were proved (for $0<\alpha\leqq 1$) by Lungu [5]. The proof for other cases is exactly the same. For the proof of the upper bounds it suffices to show (as we mentioned in the Introduction) that

$$E^r_{on}(|x|^\alpha; [-1, 1]) \leqq B_\alpha n^{-\alpha}, \qquad \alpha>0. \tag{4.1}$$

Following an idea in Newman and Reddy [6], we consider the kernel

$$\varphi_n(t) := t^{\alpha-1}\left(\frac{T_n(t)}{t}\right)^{2k} \tag{4.2}$$

where n is odd, k is the smallest integer satisfying $k\geqq\alpha$ and T_n denotes the nth degree Chebyshev polynomial of the first kind. Define

$$p(x) := \frac{1}{Cx^\alpha}\int_0^x \varphi_n(t)\,dt, \qquad x>0 \tag{4.3}$$

where $C := \int_0^1 \varphi_n(t)\,dt$.

Clearly, $p(x)$ is an even polynomial of degree $2k(n-1)$. By evenness we consider only $x\in[0, 1]$. Write

$$\frac{1}{p(x)} - x^\alpha = \frac{x^\alpha\int_x^1 \varphi_n(t)\,dt}{\int_0^x \varphi_n(t)\,dt}. \tag{4.4}$$

As in [6] we make use of the estimates $|T_n(t)/t| \leqq n$, $|T_n(t)/t| \leqq 1/t$ for $0<t\leqq 1$ and $|T_n(t)/t| \geqq 2n/\pi$ for $0 \leqq t \leqq \sin(\pi/2n)$. It follows that

$$\varphi_n(t) \leqq n^{2k} t^{\alpha-1}, \quad \varphi_n(t) \leqq t^{\alpha-1-2k} \quad \text{for } 0<t\leqq 1,$$

and

$$\varphi_n(t) \geqq \left(\frac{2}{\pi}\right)^{2k} n^{2k} t^{\alpha-1} \quad \text{for } 0<t\leqq \sin\left(\frac{\pi}{2n}\right).$$

We consider now two cases.

Case 1. Suppose $0 \leqq x \leqq \sin(\pi/2n)$. In this case, we have

$$\int_x^1 \varphi_n(t)\,dt \leqq \int_0^{1/n} + \int_{1/n}^1 \leqq \int_0^{1/n} n^{2k} t^{\alpha-1}\,dt + \int_{1/n}^1 t^{\alpha-1-2k}\,dt$$
$$\leqq \frac{1}{\alpha} n^{2k-\alpha} + \frac{1}{2k-\alpha} n^{2k-\alpha}$$
$$\leqq \frac{2}{\alpha} n^{2k-\alpha} \quad \text{since } k \geqq \alpha.$$

Also,

$$\int_0^x \varphi_n(t)\,dt \geqq \int_0^x \left(\frac{2}{\pi}\right)^{2k} n^{2k} t^{\alpha-1}\,dt = \left(\frac{2}{\pi}\right)^{2k} n^{2k} \cdot \frac{1}{\alpha} x^{\alpha}.$$

It now follows from (4.4) that

$$(4.5) \qquad 0 < \frac{1}{p(x)} - x^{\alpha} \leqq 2\left(\frac{\pi}{2}\right)^{2k} n^{-\alpha}, \qquad 0 \leqq x \leqq \sin\left(\frac{\pi}{2n}\right).$$

Case 2. Suppose $\sin(\pi/2n) \leqq x \leqq 1$. In this case, we have

$$\int_x^1 \varphi_n(t)\,dt \leqq \int_x^1 t^{\alpha-1-2k}\,dt < \int_x^{\infty} t^{\alpha-1-2k}\,dt$$
$$= \frac{1}{2k-\alpha} x^{\alpha-2k} \leqq \frac{1}{\alpha} x^{\alpha-2k} \quad \text{since } k \geqq \alpha.$$

Hence,

$$x^{\alpha} \int_x^1 \varphi_n(t)\,dt \leqq \frac{1}{\alpha} x^{2(\alpha-k)} \leqq \frac{1}{\alpha}\left(\sin\frac{\pi}{2n}\right)^{2(\alpha-k)} \leqq \frac{1}{\alpha} n^{2(k-\alpha)}.$$

Also,

$$\int_0^x \varphi_n(t)\,dt \geqq \int_0^{\sin(\pi/2n)} \varphi_n(t)\,dt \geqq \int_0^{\sin(\pi/2n)} \left(\frac{2}{\pi}\right)^{2k} n^{2k} t^{\alpha-1}\,dt$$
$$= \left(\frac{2}{\pi}\right)^{2k} n^{2k} \frac{1}{\alpha}\left(\sin\frac{\pi}{2n}\right)^{\alpha} \geqq \frac{1}{\alpha}\left(\frac{2}{\pi}\right)^{2k} n^{2k-\alpha}.$$

It follows that

$$(4.6) \qquad 0 < \frac{1}{p(x)} - x^{\alpha} \leqq \left(\frac{\pi}{2}\right)^{2k} n^{-\alpha}, \qquad \sin\left(\frac{\pi}{2n}\right) \leqq x \leqq 1.$$

From (4.5) and (4.6) it follows that

$$E^r_{0,2k(n-1)}(|x|^{\alpha}; [-1,1]) \leqq 2(\pi/2)^{2k} n^{-\alpha}$$

(recall that $p(x)$ is of degree $2k(n-1)$). Using a standard technique, the last inequality implies (4.1) with a constant B_α of the form $C(B\alpha)^\alpha$, where $B, C>1$ are absolute constants. Analyzing the proof of lower bounds given in Lungu [5], we see that A_α may be taken of the form $A^{-\alpha}$, where $A>1$ is an absolute constant. The proof of Theorem 2.3 is complete. □

An appropriate change of variable yields the following corollary.

COROLLARY 4.1. *For any $a \in [-1, 1]$ and for any $\alpha > 0$ there exist constants c_1, c_2 (depending on a, α) such that*

(4.7) $$c_1 n^{-\alpha} \leqq E^r_{on}(|x-a|^\alpha; [-1, 1]) \leqq c_2 n^{-\alpha} \quad \text{if } |a|<1,$$

(4.8) $$c_1 n^{-2\alpha} \leqq E^r_{on}(|x-a|^\alpha; [-1, 1]) \leqq c_2 n^{-2\alpha} \quad \text{if } |a|=1.$$

The same estimates hold for $E^c_{on}(|x-a|^\alpha \operatorname{sgn}(x-a); [-1, 1])$.

We conclude this section with a simple lemma. This lemma together with Corollary 4.1 enable us to obtain upper bounds for $E_{on}(f; [-1, 1])$, where f is a finite product of functions of type $|x-a|^\alpha$ or $|x-a|^\alpha \operatorname{sgn}(x-a)$.

LEMMA 4.2. *For any complex-valued continuous functions f, g, on I, there is a constant K (independent of n) such that*

(4.9) $$E_{0,2n}(fg; I) \leqq K(E_{on}(f; I) + E_{on}(g; I)),$$

where E_{om} stands for E^r_{om} or for E^c_{om}.

Proof. Choose the polynomials p_n, q_n such that

$$\left\| f - \frac{1}{p_n} \right\| = E_{on}(f; I), \qquad \left\| g - \frac{1}{q_n} \right\| = E_{on}(g; I),$$

where $\|\cdot\|$ denotes the uniform norm on I. Since

$$\left\| fg - \frac{1}{p_n q_n} \right\| = \left\| \left(f - \frac{1}{p_n}\right) g + \frac{1}{p_n}\left(g - \frac{1}{q_n}\right) \right\|$$
$$\leqq \|g\| E_{on}(f; I) + 2\|f\| E_{on}(g, I),$$

the result follows. □

5. Well-approximable functions (Proof of Theorem 2.5). The proof of Theorem 2.5 given in this section will be split into several lemmas. We shall use the following notation:

$$f(x)_+ = \begin{cases} f(x) & \text{if } f(x) \geqq 0, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 5.1. *For any $\alpha > 0$ and for any $n = 1, 2, 3, \cdots$,*

(5.1) $$E^r_{on}(x^\alpha_+; [-1, 1]) \leqq E^r_{on}(x^\alpha_+; (-\infty, 1]) \leqq B_\alpha n^{-2\alpha}.$$

Proof. We consider the kernel $\varphi_n(t)$ and the polynomial $p(x)$ as in the proof of Theorem 2.3 (see formulas (4.2), (4.3)) with α replaced by 2α. From the proof of Theorem 2.3 we obtain that

$$\left| x^{2\alpha} - \frac{1}{p(x)} \right| \leqq B_\alpha n^{-2\alpha}, \qquad 0 \leqq x \leqq 1.$$

Recall that $p(x)$ is an even polynomial of degree $2k(n-1)$, where k is the smallest integer satisfying $k \geqq 2\alpha$. Define the polynomial $Q(x)$ by $Q(x) := p(\sqrt{x})$. Then

(5.2) $$|x^\alpha - 1/Q(x)| \leqq B_\alpha n^{-2\alpha}, \qquad 0 \leqq x \leqq 1.$$

Since $T_n(t)/t$ has the form $(-1)^{(n-1)/2}\sum_{j=0}^{(n-1)/2}(-1)^j a_j t^{2j}$, $a_j>0$, the polynomial $[T_n(t)/t]^{2k}$ has a similar form (except that the factor preceding the summation is now 1). Hence the polynomial $p(x)$ is of the form $\sum_{j=0}^{k(n-1)}(-1)^j b_j x^{2j}$, $b_j>0$, and therefore $Q(x)$ has the form

$$Q(x)=\sum_{j=0}^{k(n-1)}(-1)^j b_j x^j, \qquad b_j>0.$$

It follows that $Q(x)>Q(0)$ for x negative and we obtain from (5.2) that

$$0<1/Q(x)<1/Q(0)<B_\alpha n^{-2\alpha}, \qquad -\infty<x<0.$$

Hence,

$$|x_+^\alpha-1/Q(x)|\leqq B_\alpha n^{-2\alpha}, \qquad -\infty<x\leqq 1. \qquad \square$$

LEMMA 5.2. *Let p be a real polynomial. Then*

$$E_{on}^r(p_+;[-1,1])\leqq cn^{-2}\|p_+\|(\deg p)^2, \tag{5.3}$$

where c is an absolute constant and $\|\cdot\|$ *denotes the uniform norm on* $[-1,1]$.

Proof. By the proof of Lemma 5.1, there exists a polynomial $q_m(x)$ of degree m such that

$$|x_+-1/q_m(x)|\leqq cm^{-2}, \qquad -\infty<x\leqq 1. \tag{5.4}$$

Let $\deg p=:k$ and define the polynomial $Q_{mk}(x)$ of degree mk by

$$Q_{mk}(x)=\frac{1}{\|p_+\|}\,q_m(p(x)/\|p_+\|).$$

The substitution $x\to p(x)/\|p_+\|$ in (5.4) yields:

$$\|p_+(x)-1/Q_{mk}(x)\|\leqq cm^{-2}\|p_+\|=c(mk)^{-2}\|p_+\|k^2.$$

Hence the lemma is established for n of the form mk. The result for arbitrary n follows by a standard technique. $\square$

LEMMA 5.3. *Let f be a nonvanishing real continuous function on* $[-1,1]$. *Then*

$$c_1E_{no}^r(1/f;[-1,1])\leqq E_{on}^r(f;[-1,1])\leqq c_2E_{no}^r(1/f;[-1,1]), \tag{5.5}$$

where $c_1>0$, $c_2>0$ *depend on f.*

The proof of Lemma 5.3 is straightforward.

LEMMA 5.4. *For any* $0<a<1$, *there is a constant c (depending on a) such that*

$$E_{on}^r((|x|-a)_+;[-1,1])\leqq cn^{-2}. \tag{5.6}$$

Proof. It suffices to prove that

$$E_{on}^r((\sqrt{x}-a)_+;[0,1])\leqq cn^{-2}.$$

To show this write

$$(\sqrt{x}-a)_+=(x-a^2)_+\frac{1}{\sqrt{x}+a}.$$

From Lemma 5.1 it follows (by linear transformation of the variable) that

$$E_{on}^r((x-a^2)_+;[0,1])\leqq cn^{-2}. \tag{5.7}$$

Further, we can extend the function $1/(\sqrt{x}+a)$, $x \geqq a^2$, to the interval $[0, 1]$ in such a way that the resulting function, $g(x)$ say, will belong to $C^2[0, 1]$ and will be positive on $[0, 1]$. By Lemma 5.3 and by Jackson's Theorem for differentiable functions (see e.g. [4, p. 57]) we obtain that

$$E^r_{on}(g; [0, 1]) \leqq cn^{-2}. \tag{5.8}$$

Since $(x-a^2)_+ g(x) = (\sqrt{x}-a)_+$ on $[0, 1]$, the inequalities (5.7), (5.8) and Lemma 4.2 yield the desired estimate. □

LEMMA 5.5. *For any $0 < a < 1$ there is a constant c (depending on a) such that*

$$E^c_{on}((|x|-a)_+ \operatorname{sgn}(x); [-1, 1]) \leqq cn^{-2}. \tag{5.9}$$

Proof. Write

$$(|x|-a)_+ \operatorname{sgn}(x) = x^3(|x|-a)_+|x|^{-3},$$

and extend the function $|x|^{-3}$, $|x| \geqq a$, to the interval $[-1, 1]$ as a twice differentiable positive function. The lemma now follows (as in the proof of Lemma 5.4) from Theorem 2.3, Lemma 4.2 and Lemma 5.4. □

Using the proof similar to that of Lemma 5.2 we obtain the following.

LEMMA 5.6. *Let $p(x)$ be a real polynomial and let $0 < a < \|p\|$, where $\|\cdot\|$ denotes the uniform norm on $[-1, 1]$. Then*

$$E^c_{on}((|p(x)|-a)_+ \operatorname{sgn} p(x); [-1, 1]) \leqq cn^{-2}\|p\|(\deg p)^2, \tag{5.10}$$

where c depends only on a.

Proof of Theorem 2.5. We first consider the case when all functions f_j are of the same sign (positive, say). Define the polynomial $p(x)$ by

$$p(x) := -\prod_{j=1}^{N} (x-a_j)(x-b_j).$$

Then $p(x) > 0$ on each interval (a_j, b_j). It follows that the function

$$g_j(x) := f_j(x)/p(x), \qquad x \in [a_j, b_j],$$

is positive on $[a_j, b_j]$ and belongs to $C^2[a_j, b_j]$. We can find now a function $G(x) \in C^2[-1, 1]$ that is positive on $[-1, 1]$ and coincides with g_j on $[a_j, b_j]$, $j = 1, 2, \cdots, N$. Since $p(x) \leqq 0$ whenever $f(x) = 0$, we can write

$$f(x) = p_+(x)G(x).$$

By Lemma 5.2,

$$E^r_{on}(p_+; [-1, 1]) \leqq cn^{-2}$$

(c depends on f) and by Lemma 5.3

$$E^r_{on}(G; [-1, 1]) \leqq cn^{-2},$$

since $1/G$ is twice differentiable. Applying Lemma 4.2 we obtain that

$$E^r_{on}(f; [-1, 1]) \leqq c(f)n^{-2}.$$

For the general case, when the f_j are of arbitrary signs, define the function φ on $\bigcup_{j=1}^N [a_j, b_j]$ by

$$\varphi(x) := \begin{cases} f_j(x) + \frac{1}{2} & \text{if } f_j > 0, \\ f_j(x) - \frac{1}{2} & \text{if } f_j < 0. \end{cases}$$

Then

$$(|\varphi(x)| - \tfrac{1}{2}) \operatorname{sgn} \varphi(x) = f_j(x), \qquad x \in [a_j, b_j], \quad j = 1, 2, \cdots, N. \tag{5.11}$$

Next, we claim that there is a polynomial $P(x)$ of some fixed but large degree, such that

$$\text{(i)} \qquad |P(x)| < \tfrac{1}{2} \quad \text{for } x \in [-1, 1] \setminus \bigcup_{j=1}^{N} [a_j, b_j],$$

$$\text{(ii)} \qquad |P(x)| > \tfrac{1}{2} \quad \text{for } x \in (a_j, b_j), \quad j = 1, 2, \cdots N,$$

$$\text{(iii)} \qquad P(a_j) = P(b_j) = \begin{cases} \tfrac{1}{2} & \text{if } f_j > 0, \\ -\tfrac{1}{2} & \text{if } f_j < 0. \end{cases}$$

This can be seen as follows. The function φ satisfies conditions (ii), (iii). Extend it to $[-1, 1]$ in such a way that it will satisfy (i) and will belong to $C^3[-1, 1]$. Now approximate φ simultaneously with φ' by a polynomial P that interpolates φ, φ' at a_j, b_j, $j = 1, 2, \cdots, N$. If the degree of P is large enough, the norms $\|\varphi - P\|$, $\|\varphi' - P'\|$ will be arbitrarily small (see Chalmers and Taylor [1, pp. 55-56]). From this it follows easily that P will satisfy (i)-(iii).

From this construction we obtain that the function

$$g(x) = \frac{(|\varphi(x)| - \frac{1}{2}) \operatorname{sgn} \varphi(x)}{(|P(x)| - \frac{1}{2}) \operatorname{sgn} P(x)}$$

is positive on $[a_j, b_j]$, $j = 1, 2, \cdots, N$ and has there two continuous derivatives. Extend g to $[-1, 1]$ preserving its sign and the differentiability. From (5.11) and from the definition of g we obtain that

$$[(|P(x)| - \tfrac{1}{2})_+ \operatorname{sgn} P(x)] g(x) = f(x), \qquad -1 \leqq x \leqq 1.$$

By Lemma 5.3 and Jackson's Theorem,

$$E^c_{on}(g; [-1, 1]) \leqq cn^{-2}.$$

Also, by Lemma 5.6,

$$E^c_{on}((|P(x)| - \tfrac{1}{2})_+ \operatorname{sgn} P(x); [-1, 1]) \leqq cn^{-2}.$$

Finally, Lemma 4.2 yields

$$E^c_{on}(f; [-1, 1]) \leqq cn^{-2}.$$

□

6. Approximation of analytic functions (Proofs of Theorems 2.4 and 2.6).

Proof of Theorem 2.4. If $f(\neq 0)$ is real analytic on $[-1, 1]$, we can write

$$f(x) = (x+1)^{a_1}(x-1)^{a_2} \prod_{j=1}^{N} (x - x_j)^{b_j} \cdot g(x), \tag{6.1}$$

where $a_1, a_2, b_1, \cdots, b_N$ are nonnegative integers, $|x_j| < 1$ for $j = 1, 2, \cdots, N$ and g is real analytic and nonvanishing on $[-1, 1]$. By Corollary 4.1, we have $E^r_{on}((x+1)^{a_1}; [-1, 1]) \leqq cn^{-2a_1}$, $E^r_{on}((x-1)^{a_2}; [-1, 1]) \leqq cn^{-2a_2}$, $E^c_{on}((x-x_j)^{b_j}; [-1, 1]) \leqq cn^{-b_j}$. Also, by Lemma 5.3 and by Bernstein's Theorem (cf. [4, p. 76]),

$$\limsup_{n \to \infty} [E^c_{on}(g; [-1, 1])]^{1/n} < 1.$$

Applying Lemma 4.2 we obtain the estimate

$$E^c_{on}(f; [-1, 1]) \leqq cn^{-k},$$

where k is defined in Theorem 2.4.

For the lower bound in (2.8) we write $f(x) = (x - x_j)^{b_j} \varphi_j(x)$, where $\varphi_j(x_j) \neq 0$ and apply the argument in Lungu [5] to obtain $E^c_{on}(f; [-1, 1]) \geqq cn^{-b_j}$, $j = 1, 2, \cdots, N$. We omit the details. □

Concerning Theorem 2.6, Walsh [8] proves the corresponding result for approximation on a Jordan region. He asserts that the result is also true for Jordan arcs, but does not provide the proof. For completeness we provide the details.

Proof of Theorem 2.6. The implication (ii)⇒(i) is trivial (apply Lemma 5.3 and Bernstein's Theorem). Assume now that

$$\limsup_{n\to\infty} (E^c_{on}(f; [-1, 1]))^{1/n} = q < 1,$$

and let $P_n(x)$, $n = 1, 2, 3, \cdots$, be polynomials for which

$$\|f - 1/P_n\| = E^c_{on}(f; [-1, 1]).$$

It suffices to prove that $f \neq 0$ on $[-1, 1]$, since then Lemma 5.3 and Bernstein's Theorem will imply the analyticity of f on $[-1, 1]$. Suppose that f vanishes somewhere on $[-1, 1]$. Since $f \not\equiv 0$, we can find an interval $I \subset [-1, 1]$ such that $f \neq 0$ inside I but vanishes at one of its endpoints. Assume, for simplicity, that $I = [-\delta, \delta]$, $\delta < 1$, and $f(\delta) = 0$. Then

$$\limsup_{n\to\infty} |1/P_n(\delta)|^{1/n} \leqq q,$$

which implies that

$$\liminf_{n\to\infty} |P_n(\delta)|^{1/n} \geqq 1/q. \tag{6.2}$$

Pick $\delta_1 < \delta$. Since $f \neq 0$ on $[-\delta_1, \delta_1]$, there exists a constant $M = M(\delta_1)$ such that $|P_n(x)| \leqq M$ for $n = 1, 2, 3, \cdots$, and for $x \in [-\delta_1, \delta_1]$. Then (cf. [4, p. 43])

$$|P_n(\delta)| \leqq M\left(\frac{1+\sqrt{1-(\delta_1/\delta)^2}}{\delta_1/\delta}\right)^n.$$

It follows that

$$\liminf_{n\to\infty} |P_n(\delta)|^{1/n} \leqq \frac{1+\sqrt{1-(\delta_1/\delta)^2}}{\delta_1/\delta} < \frac{1}{q},$$

provided δ_1 is close enough to δ. This contradicts (6.2). □

7. Approximation on the real line.

Proof of Theorem 2.7. Consider the polynomial $p(x)$ defined by formulae (4.2) and (4.3) with $\alpha = 1$, $k = 1$. By the proof in § 4, we obtain that p is a polynomial of degree $2(n-1)$ satisfying

$$||x| - 1/p(x)| \leqq An^{-1} \quad \text{for } -1 \leqq x \leqq 1. \tag{7.1}$$

From (4.4) we also obtain that

$$0 < |x| - 1/p(x) < |x| \quad \text{for } |x| > 1. \tag{7.2}$$

For $T > 0$, set

$$F_T := \{x \in \mathbb{R}: |K(x)| \leqq \|K\|_{[-T,T]}\}.$$

Then we obtain by the substitution $x \to |K(x)|/\|K\|_{[-T,T]}$ in (7.1), (7.2) that

$$\left| |K(x)| - \frac{1}{q(x)} \right| \leqq \begin{cases} An^{-1}\|K\|_{[-T,T]} & \text{if } x \in F_T, \\ |K(x)| & \text{if } x \in \mathbb{R}\backslash F_T, \end{cases} \tag{7.3}$$

where $q(x) := \|K\|_{[-T,T]}^{-1} p(K(x)/\|K\|_{[-T,T]})$ is a polynomial of degree $\leqq 2k(n-1)$. Dividing (7.3) by $L(x)$ and setting $a := \min_{\mathbb{R}} |L(x)| > 0$, we obtain that

$$\left|\frac{|K(x)|}{L(x)} - \frac{1}{r(x)}\right| \leqq \begin{cases} Aa^{-1}n^{-1}\|K\|_{[-T,T]} & \text{if } x \in F_T, \\ |K(x)|/L(x) & \text{if } x \in \mathbb{R}\backslash F_T, \end{cases}$$

where $r(x) := q(x)L(x)$ is a polynomial of degree $2k(n-1)+l$. For T large enough we have $\|K\|_{[-T,T]} = O(T^k)$ and $|K(x)/L(x)| = O(T^{k-l})$ uniformly for $x \in \mathbb{R}\backslash F_T$ and consequently

$$E^r_{0,2k(n-1)+l}(|K(x)|/L(x); \mathbb{R}) = O(T^k)n^{-1} + O(T^{k-l}).$$

When we choose $T = n^{1/l}$ we obtain the first assertion of Theorem 2.7.

For the third assertion we assume that $K(x) \geqq 0$, $L(x) > 0$ on $\mathbb{R}$ and make use of the polynomial $\tilde{p}(x) := p(\sqrt{x})$, where p is defined by (4.2) and (4.3) with $\alpha = 2$, $k = 2$. Then

$$|x - 1/p(x)| \leqq \begin{cases} An^{-2} & \text{if } 0 \leqq x \leqq 1, \\ x & \text{if } x > 1, \end{cases}$$

and we can repeat the above proof choosing eventually $T = n^{2/l}$.

It remains to prove the second assertion of Theorem 2.7 (formula (2.11)). Let p be the polynomial satisfying (7.1)-(7.2). Using the method of the proof of Lemma 3.1 we see that when we choose

$$\tilde{p}(x) = (x - i\varepsilon)p^2(x), \qquad \varepsilon := E^r_{on}(|x|; [-1, 1]), \tag{7.4}$$

we obtain

$$|x - 1/\tilde{p}(x)| \leqq An^{-1} \quad \text{for } |x| \leqq 1. \tag{7.5}$$

For $|x| > 1$ we have

$$|1/\tilde{p}(x)| = \frac{1}{|x - i\varepsilon|}\frac{1}{p^2(x)} \leqq \frac{|x|^2}{|x - i\varepsilon|} \leqq |x| \quad \text{(by (7.2))}.$$

Hence

$$|x - 1/\tilde{p}(x)| \leqq 2|x| \quad \text{for } |x| > 1. \tag{7.6}$$

Using (7.5) and (7.6) the proof can be completed as above. □

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