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# WHERE DOES THE $L^p$-NORM OF A WEIGHTED POLYNOMIAL LIVE?

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ABSTRACT. For a general class of nonnegative weight functions $w(x)$ having bounded or unbounded support $\Sigma \subset \mathbf{R}$, the authors have previously characterized the smallest compact set $\mathfrak{S}_w$ having the property that for every $n = 1, 2, \ldots$ and every polynomial $P$ of degree $\leq n$,

$$\|[w(x)]^n P(x)\|_{L^\infty(\Sigma)} = \|[w(x)]^n P(x)\|_{L^\infty(\mathfrak{S}_w)}.$$

In the present paper we prove that, under ~~mild~~ certain conditions on $w$, the $L^p$-norms $(0 < p < \infty)$ of such weighted polynomials also "live" on $\mathfrak{S}_w$ in the sense that for each $\eta > 0$ there exist a compact set $\Delta$ with Lebesgue measure $m(\Delta) < \eta$ and positive constants $c_1, c_2$ such that

$$\|w^n P\|_{L^p(\Sigma)} \leq (1 + c_1 \exp(-c_2 n))\|w^n P\|_{L^p(\mathfrak{S}_w \cup \Delta)}.$$

As applications we deduce asymptotic properties of certain extremal polynomials that include polynomials orthogonal with respect to a fixed weight over an unbounded interval. Our proofs utilize potential theoretic arguments along with Nikolskii-type inequalities.

**1. Introduction.** In 1974, G. Freud [**3**] proved the following "infinite-finite range inequality" for weighted polynomials.

Suppose that $Q$ is an even, convex, positive function on $\mathbf{R}$, differentiable on $(0, \infty)$ and $Q'(t)$ is positive and increasing to $\infty$ for $0 < t < \infty$. Then there exist positive constants $c_1, c_2, c_3$ depending only on $Q$ with the following property: For every integer $n \geq 1$ and every polynomial $P$ of degree not more than $n$,

$$\int_{-\infty}^{\infty} [P(x)\exp(-Q(x))]^2\,dx \leq (1 + c_1\exp(-c_2 n)) \cdot \int_{|t| \leq c_3 q_{2n}} [P(t)\exp(-Q(t))]^2\,dt, \tag{1.1}$$

where $q_{2n}$ is defined by the equation

$$q_{2n} Q'(q_{2n}) = 2n. \tag{1.2}$$

This inequality has been generalized or investigated in further detail for specific weight functions by several authors including Bonan [**1**], Lubinsky [**7**], Zalik [**15**]

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and the present authors [**9**, **10**]. In [**11**], we obtained the following sharp result for the sup norm, under less restrictive conditions on $Q$.

Let $a_n > 0$ be defined by the equation

$$\frac{2}{\pi}\int_0^1 \frac{a_n x Q'(a_n x)}{\sqrt{1-x^2}}\,dx = n. \tag{1.3}$$

Then for every integer $n \geq 1$ and polynomial $P$ of degree not exceeding $n$,

$$\max_{x\in\mathbf{R}} |W(x)P(x)| = \max_{|t|\leq a_n} |W(t)P(t)|, \tag{1.4}$$

where $W(x) := \exp(-Q(x))$. Moreover, (1.4) cannot be improved in the sense that the sequence $\{a_n\}$ cannot be replaced by $\{a_n(1-\delta)\}$ for any positive $\delta$.

In this paper, our aim is to obtain similar precise results for the $L^p$-norms of the "weighted polynomials," i.e. expressions of the form $W(x)P(x)$, where $W$ is a weight function and $P$ is a polynomial. Our theorems are general in that they apply to weights $W$ with bounded or unbounded support (not necessarily an interval) and allow $W$ to have zeros at interior points. Of particular interest are the cases when $W$ is supported on $\mathbf{R}$, $[0,\infty)$, or on a finite union of disjoint closed intervals. In our investigations, we also obtain new results concerning the $L^\infty$-norm of weighted polynomials that complement those in [**11**].

In the next section we state and discuss our main results. The proofs are given in §3.

**2. Main results.** We begin by recalling some definitions and theorems that appear in [**11**].

DEFINITION 2.1. *Let $w\colon \mathbf{R} \to [0,\infty)$. We say that $w$ is an admissible weight function if each of the following properties holds.*

(i) *$\Sigma := \operatorname{supp}(w)$ has positive capacity.*

(ii) *$Z := \{x \in \Sigma\colon w(x) = 0\}$ has capacity zero.*

(iii) *The restriction of $w$ to $\Sigma$ is continuous on $\Sigma$.*

(iv) *If $\Sigma$ is unbounded, then $|x|w(x) \to 0$ as $|x| \to \infty$, $x \in \Sigma$.*

By $\operatorname{supp}(w)$ we mean the closure of the set where $w > 0$ and by *capacity* we mean the inner logarithmic capacity (cf. [**14**, p. 55]). We use $C(E)$ to denote the capacity of a set $E \subset \mathbf{R}^2$. The class of all polynomials of degree at most $n$ is denoted by $\Pi_n$. We also adopt the convention that $c, c_1, c_2$, etc. will denote positive constants that are independent of $n$, but may depend on $w$ and other relevant parameters. Furthermore, the same symbol may denote different values even within a single formula. Constants that retain their values will be denoted by capital letters.

If $K$ is a compact set with positive capacity, then $\nu_K$ will denote the unique unit equilibrium measure on $K$ with the property that

$$\int_K \log|x-t|\,d\nu_K(t) = \log C(K) \tag{2.1}$$

quasi-everywhere (q.e.) on $K$ (cf. [**14**, p. 60]). A property is said to hold q.e. on a set $A$ if the subset $E \subset A$ where it does not hold satisfies $C(E) = 0$.

For an admissible weight $w$, we always set

$$Q(x) := \log[1/w(x)]. \tag{2.2}$$

Finally, if $K \subset \Sigma \backslash Z$ is compact and $C(K) > 0$, the *F-functional* of $K$ is defined as in [**11**] by the formula

$$F(K) := \log C(K) - \int_K Q d\nu_K. \tag{2.3}$$

For admissible weight functions, we proved

THEOREM 2.2 [**11**]. *There exists a unique compact set* $\mathfrak{S}_w \subset \Sigma \backslash Z$ *with* $C(\mathfrak{S}_w) > 0$ *that has the following properties:*

(a) *For every compact set* $K \subset \Sigma \backslash Z$ *with* $C(K) > 0$,

$$F(K) \le F(\mathfrak{S}_w) \tag{2.4}$$

*where* $F$ *is defined in* (2.3).

(b) *If equality holds in* (2.4), *then* $\mathfrak{S}_w \subset K$.

(c) *For any positive integer* $n$, *if* $P \in \Pi_n$ *and the inequality*

$$|[w(x)]^n P(x)| \le 1 \tag{2.5}$$

*holds q.e. on* $\mathfrak{S}_w$, *then it holds q.e. on* $\Sigma$.

(d) *If* $\Sigma$ *is regular, i.e. for all* $k$ *large,* $\Sigma \cap [-k, k]$ *is regular with respect to the Dirichlet problem for its complement on the Riemann sphere, then for every* $P \in \Pi_n$ *and every* $n = 1, 2, \ldots,$

$$\|[w(x)]^n P(x)\|_{\infty, \Sigma} = \|[w(x)]^n P(x)\|_{\infty, \mathfrak{S}_w}, \tag{2.6}$$

*where* $\| \cdot \|_{\infty, A}$ *denotes the sup norm over a set* $A$.

(e) *In particular, when* $\Sigma \backslash Z$ *is a finite union of disjoint nondegenerate intervals and* $Q$ *is convex in each of the components of* $\Sigma \backslash Z$, *then* $\mathfrak{S}_w$ *is itself a finite union of nondegenerate disjoint closed intervals, at most one in each component of* $\Sigma \backslash Z$; *moreover, if* $K \subset \Sigma \backslash Z$ *is compact with* $C(K) > 0$, *then* $F(K) < F(\mathfrak{S}_w)$ *unless* $\mathfrak{S}_w \subset K$ *and* $C(K \backslash \mathfrak{S}_w) = 0$.

The major theorems of this paper can now be formulated as follows.

THEOREM 2.3. *Let* $w^\lambda$ *be admissible for every* $\lambda \in (0, 1]$, $n \ge 1$ *be an integer and* $P \in \Pi_n$. *Suppose that*

$$|[w(x)]^n P(x)| \le 1 \quad \text{q.e. on } \mathfrak{S}_w, \tag{2.7}$$

*where* $\mathfrak{S}_w$ *is given by Theorem 2.2. Then*

$$|[w(x)]^n P(x)| \le e^{-cn} < 1 \quad \text{q.e. on } \Sigma \backslash \mathfrak{S}^* \tag{2.8}$$

*where the constant* $c := c(w, x) > 0$ *is independent of* $n$ *and* $P$. *Moreover, if* $\Sigma$ *is regular, then* [redacted] *for every compact set* $K \subset \Sigma \backslash \mathfrak{S}^*$,

$$\|[w(x)]^n P(x)\|_{\infty, K} \le e^{-cn} < 1, \tag{2.8a}$$

*where* $c := c(w, K) > 0$ *is independent of* $P$ *and* $n$.

We will show that the set $\mathfrak{S}^*$ in Theorem 2.3 can be taken as $\mathfrak{S}^* = \bigcap_{n=1}^{\infty} \mathfrak{S}_{1/n}$, where $\mathfrak{S}_{1/n}$ is the extremal set of Theorem 2.2 corresponding to the weight $[w(x)]^{1/(1+\delta)}$, with $\delta = 1/n$ (see Lemma 3.4).

For our new results for $L^p$-norms, we need the following definitions.

DEFINITION 2.4. *Let $E \subset \mathbf{R}$ be Lebesgue measurable. We say that $E$ is interval-like if for every $c > 0$ there is a sequence $\{\delta_n\}$ of positive numbers (depending upon $E$ and $c$) with the following properties:*

(i) $\delta_n \to 0$ as $n \to \infty$;

(ii) $\liminf \delta_n^{1/n} = 1$;

(iii) *For quasi-all $x \in E$,*

$$m(E \cap I_n(x)) \geq (1 - c/n^2)\delta_n, \qquad n \geq 1,$$

*where, for each $n$, $I_n(x)$ is one of the intervals $[x, x+\delta_n], [x-\delta_n, x]$, and $m$ denotes the Lebesgue measure.*

DEFINITION 2.5. *We say that $w$ is strongly admissible if*

(i) *$w^\lambda$ is admissible for every $\lambda$, $0 < \lambda \leq 1$,*

(ii) *$\Sigma$ is regular, and*

(iii) *$\Sigma \backslash Z$ is interval-like.*

If $A \subseteq \mathbf{R}$ is Lebesgue measurable, $g\colon \mathbf{R} \to \mathbf{R}$ is Lebesgue measurable, and $0 < p < \infty$, we set

$$\|g\|_{p,A} := \left(\int_A |g(x)|^p dx\right)^{1/p}. \tag{2.9}$$

For strongly admissible weight functions, the following theorem states that, in a sense, the $L^p$-norms of $w^n P$ "live" on $\mathfrak{S}^*$.

THEOREM 2.6. *Suppose that $w$ is strongly admissible and $0 < p < \infty$.*

(a) *Let $\eta > 0$. Then there are constants $c_1 := c_1(w, \eta, p) > 0$, $c_2 := c_2(w, \eta, p) > 0$ and a compact set $\Delta := \Delta(w, \eta, p)$ with $m(\Delta) < \eta$ such that for every integer $n \geq 1$ and $P \in \Pi_n$,*

$$\|w^n P\|_{p,\Sigma} \leq (1 + c_1 \exp(-c_2 n))\|w^n P\|_{p,\mathfrak{S}^* \cup \Delta}. \tag{2.10}$$

(b) *Let $0 < p, r \leq \infty$ and $\eta > 0$. Then, there exists a set $\Delta := \Delta(w, \eta, p, r)$ with $m(\Delta) < \eta$ such that whenever a polynomial $P \in \Pi_n$ satisfies*

$$\|[w(x)]^n P(x)\|_{p,\mathfrak{S}^* \cup \Delta} \leq 1, \tag{2.11}$$

*we have*

$$\|[w(x)]^n P(x)\|_{r,\Sigma\backslash(\mathfrak{S}^* \cup \Delta)} \leq c_1 \exp(-c_2 n), \tag{2.12}$$

*where $c_1 := c_1(w, \eta, p, r)$ and $c_2 := c_2(w, \eta, p, r)$ are positive constants independent of $n$ and $P$.*

THEOREM 2.7. *Let $\Sigma \backslash Z$ be a finite union of nondegenerate disjoint intervals and $Q$ be convex in each component of $\Sigma \backslash Z$. Assume that $w$ is strongly admissible. Then $\mathfrak{S}_w =: \bigcup_{j=1}^l [a_j, b_j]$ (cf. Theorem 2.2(e)). Let $\{\varepsilon_j\}_{j=1}^l$ be arbitrary positive numbers.*

(a) *Then inequality (2.10) holds with $\mathfrak{S}_w \cup \Delta = \bigcup_{j=1}^l [a_j - \varepsilon_j, b_j + \varepsilon_j]$ for every $p > 0$. (The constants $c_1, c_2$ will now depend upon $w, p$ and $\{\varepsilon_j\}_{j=1}^l$.)*

(b) *If $0 < p, r \leq \infty$, then with $\mathfrak{S}_w \cup \Delta = \bigcup_{j=1}^l [a_j - \varepsilon_j, b_j + \varepsilon_j]$, any polynomial $P \in \Pi_n$ that satisfies (2.11) also satisfies (2.12).*

To illustrate the result of Theorem 2.7 we discuss the special case of an exponential weight on $[0, +\infty)$.

EXAMPLE. Let $w(x) := \exp(-x^\alpha)$, $\alpha > 0$, with support $\Sigma := [0, +\infty)$. Then $Z = \varnothing$ and $Q(x) = \log[1/w(x)] = x^\alpha$ is convex for $\alpha \geq 1$. Hence, by Theorem 2.2 (e), the set $\mathfrak{S}_w$ for $\alpha \geq 1$ consists of a single compact interval $[a, b] \subset [0, +\infty)$. For $0 < \alpha < 1$, the function $Q(x)$ is no longer convex, but does possess the property that $xQ'(x)$ is increasing on $[0, +\infty)$. It is not difficult to show that this property again implies that $\mathfrak{S}_w$ is a single compact interval. To explicity determine $\mathfrak{S}_w$ we consider the $F$-functional (cf. (2.3)) for intervals $K = [c, d] \subset [0, +\infty)$. Since

$$C([c,d]) = \frac{d-c}{4} \quad \text{and} \quad d\nu_{[c,d]} = \frac{1}{\pi}\frac{dx}{\sqrt{(d-x)(x-c)}},$$

we find

$$\begin{aligned} F([c,d]) &= \log\left(\frac{d-c}{4}\right) - \frac{1}{\pi}\int_c^d \frac{x^\alpha dx}{\sqrt{(d-x)(x-c)}} \\ &= \log\left(\frac{d-c}{4}\right) - \frac{1}{\pi 2^\alpha}\int_0^\pi [d + c - (d-c)\cos\,\theta]^\alpha d\theta. \end{aligned}$$

On computing the partial derivatives $\partial F/\partial c, \partial F/\partial d$, it is straightforward to show that $F$ is maximized when $c = 0$ and

$$d = d_\alpha := \left[2\alpha\pi^{-1}\int_0^{\pi/2} \sin^{2\alpha}\theta d\theta\right]^{-1/\alpha} = \left[\frac{\sqrt{\pi}\Gamma(\alpha+1)}{\alpha\Gamma(\alpha+\frac{1}{2})}\right]^{1/\alpha}.$$

Hence $\mathfrak{S}_w = [0, d_\alpha]$.

As a consequence of Theorem 2.7(a), for each $p > 0$ and $\varepsilon > 0$, there exist positive constants $c_1$, $c_2$ depending on $\alpha, p$, and $\varepsilon$ such that for every $n \geq 1$ and $P \in \Pi_n$,

$$\int_0^\infty |e^{-nx^\alpha} P(x)|^p dx \leq (1 + c_1 \exp(-c_2 n))^p \int_0^{d_\alpha+\varepsilon} |e^{-nx^\alpha} P(x)|^p dx.$$

Furthermore, suppose that $P_n \in \Pi_n$, $n = 1, 2, \ldots$, is a sequence of polynomials such that for some $p > 0$ and $\varepsilon > 0$

$$\int_0^{d_\alpha+\varepsilon} |e^{-nx^\alpha} P_n(x)|^p dx \leq 1, \qquad n = 1, 2, \ldots.$$

Then, from Theorem 2.7(b) with $r = \infty$, we deduce that

$$e^{-nx^\alpha} P_n(x) \to 0 \quad \text{for all } x > d_\alpha.$$

For other applications of Theorem 2.7, see [**8** and **12**].

The proof of Theorem 2.3 uses potential theoretic arguments while the proofs of Theorems 2.6 and 2.7 utilize a general Nikolskii-type inequality (Lemma 3.7) in addition to Theorem 2.3. Using Nikolskii-type inequalities, we can also deduce asymptotic properties of certain extremal polynomials. These polynomials, in particular, include the orthogonal polynomials with respect to the Freud weights $\exp(-|x|^\alpha)$. Similar extremal problems have been studied by Gonchar and Rakhmanov [**5**].

In order to state our applications to polynomial extremal problems, we define

$$E_{n,p}(w) := \inf\left\{\|[w(x)]^n[x^n - P(x)]\|_{\Sigma,p} \colon P \in \Pi_{n-1}\right\}, \tag{2.13}$$

$n = 1, 2, \ldots,\ 0 < p \le \infty$. The extremal polynomials $T_n(x; w, p)$ are defined by the properties

$$T_n(x; w, p) = x^n + \cdots \in \Pi_n, \tag{2.14a}$$

$$\|[w(x)]^n T_n(x; w, p)\|_{\Sigma,p} = E_{n,p}(w). \tag{2.14b}$$

In particular, $T_n(x; w, 2)$ is the $n$th member of the system of monic orthogonal polynomials with respect to the weight function $w^{2n}$ and $[E_{n,2}(w)]^{-1}T_n(x; w, 2)$ is the corresponding orthonormalized polynomial.

THEOREM 2.8. *Let $w$ be strongly admissible. Then*

$$\lim_{n\to\infty} [E_{n,p}(w)]^{1/n} = \exp(F(\mathfrak{S}_w)); \qquad 0 < p \le \infty, \tag{2.15}$$

*where $\mathfrak{S}_w$ is defined in Theorem 2.2.*

To illustrate Theorem 2.8 we again consider the weight $w = w_\alpha(x) := \exp(-x^\alpha)$, $\alpha > 0$, on $\Sigma = [0, +\infty)$. Referring to the example following Theorem 2.7, a simple computation yields

$$F(\mathfrak{S}_w) = F([0, d_\alpha]) = \log(d_\alpha/4) - 1/\alpha.$$

Hence, by Theorem 2.8, the minimal errors

$$E_{n,p}(w_\alpha) = \inf\left\{\left[\int_0^\infty e^{-npx^\alpha}|x^n - P(x)|^p dx\right]^{1/p} : P \in \Pi_{n-1}\right\}$$

satisfy for each $p > 0$

$$\lim_{n\to\infty}[E_{n,p}(w_\alpha)]^{1/n} = \exp(F(\mathfrak{S}_w)) = d_\alpha/(4e^{1/\alpha}).$$

In order to describe the distribution of zeros of the extremal polynomials $T_n(x; w, p)$, we recall our previous results [**11**] concerning the solution of a generalized energy problem. Let $\mathfrak{M}(\Sigma)$ denote the collection of all positive unit Borel measures $\mu$ with $\operatorname{supp}(\mu) \subset \Sigma$. For $\mu \in \mathfrak{M}(\Sigma)$, and $Q(x) = \log[1/w(x)]$, we put

$$I_w(\mu) := \iint [\log|x - t| - Q(x) - Q(t)]\, d\mu(x)\, d\mu(t). \tag{2.16}$$

Let

$$V_w := \sup\{I_w(\mu) : \mu \in \mathfrak{M}(\Sigma)\}. \tag{2.17}$$

We proved in [**11**] that $V_w$ is a (finite) real number and that there exists a unique $\mu_w \in \mathfrak{M}(\Sigma)$ such that

$$V_w = I_w(\mu_w). \tag{2.18}$$

The measure $\mu_w$ was shown to be the limiting distribution of the zeros of $T_n(x; w, \infty)$ under certain conditions on $w$ [**11**, Corollary 2.5]. Various other interesting properties of $\mu_w$, also proved in [**11**] are summarized in Lemma 3.1. The following theorem is a generalization, in an $L^p$ sense, of Theorem 2.4 of [**11**].

THEOREM 2.9. *Let $w$ be strongly admissible. Suppose that $I \subset \mathbf{R}$ is a closed bounded interval containing $\mathfrak{S}_w$. Let $\{t_{k,n}\}_{k=1}^n$, $n = 1, 2, \ldots$, be a triangular scheme of points lying in $I$. With this scheme, associate the sequence of polynomials*

$$q_n(x) := \prod_{k=1}^{n} (x - t_{k,n}), \qquad n = 1, 2, \ldots,$$

*and the sequence of unit measures $\{\nu^{(n)}\}_{n=1}^{\infty}$, where for any Borel set $\mathcal{B}$*

$$\nu^{(n)}(\mathcal{B}) := (1/n)|\{k\colon t_{k,n} \in \mathcal{B}\}|, \qquad n = 1, 2, \ldots. \tag{2.19}$$

*Assume that for some $p$ $(0 < p \le \infty)$*

$$\limsup_{n\to\infty} \|w^n q_n\|_{\Sigma,p}^{1/n} \le \exp(F(\mathfrak{S}_w)). \tag{2.20}$$

*Then, in the weak-star topology,*

$$\lim_{n\to\infty} \nu^{(n)} = \mu_w. \tag{2.21}$$

*Moreover,*

$$\lim_{n\to\infty} |q_n(z)|^{1/n} = \exp\left[\int \log|z - t|d\mu_w(t)\right], \tag{2.22}$$

*uniformly on every compact set of the complex plane disjoint from $I$.*

COROLLARY 2.10. *Let $w$ be strongly admissible and $0 < p \le \infty$. Let $\{t_{k,n}\}_{k=1}^n$ be the zeros of the extremal polynomial $T_n(x; w, p)$ of (2.14). Then there exists a closed bounded interval $I$ containing $\mathfrak{S}_w$ and all the zeros $\{t_{k,n}\}_{k=1}^n$, $n = 1, 2, \ldots$. Moreover, the relations (2.21) and (2.22) hold with $q_n(z) = T_n(z; w, p)$.*

**3. Proofs.** Before providing the proofs of our theorems, we need to recall certain properties of the extremal measure $\mu_w$ defined in (2.18). These are summarized in Lemma 3.1. In the statement of this lemma and in the sequel, we assume, without loss of generality, that $Q(x) \ge 0$ for all $x \in \Sigma$.

LEMMA 3.1 [**11**]. *Let $w$ be admissible. Then*
(a) *The measure $\mu_w$ has finite logarithmic energy.*
(b) *The set $\mathfrak{S}_w$ of Theorem 2.2 is given by $\mathfrak{S}_w = \operatorname{supp}(\mu_w)$.*
(c) *The inequality*

$$\int \log|x - t|d\mu_w(t) \le Q(x) + F(\mathfrak{S}_w) \tag{3.1}$$

*holds q.e. on $\Sigma$.*
(d) *The inequality*

$$\int \log|x - t|d\mu_w(t) \ge Q(x) + F(\mathfrak{S}_w) \tag{3.2}$$

*holds for all $x \in \mathfrak{S}_w$.*
(e) *The $F$-functional for $\mathfrak{S}_w$ is given by*

$$F(\mathfrak{S}_w) = V_w + \int Q d\mu_w, \tag{3.3}$$

*where $V_w$ is defined in (2.17).*

(f) *For any positive integer* $n$, *if* $P \in \Pi_n$ *and*

$$|[w(x)]^n P(x)| \le 1 \quad q.e. \text{ on } \mathfrak{S}_w, \tag{3.4}$$

*then for all* $z \in \mathbf{C}$

$$|P(z)| \le \exp\left\{n\left[\int \log|z-t|d\mu_w(t) - F(\mathfrak{S}_w)\right]\right\}. \tag{3.5}$$

(g) *If* $\Sigma$ *is regular, then* (3.1) *holds for all points of* $\Sigma \backslash \mathfrak{S}_w$.

Assertions (a)–(f) are contained in Theorem 2.3 of [**11**] while part (g) is an observation in the proof of Theorem 2.1(c′) in [**11**, p. 84].

In what follows, we shall assume that $w^\lambda$ is admissible for every $\lambda$, $0 < \lambda \le 1$. For brevity, we denote the extremal measure $\mu_w$ by $\mu$, its support $\mathfrak{S}_w$ by $\mathfrak{S}$ and $F(\mathfrak{S})$ by $F$. Next, we define, for $\delta > 0$,

$$w_\delta(x) := \exp(-Q_\delta(x)) := \exp\left(-\frac{1}{1+\delta}Q(x)\right). \tag{3.6}$$

Since $w_\delta$ is admissible, we may apply our results in [**11**] to $w_\delta$ and get the corresponding extremal measure $\mu_\delta$ with $\operatorname{supp}(\mu_\delta) =: \mathfrak{S}_\delta$. Thus, $\mathfrak{S}_\delta$ will maximize the corresponding $F$-functional defined for compact $K$ with $C(K) > 0$ by the formula

$$F_\delta(K) := \log C(K) - \int Q_\delta d\nu_K. \tag{3.7}$$

The quantity $F_\delta(\mathfrak{S}_\delta)$ will be denoted by $F_\delta$.

The following two lemmas will play a central role in the proof of Theorem 2.3.

LEMMA 3.2. *Suppose that* $\mu$ *is a nonnegative measure with finite logarithmic energy and* $\nu$ *is any measure with*

$$\|\nu\| \le \|\mu\|. \tag{3.8}$$

*Then, if the inequality*

$$\int \log|x-t|d\nu(t) \le \int \log|x-t|d\mu(t) + c \tag{3.9}$$

*holds* $\mu$*-almost everywhere, it must hold everywhere in the complex plane* $\mathbf{C}$.

Lemma 3.2 is a variant of the Second Maximum Principle. Landkof [**6**] gives a proof of this principle for the case of Riesz potentials. Analogous arguments for the logarithmic potential in the plane lead to the version stated in Lemma 3.2.

LEMMA 3.3. *Let* $\delta > 0$ *and suppose that* $x_0 \in \Sigma \backslash \mathfrak{S}_\delta$ *satisfies*

$$\int \log|x_0 - t|d\mu_\delta(t) \le Q_\delta(x_0) + F_\delta. \tag{3.10}$$

*Then*

$$\int \log|x_0 - t|d\mu(t) < Q(x_0) + F. \tag{3.11}$$

PROOF. Since $x_0 \notin \mathfrak{S}_\delta$ and $\mathfrak{S}_\delta$ is compact, there exists a polynomial $P$ such that

$$|P(x_0)| > 3/4 \quad \text{and} \quad |P(x)| < 1/4 \qquad \text{for all } x \in \mathfrak{S}_\delta. \tag{3.12}$$

Let $r := N/\delta$, where $N$ is the degree of $P$. Then (3.12) and Lemma 3.1(c) imply that

$$\log|P(x)| + r\int \log|x-t|d\mu(t) \le rQ(x) + rF + \log(1/4) \quad \text{q.e. on } \mathfrak{S}_\delta. \tag{3.13}$$

Also, from Lemma 3.1(c), (d), we have

$$Q_\delta(x) = \int \log|x-t|d\mu_\delta(t) - F_\delta \quad \text{q.e. on } \mathfrak{S}_\delta. \tag{3.14}$$

Since $1+\delta = (N+r)/r$, we see from (3.13) and (3.14) that quasi-everywhere on $\mathfrak{S}_\delta$

$$\begin{aligned} &\log|P(x)| + r\int \log|x-t|d\mu(t) \\ &\quad \le (N+r)\int \log|x-t|d\mu_\delta(t) - (N+r)F_\delta + rF + \log(\tfrac{1}{4}). \end{aligned} \tag{3.15}$$

Note that, by Lemma 3.1(a), the measure $\mu_\delta$ has finite logarithmic energy. Hence the maximum principle of Lemma 3.2 implies that (3.15) holds for all $x \in \mathbf{C}$. In particular, with $x = x_0$, we obtain from (3.15) and (3.10),

$$\log|P(x_0)| + r\left\{\int \log|x_0-t|d\mu(t) - Q(x_0) - F\right\} \le \log(\tfrac{1}{4}). \tag{3.16}$$

Finally, since $\log|P(x_0)| > \log(\frac{3}{4})$, we see from (3.16) that

$$\int \log|x_0-t|d\mu(t) - Q(x_0) - F \le r^{-1}\log(\tfrac{1}{3}) < 0. \quad \square \tag{3.17}$$

In the next lemma, we summarize certain relationships between $\mathfrak{S}_\delta$'s.

LEMMA 3.4. *Let* $\mathfrak{S}^* := \bigcap_{n=1}^\infty \mathfrak{S}_{1/n}$. *Then*

$$\mathfrak{S} \subset \mathfrak{S}^* \tag{3.18}$$

*and*

$$\lim_{n\to\infty} F_{1/n} = F, \quad \lim_{n\to\infty} \mu_{1/n} = \mu, \tag{3.19}$$

*where the limit of the measures is the weak limit.*

PROOF. Let
$E_n := \{x_0 \in \Sigma : (3.10) \text{ does not hold with } \delta = 1/n\}$, $n = 1, 2, \dots,$
$E_0 := \{x \in \Sigma : (3.1) \text{ does not hold}\}$,
$E := E_0 \cup (\bigcup_{n=1}^\infty E_n)$.
Since each of the $E_n$'s and $E_0$ has capacity zero, it follows that $C(E) = 0$. Let $x \in \mathfrak{S}\backslash\mathfrak{S}^*$. Then (3.2) holds and $x \notin \mathfrak{S}_{1/N}$ for some $N$. If $x \notin E$, then Lemma 3.3 yields a contradiction to (3.2). Thus, $\mathfrak{S}\backslash\mathfrak{S}^* \subset E$ and so $C(\mathfrak{S}\backslash\mathfrak{S}^*) = 0$. A similar application of Lemma 3.3 to $Q_{1/n}$ in place of $Q$ shows also that $C(\mathfrak{S}_{1/m}\backslash\mathfrak{S}_{1/n}) = 0$ if $m \ge n$.

Now, if $K$ is any compact set with $C(K) > 0$, then

$$\begin{aligned} F_{1/n}(K) &= \log C(K) - \frac{n}{n+1}\int_K Q\,d\nu_K \\ &= \log C(K) - \int_K Q\,d\nu_K + \frac{1}{n+1}\int_K Q\,d\nu_K. \end{aligned} \tag{3.20}$$

In view of our assumption that $Q \geq 0$ on $\mathbf{R}$, we now see that $F_{1/n}(K) \geq F(K)$ for every compact set $K$ with $C(K) > 0$. Hence (cf. Theorem 2.2(a))

$$F_{1/n} := F_{1/n}(\mathfrak{S}_{1/n}) \geq F_{1/n}(\mathfrak{S}) \geq F(\mathfrak{S}) =: F. \tag{3.21}$$

If $K \subset \mathfrak{S}_1$ and $Q(x) \leq M$ for $x \in \mathfrak{S}_1$, then (3.20) also shows that

$$F_{1/n}(K) - F(K) \leq M/(n+1). \tag{3.22}$$

Thus, since $F(\mathfrak{S}) \geq F(\mathfrak{S}_{1/n})$ and $C(\mathfrak{S}_{1/n} \backslash \mathfrak{S}_1) = 0$,

$$\begin{aligned} F_{1/n}(\mathfrak{S}_{1/n}) - F(\mathfrak{S}) &\leq F_{1/n}(\mathfrak{S}_{1/n}) - F(\mathfrak{S}_{1/n}) \\ &= F_{1/n}(\mathfrak{S}_{1/n} \cap \mathfrak{S}_1) - F(\mathfrak{S}_{1/n} \cap \mathfrak{S}_1) \leq M/(n+1). \end{aligned} \tag{3.23}$$

Inequalities (3.21) and (3.23) give the first part of (3.19).

Next, for $\delta > 0$, $\nu \in \mathfrak{M}(\Sigma)$ and $n = 1, 2, \ldots$, we put

$$I_\delta(\nu) := \iint [\log|x-t| - Q_\delta(x) - Q_\delta(t)]\, d\nu(x)\, d\nu(t), \tag{3.24a}$$

$$V_\delta := \sup\{I_\delta(\nu) \colon \nu \in \mathfrak{M}(\Sigma)\}, \tag{3.24b}$$

$$I_0(\nu) := I_w(\nu), \quad V_0 := V_w. \tag{3.24c}$$

Since $C(\mathfrak{S}_{1/n} \backslash \mathfrak{S}_1) = 0$ and each of the measures $\mu_{1/n}, \mu$ has finite logarithmic energy, we may assume that each of these measures is supported on $\mathfrak{S}_1$. Moreover, on $\mathfrak{S}_1, 0 \leq Q(x) \leq M$. So,

$$\begin{aligned} V_{1/n} \geq I_{1/n}(\mu) &= \iint \left[\log|x-t| - \frac{n}{n+1}Q(x) - \frac{n}{n+1}Q(t)\right] d\mu(x)d\mu(t) \\ &\geq I_0(\mu) = V_0 \geq I_0(\mu_{1/n}) \\ &= I_{1/n}(\mu_{1/n}) - \frac{2}{n+1}\int Q d\mu_{1/n} \geq V_{1/n} - \frac{2M}{n+1}. \end{aligned} \tag{3.25}$$

Thus,

$$\lim_{n\to\infty} V_{1/n} = V_0. \tag{3.26}$$

We shall use this fact to show the second half of (3.19), concerning the weak limit of $\{\mu_{1/n}\}$. Using Helley's theorem, every subsequence of $\{\mu_{1/n}\}$ has a weakly convergent subsequence. Therefore, it suffices to show that if $\{\sigma_k\}$ is any weakly convergent subsequence of $\{\mu_{1/n}\}$ and $\lim_{k\to\infty} \sigma_k =: \sigma$ then $\sigma = \mu$. We may assume further that $\{\sigma_k \times \sigma_k\}$ converges to $\sigma \times \sigma$, and that $\sigma$ is supported on $\mathfrak{S}_1$. Suppose $\sigma_k =: \mu_{1/n_k}$ and $\varepsilon > 0$. Then for sufficiently large $R > 0$ and large $k$, we have

$$\begin{aligned} I_0(\sigma) &\geq \iint [\log_R|x-t| - Q(x) - Q(t)]\, d\sigma(x)\, d\sigma(t) - \varepsilon/4 \\ &\geq \iint [\log_R|x-t| - Q(x) - Q(t)]\, d\sigma_k(x)\, d\sigma_k(t) - \varepsilon/2 \\ &\geq \iint \left[\log|x-t| - Q_{1/n_k}(x) - Q_{1/n_k}(t)\right] d\sigma_k(x)\, d\sigma_k(t) - 3\varepsilon/4 \\ &= V_{1/n_k} - 3\varepsilon/4 \geq V_0 - \varepsilon, \end{aligned} \tag{3.27}$$

where $\log_R y := \max(\log y, -R)$, $y > 0$. Thus, $I_0(\sigma) \geq V_0$. But, the definition of $V_0$ then gives $I_0(\sigma) = V_0 = I_0(\mu)$. Since $\mu$ is the unique measure satisfying this equation, we have $\sigma = \mu$. This proves that $\mu_{1/n} \to \mu$ as $n \to \infty$.

Finally, we need to show that $\mathfrak{S} \subset \mathfrak{S}^*$. Since $\mu_{1/n} \to \mu$ as $n \to \infty$, it follows from Lemma 3.1(d) and the principle of descent [**6**, p. 62] that

$$\begin{aligned} \int \log|x-t|\,d\mu(t) &\geq \limsup_{n\to\infty} \int \log|x-t|d\mu_{1/n}(t) \\ &\geq \limsup_{n\to\infty} \left[\frac{n}{n+1}Q(x) + F_{1/n}\right] \\ &= Q(x) + F, \qquad \text{for } x \in \mathfrak{S}^*. \end{aligned} \tag{3.28}$$

Next we integrate both sides of (3.28) with respect to $d\nu_{\mathfrak{S}^*}(x)$. Interchanging the order of integration and using the fact that $C(\mathfrak{S}\backslash\mathfrak{S}^*) = 0$, we see that $F(\mathfrak{S}^*) \geq F(\mathfrak{S})$. Theorem 2.2(b) then gives $\mathfrak{S} \subset \mathfrak{S}^*$. □

PROOF OF THEOREM 2.3. Let $\mathfrak{S}^*$ be defined as in Lemma 3.4 and assume that $P \in \Pi_n$ satisfies

$$|[w(x)]^n P(x)| \leq 1 \quad \text{q.e. on } \mathfrak{S}. \tag{3.29}$$

Then Lemma 3.1(f) gives, for $x \in \Sigma$,

$$|[w(x)]^n P(x)| \leq \exp\left\{n\left[\int \log|x-t|\,d\mu(t) - Q(x) - F(\mathfrak{S})\right]\right\}. \tag{3.30}$$

In view of Lemma 3.3, for quasi-all $x \in \Sigma\backslash\mathfrak{S}^*$ and hence, for quasi-all $x \in \Sigma\backslash\mathfrak{S}$, this gives

$$|[w(x)]^n P(x)| \leq e^{-cn} < 1, \qquad c := c(w,x) > 0. \tag{3.31}$$

When $\Sigma$ is regular, it follows from Lemma 3.1(g) and the continuity of the logarithmic potential that (3.31) holds for all $x$ in any compact subset $K \subset \Sigma\backslash\mathfrak{S}^*$, with $c := c(K) > 0$ independent of $x$ in $K$. □

REMARK. When $\Sigma\backslash Z$ is a finite union of nondegenerate disjoint intervals and $Q$ is convex in each component of $\Sigma\backslash Z$ then each of the sets $\mathfrak{S}_{1/n}$ is a finite union of nondegenerate disjoint intervals, at most one in each component of $\Sigma\backslash Z$. This, together with the fact that $\mathfrak{S} \subset \mathfrak{S}^*$ and $C(\mathfrak{S}^*\backslash\mathfrak{S}) = 0$ shows that in this important special case, $\mathfrak{S}^* = \mathfrak{S}$. This fact generalizes our earlier results in [**9** and **10**].

A major step in the proof of Theorem 2.6 is to obtain Nikolskii-type inequalities relating the various $L^p$-metrics of weighted polynomials. This, in turn, requires an estimation of Christoffel functions. When $\Sigma\backslash Z$ is a union of finitely many disjoint nondegenerate intervals, this is easily done using the now classical ideas of Freud in [**3** or **4**]. For the more general case we need the following lemma.

LEMMA 3.5. *Let $0 < p < \infty$. Then there exists a constant $A_1 > 0$ depending upon $p$ alone with the following property: If $n \geq 1$, $P \in \Pi_n$, $B \subset [-1,1]$ is measurable and*

$$m([-1,1]\backslash B) \leq A_1/n^2, \tag{3.32}$$

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$$\|P\|_{p,[-1,1]} \leq 2\|P\|_{p,B}. \tag{3.33}$$

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Let $r := N/\delta$, where $N$ is the degree of $P$. Then (3.12) and Lemma 3.1(c) imply that

$$\log|P(x)| + r\int \log|x-t|d\mu(t) \le rQ(x) + rF + \log(1/4) \quad \text{q.e. on } \mathfrak{S}_\delta. \tag{3.13}$$

Also, from Lemma 3.1(c), (d), we have

$$Q_\delta(x) = \int \log|x-t|d\mu_\delta(t) - F_\delta \quad \text{q.e. on } \mathfrak{S}_\delta. \tag{3.14}$$

Since $1+\delta = (N+r)/r$, we see from (3.13) and (3.14) that quasi-everywhere on $\mathfrak{S}_\delta$

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REMARK. When $\Sigma\backslash Z$ is a finite union of nondegenerate disjoint intervals and $Q$ is convex in each component of $\Sigma\backslash Z$ then each of the sets $\mathfrak{S}_{1/n}$ is a finite union of nondegenerate disjoint intervals, at most one in each component of $\Sigma\backslash Z$. This, together with the fact that $\mathfrak{S} \subset \mathfrak{S}^*$ and $C(\mathfrak{S}^*\backslash\mathfrak{S}) = 0$ shows that in this important special case, $\mathfrak{S}^* = \mathfrak{S}$. This fact generalizes our earlier results in [**9** and **10**].

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*then*

$$\|P\|_{p,[-1,1]} \leq 2\|P\|_{p,B}. \tag{3.33}$$

PROOF. Let

$$m_B(P,y) := m\{x \in B \colon |P(x)| > y\}. \tag{3.34}$$

Then it is well known [**16**, Vol. II, p. 112] that

$$\|P\|_{p,B}^p = p\int_0^\infty y^{p-1} m_B(P,y)\,dy, \tag{3.35a}$$

$$\|P\|_{p,[-1,1]}^p = p\int_0^\infty y^{p-1} m_{[-1,1]}(P,y)\,dy. \tag{3.35b}$$

But $m_{[-1,1]}(P,y) = 0$ if $y > \|P\| := \|P\|_{\infty,[-1,1]}$. So,

$$\begin{aligned}\|P\|_{p,[-1,1]}^p &= p\int_0^{\|P\|} y^{p-1}\left\{m_B(P,y) + m_{[-1,1]\backslash B}(P,y)\right\} dy \\ &\le \|P\|_{p,B}^p + \left(A_1/n^2\right)\|P\|^p\end{aligned} \tag{3.36}$$

provided $m([-1,1]\backslash B) \le A_1/n^2$.

Now in view of Corollary 16 in [**13**, p. 114],

$$\|P\|^p \le A_2 n^2 \|P\|_{p,[-1,1]}^p. \tag{3.37}$$

Thus, if we choose $A_1$ so that $0 < (1 - A_1A_2)^{-1} < 2^p$, then (3.36) gives (3.33). □

In the case when $\Sigma\backslash Z$ is interval-like, we can now obtain an estimation of the Christoffel functions.

LEMMA 3.6. *Let $w$ be strongly admissible in the sense of Definition* 2.5 *and $\mathfrak{S} := \mathfrak{S}_w$ be the unique compact set of Theorem* 2.2. *Put*

$$\lambda_n(w^{2n},x) := \min_{P\in\Pi_n} [P(x)]^{-2}\int_\Sigma [P(t)w^n(t)]^2\,dt, \tag{3.38}$$

$$\omega(Q,\delta) := \max\{|Q(t) - Q(y)| \colon y \in \mathfrak{S},\ t \in \Sigma,\ |y-t| \le \delta\}, \tag{3.39}$$

$$d := \inf\{|y - z| \colon y \in \mathfrak{S},\ z \in Z\}. \tag{3.40}$$

*Then, for $n$ sufficiently large, we have for all $x \in \Sigma$*

$$\lambda_n(w^{2n},x) \ge c_0\delta_n n^{-2}\exp\{-2n\omega(Q,\delta_n)\}[w(x)]^{2n}, \tag{3.41}$$

*where the sequence $\{\delta_n\}$ satisfies the conditions of Definition* 2.4 *with $E = \Sigma\backslash Z$ and $c = A_1/2$ with $A_1$ defined in Lemma* 3.5.

PROOF. First, let $x \in \mathfrak{S}$ be such that the condition (iii) in Definition 2.4 holds with $A_1/2$ in place of $c$. Choose $n$ so large that $\delta_n \le d/2$. Then,

$$\lambda_n(w^{2n},x) \ge \min_{P\in\Pi_n}[P(x)]^{-2}\int_{(\Sigma\backslash Z)\cap I_n(x)} [P(t)w^n(t)]^2\,dt,$$

and so in view of Lemma 3.5, we have

(3.42)

$$\begin{aligned}\lambda_n(w^{2n},x) &\ge cw^{2n}(x)\exp(-2n\omega(Q,\delta_n))\cdot\min_{P\in\Pi_n}\left\{[P(x)]^{-2}\int_{I_n(x)} P(t)^2\,dt\right\} \\ &\ge c\delta_n w^{2n}(x)\exp(-2n\omega(Q,\delta_n))\cdot\min_{R\in\Pi_n}\left\{[R(0)]^{-2}\int_0^1 R(t)^2 dt\right\}.\end{aligned}$$

Using standard estimations for the Legendre polynomials, we then get

$$\lambda_n(w^{2n}, x) \geq c_0\delta_n n^{-2} \exp(-2n\omega(Q, \delta_n))w^{2n}(x). \tag{3.43}$$

Setting

$$M_n := [\delta_n n^{-2} \exp(-2n\omega(Q, \delta_n))]^{-1}, \tag{3.44}$$

we have then proved

$$w^{2n}(x)\lambda_n^{-1}(w^{2n}, x) \leq c_0^{-1} \cdot M_n \quad \text{q.e. on } \mathfrak{S}. \tag{3.45}$$

Since $\lambda_n^{-1}(w^{2n}, x)$ is a polynomial of degree $2n$, inequality (3.45) holds everywhere on $\Sigma$ in view of Theorem 2.2(d). □

Using Lemma 3.6, we may now proceed exactly as in **[9]** to get the following inequalities.

LEMMA 3.7. *Let $w$ be strongly admissible and $M_n$ be as in* (3.44), $0 < p < r \leq \infty$ *and $P \in \Pi_n$. Then there exists a constant $c > 0$ independent of $n$ and $P$ such that*

$$\|[w(x)]^n P(x)\|_{r,\Sigma} \leq c \cdot M_n^{1/p-1/r} \|[w(x)]^n P(x)\|_{p,\Sigma}. \tag{3.46}$$

Using the fact (cf. Definition 2.4) that

$$\lim_{n\to\infty} M_n^{1/n} = 1 \tag{3.47}$$

it is now easy to see that even if $\Sigma$ is unbounded, the $L^p$-norm of a weighted polynomial on $\Sigma$ almost "lives" on a fixed, compact interval. The following lemma makes this precise.

LEMMA 3.8. *Let $w$ be strongly admissible and $0 < p < \infty$. Then there is a fixed compact interval $J$, and constants $c_1, c_2 > 0$, depending only on $p, w$ and $\Sigma$ with the following property:*

*If $P \in \Pi_n$, then*

$$\|w^n P\|_{p,\Sigma} \leq (1 + c_1 e^{-c_2 n})\|w^n P\|_{p,J\cap\Sigma}. \tag{3.48}$$

PROOF. First, let $\delta$ be an integer such that $\delta > 1/p$ and choose $A$ such that (cf. (3.6))

$$\mathfrak{S}_\delta \subset [-A, A]. \tag{3.49}$$

Then for $x \in [-A, A] \cap \Sigma$

$$\begin{aligned} |x^{n\delta} w(x)^n P(x)| &\leq A^{n\delta} \|w(x)^n P(x)\|_{\infty,[-A,A]\cap\Sigma} \\ &\leq A^{n\delta} \|w(x)^n P(x)\|_{\infty,\Sigma} \\ &\leq A^{n\delta} c M_n^{1/p} \|w^n P\|_{p,\Sigma}. \end{aligned} \tag{3.50}$$

Here the last inequality follows from Lemma 3.7. Now, in view of (3.47), let $n \geq 2$ be so large that

$$M_n^{1/n} \leq 2^{\delta p}. \tag{3.51}$$

Then, on writing $x^{n\delta} P(x) w(x)^n = x^{n\delta} P(x) w_\delta(x)^{n(1+\delta)}$ and using Theorem 2.2(d) and (3.50), we see that

$$|P(x) w(x)^n| \leq c(2A)^{n\delta} \|w^n P\|_{p,\Sigma} \cdot |x|^{-n\delta}, \qquad x \in \Sigma. \tag{3.52}$$

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