Constr. Approx. (1987) 3: 331-361

CONSTRUCTIVE
APPROXIMATION

Convergence of Padé Approximants of Partial Theta Functions and the Rogers–Szegö Polynomials

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Abstract. We investigate the convergence of sequences of Padé approximants for the partial theta function

$$h_q(z) := \sum_{j=0}^{\infty} q^{j(j-1)/2} z^j, \qquad q = e^{i\theta}, \quad \theta \in [0, 2\pi).$$

When $\theta/(2\pi)$ is irrational, this function has the unit circle as its natural boundary. We determine subregions of $|z|<1$ in which sequences of Padé approximants converge uniformly, and subregions in which they converge in capacity, but not uniformly. In particular, we show that only a proper subsequence of the diagonal sequence $\{[n/n]\}_{n=1}^{\infty}$ converges locally uniformly in all of $|z|<1$; in contrast, no subsequence of any Padé row $\{[m/n]\}_{m=1}^{\infty}$ (with $n \geq 2$ fixed) can converge locally uniformly in all of $|z|<1$. Further, we obtain the zero and pole distributions of sequences of Padé approximants by analyzing the zero distribution of the Rogers–Szegö polynomials

$$G_n(z) := \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} z^j, \qquad n = 0, 1, 2, \ldots.$$

1. Introduction

In a number of contexts, the function

$$(1.1) \qquad h_q(z) := \sum_{j=0}^{\infty} q^{j(j-1)/2} z^j = 1 + z + qz^2 + q^3 z^3 + \cdots$$

arises as an extremal or limit function. For example, in investigating convergence of the rows of the Padé table for a Maclaurin series with "smooth coefficients" ($a_{j-1}a_{j+1}/a_j^2$ has a finite limit as $j \to \infty$), the power series (1.1) and its Padé denominators play a pivotal role (see Lubinsky [24], [25]).

Also, if $\mathcal{S}$ denotes the shift operator

$$(1.2) \qquad \mathcal{S}(a_0 + a_1 z + a_2 z^2 + \cdots) = a_1 + a_2 z + a_3 z^2 + \cdots,$$

Date received: February 3, 1986. Date revised: October 10, 1986. Communicated by Paul Nevai.
AMS classification: Primary, 41A21, 33A65; Secondary, 30E05, 30E10.
Key words and phrases: Padé approximant, Theta function, Rogers–Szegö polynomial, Convergence regions, Distribution of zeros and poles.

then h_q is the unique solution of the initial value problem

$$(\mathcal{S}h)(z)=h(qz), \qquad h(0)=1. \tag{1.3}$$

Because of the form of this equation, h_q plays a role in the study of zeros of normalized power series remainders $\mathcal{S}^n h$, that is analogous to the role played by the Macintyre functions in the study of the Whittaker constant (see Varga [42]).

The functions h_q are known as *partial theta functions* in the theory of special functions, because there is a close relationship between h_q and classical theta functions—see Andrews [1] and Askey and Ismail [2]. We use the notation $h_q(z)$ because the continued fraction expansion

$$h_q(z)\sim 1+\frac{z|}{|1}-\frac{qz|}{|1}+\frac{q(1-q)z|}{|1}-\frac{q^3z|}{|1}+\frac{q^2(1-q^2)z|}{|1}-\cdots \tag{1.4}$$

was originally studied by Heine (see Perron [30]). For $|q|<1$, it is known that this continued fraction converges for all $z\in\mathbf{C}$ (see Perron [30] and Balk [5]). However, for $|q|=1$, the validity of the continued fraction representation does not seem to have been considered.

The main goal of this paper is to study the convergence of Padé approximants for h_q in the case when $|q|=1$. If $q=e^{i\theta}$, $\theta\in[0,2\pi)$, then it is easily seen that $\theta/(2\pi)$ rational implies that $h_q(z)$ is a rational function. However, if $\theta/(2\pi)$ is irrational, then $h_q(z)$ has the unit circle as its natural boundary (see Theorem 6.1).

In investigating the Padé table of h_q, we shall use the explicit formulae available for the Padé denominators, and the coefficients in the associated continued fraction (Balk [5], Gragg [16], Perron [30], Wynn [45]). These lead naturally to a consideration of the *Rogers-Szegö polynomials*:

$$G_n(z):=\sum_{j=0}^{n}\begin{bmatrix} n\\ j\end{bmatrix} z^j, \qquad n=0,1,2,\ldots, \tag{1.5}$$

where

$$\begin{bmatrix} n\\ j\end{bmatrix}:=\begin{cases}\dfrac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-j+1})}{(1-q)(1-q^2)\cdots(1-q^j)}, & j=1,2,\ldots,n,\\ 1, & j=0,\end{cases} \tag{1.6}$$

is the *Gaussian binomial coefficient.* Despite the apparent singularity when q is a kth root of unity for some $1\le k\le j$, the coefficients (1.6) are polynomials in q, which may be defined, by induction on n, with the aid of the identity

$$\begin{bmatrix} n\\ j\end{bmatrix}=\begin{bmatrix} n-1\\ j\end{bmatrix}+q^{n-j}\begin{bmatrix} n-1\\ j-1\end{bmatrix}, \qquad j=1,2,\ldots,n. \tag{1.7}$$

For $0<q<1$, the Rogers-Szegö polynomials (when suitably normalized) are orthogonal with respect to a nonnegative weight on the unit circle (see Szegö [41] and Askey and Ismail [2]), but have not apparently been considered before for $|q|=1$.

In this paper we obtain the zero distribution of the Rogers-Szegö polynomials for $|q|=1$, and also obtain regions omitting all zeros of $\{G_n(z)\}_{n=1}^{\infty}$. We shall see that the largest disk, center 0, in which $\{G_n(z)\}_{n=1}^{\infty}$ is a normal family, is the disk $|z|<R(q)$, where

$$R(q) := \liminf_{n\to\infty} |1-q^n|^{1/n}. \tag{1.8}$$

Further, if q is not a root of unity, then regardless of whether $R(q)$ is positive or zero, there exists an increasing sequence $\mathcal{J}$ of positive integers such that

$$\lim_{n\in\mathcal{J}} G_n(z) = 1, \tag{1.9}$$

locally uniformly in $|z|<1$ and, more generally, for each fixed positive integer l,

$$\lim_{n\in\mathcal{J}} G_{n+l}(z) = G_l(z), \tag{1.10}$$

locally uniformly in $|z|<1$. All these results are stated in Section 2.

In Section 3 we present our results on convergence of Padé approximants to the function $h_q(z)$ for $q=e^{i\theta}$, with $\theta/(2\pi)$ irrational. In particular, we prove that any sequence $\{[m_j/n_j]\}_1^{\infty}$ with

$$m_j \ge n_j - 1, \qquad j=1,2,3,\ldots, \tag{1.11}$$

and

$$\lim_{j\to\infty} n_j = \infty, \tag{1.12}$$

converges geometrically in capacity in $|z|<1$. Further, if

$$\Delta_q := \inf\{|z|\colon G_n(z)=0, \text{ some } n\ge 1\}, \tag{1.13}$$

we show that $1>\Delta_q \ge 3-2\sqrt{2}$, and that $\{[m_j/n_j]\}_1^{\infty}$ converges locally uniformly in $|z|<\Delta_q$. This latter result is best possible in the sense that there exists a sequence of integers $\{n_j\}_1^{\infty}$ satisfying (1.12), such that for any sequence of integers $\{m_j\}_1^{\infty}$ satisfying (1.11), $\{[m_j/n_j]\}_1^{\infty}$ has a limit point of poles on $|z|=\Delta_q$. However, there exists another sequence of integers $\mathcal{J}=\{n_j\}_1^{\infty}$ satisfying (1.12), such that for any $\{m_j\}_1^{\infty}$ satisfying (1.11), $\{[m_j/n_j]\}_1^{\infty}$ converges locally uniformly in $|z|<1$.

Concerning the rows of the Padé table, it is shown that whenever $\theta/(2\pi)$ is irrational, $q=e^{i\theta}$ and $n\ge 2$, there exists $0<\Delta_{nq}<1$, such that $[m/n](z)$ has a pole on $|z|=\Delta_{nq}$ for $m\ge n-1$, so that no subsequence of $\{[m/n]\}_{m=1}^{\infty}$ converges locally uniformly in all of $|z|<1$. This provides another example of a phenomenon recently discovered by Buslaev, Gončar, and Suetin [8]. Finally, we analyze the zero distribution of sequences of normalized Padé numerators and denominators.

The results of Section 2 are proved in Sections 4 and 5, and those of Section 3 are proved in Section 6.

2. The Rogers–Szegö Polynomials

Throughout, we let

$$G_0(z) = G_0(z;\, q) := 1$$

and

$$(2.1)\qquad G_n(z)=G_n(z;q):=\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix} z^j, \qquad n=1,2,3,\ldots,$$

where the Gaussian binomial coefficients are defined by (1.6) and (1.7). We shall use the abbreviated notation $G_n(z)$ except when it is necessary to distinguish $G_n(z;q)$ for different values of q. We also assume that q and $\theta\in[0,2\pi)$ are related by

$$(2.2)\qquad q=e^{i\theta}.$$

In analyzing the zero distribution of $\{G_n\}$, the following result plays a crucial role.

Theorem 2.1. *Let $\|\cdot\|$ denote the sup norm on the unit circle $|z|=1$.*

(i) *If $\theta/(2\pi)$ is a rational number μ/ν, where μ, ν are positive integers whose greatest common divisor is* 1, *then*

$$(2.3)\qquad \lim_{n\to\infty}\|G_n\|^{1/n}=2^{1/\nu}.$$

(ii) *If $\theta/(2\pi)$ is irrational, then*

$$(2.4)\qquad \lim_{n\to\infty}\|G_n\|^{1/n}=1.$$

The significance of (2.4) becomes more apparent if one applies results of Rosenbloom [34] or Blatt and Saff [7] on the zero distribution of polynomials. Together with the observations that $G_n(z)$ is monic, and *inversive*; that is,

$$(2.5)\qquad z^nG_n(1/z)=G_n(z), \qquad n=0,1,2,\ldots$$

(see Lemma 4.1 below), these results imply that all but a negligible proportion of the zeros of $G_n(z)$ approach the unit circle as $n\to\infty$. Furthermore, the arguments of the zeros are uniformly distributed in angle. In order to make these assertions more precise, we need:

Definition 2.2. Let $\mathcal{J}$ be an increasing sequence of positive integers, and for $n\in\mathcal{J}$, let $p_n(z)=A_nz^n+\cdots$ be a polynomial of degree n and assume that

$$(2.6)\qquad \lim_{n\in\mathcal{J}}|A_n|^{1/n}=1$$

and

$$(2.7)\qquad \lim_{n\in\mathcal{J}}\|p_n\|^{1/n}=1.$$

where $\lim_{n\in\mathcal{J}}$ means that n tends to ∞ through $\mathcal{J}$. Then we say that $\{p_n\}_{n\in\mathcal{J}}$ has *property U.* If $\{p_n\}_{n\in\mathcal{J}}$ has property U and there exist numbers v_n and w_n of unit modulus such that

$$(2.8)\qquad v_nz^np_n(w_n/z)=p_n(z), \qquad n\in\mathcal{J},$$

then we say that $\{p_n\}_{n\in\mathcal{J}}$ has *property W.*

It is clear from Theorem 2.1 and from (2.5) that *when* $\theta/(2\pi)$ *is irrational,* $\{G_n\}_0^\infty$ *has property W.* The consequences of property W are summarized in the following result:

Theorem 2.3. *Suppose* $\{p_n\}_{n\in\mathcal{J}}$ *has property W and let* $\varepsilon>0$.

(i) *If* N_n *is the number of zeros of* $p_n(z)$ *outside the annulus* $\mathcal{A}_\varepsilon := \{z: 1-\varepsilon<|z|<1+\varepsilon\}$, *then*

$$\lim_{n\in\mathcal{J}} N_n/n = 0. \tag{2.9}$$

(ii) *Write*

$$p_n(z) = \tilde{p}_n(z)\tilde{q}_n(z), \qquad n\in\mathcal{J}, \tag{2.10}$$

where $\tilde{q}_n(z)$ *is the polynomial of degree* N_n *with leading coefficient* A_n, *whose zeros are the zeros of* $p_n(z)$ *outside* $\mathcal{A}_\varepsilon$. *Then, with a suitable choice of branches,*

$$\lim_{n\in\mathcal{J}} \tilde{p}_n(z)^{1/(n-N_n)} = z, \tag{2.11}$$

locally uniformally in $|z|>1+\varepsilon$, *and for a suitable sequence of numbers* $\{\hat{w}_n\}_{n\in\mathcal{J}}$ *of unit modulus,*

$$\lim_{n\in\mathcal{J}} (\hat{w}_n\tilde{p}_n(z))^{1/(n-N_n)} = 1, \tag{2.12}$$

locally uniformly in $|z|<1-\varepsilon$.

(iii) *If* $0\le\alpha<\beta<2\pi$, *and* $Z_n[\alpha,\beta]$ *denotes the number of zeros of* $p_n(z)$ *in* $\mathcal{A}_\varepsilon\cap\{z: \arg(z)\in[\alpha,\beta]\}$, *then*

$$\lim_{n\in\mathcal{J}} n^{-1}Z_n[\alpha,\beta] = (\beta-\alpha)/(2\pi). \tag{2.13}$$

If $\{p_n\}_{n\in\mathcal{J}}$ *has only property U, then* (2.9) *of* (i) *and* (2.11) *of* (ii) *remain valid if* $\mathcal{A}_\varepsilon$ *is replaced by* $\{z: |z|<1+\varepsilon\}$.

When $\theta/(2\pi)$ is irrational, more can be said concerning (2.11), namely

$$\lim_{n\to\infty} G_n(z)^{1/n} = z, \qquad |z|>3+2\sqrt{2}. \tag{2.14}$$

This follows from the above result and Theorem 2.7. One might hope for something sharper than (2.14), such as a ratio asymptotic

$$G_{n+1}(z)/G_n(z)\to z, \qquad n\to\infty;$$

asymptotics of this type are valid for polynomials orthogonal on the unit circle with respect to a measure positive almost everywhere on $|z|=1$ (Rakhmanov [33], Máté, Nevai, and Totik [27]). However, it is a consequence of Theorem 2.5 below that, for $|z|>1$ such that $|1+z^{-1}|\neq 1$, even $\lim_{n\to\infty}|G_{n+1}(z)/G_n(z)|$ cannot exist. This latter statement is proved in Lemma 4.7.

For *subsequences* of $\{G_n(z)\}_1^\infty$, one can say much more than Theorem 2.3. In describing the behavior of these subsequences, we need the series

$$(2.15)\qquad I_w(z)=I_w(z;q):=\sum_{j=0}^{\infty}\frac{(1-w)(1-wq^{-1})\cdots(1-wq^{-j+1})z^j}{(1-q)(1-q^2)\cdots(1-q^j)}$$

for $|w|=1$.

Theorem 2.4. *Let $\theta/(2\pi)$ be irrational, and let*

$$(2.16)\qquad R(q):=\liminf_{n\to\infty}|1-q^n|^{1/n}.$$

(i) *The sequence of functions $\{G_n(z)\}_1^\infty$ is uniformly bounded in each compact subset of $|z|<R(q)$, but not uniformly bounded in the annulus*

$$\{z\colon R(q)<|z|<R(q)+\varepsilon\}\qquad \textit{for any}\quad \varepsilon>0.$$

(ii) *Let $\mathcal{J}$ be an increasing sequence of positive integers such that*

$$(2.17)\qquad \lim_{n\in\mathcal{J}} q^n=w.$$

Then, locally uniformly in $|z|<R(q)$,

$$(2.18)\qquad \lim_{n\in\mathcal{J}} G_n(z)=I_w(z).$$

In particular, if l is a fixed nonnegative integer and $w=q^l$,

$$(2.19)\qquad \lim_{n\in\mathcal{J}} G_n(z)=G_l(z),$$

locally uniformly in $|z|<R(q)$.

When $R(q)=0$, the above theorem yields very little information about $\{G_n(z)\}_1^\infty$, and when $0<R(q)<1$, it does not say much about what happens in the annulus $\{z\colon R(q)\le|z|<1\}$. In order to fill this gap, we need to define a certain "thin" set $\mathcal{G}$. Let

$$(2.20)\qquad \mathcal{G}:=\{q\colon \theta/(2\pi)\text{ is irrational and } R(q)<1\}.$$

Since $|1-q^n|=2|\sin(n\theta/2)|$, it is easily seen that $q\in\mathcal{G}$ iff $\theta/(2\pi)\in\mathcal{H}$, where $\mathcal{H}$ is the set of all irrational numbers $\chi\in(0,1)$ that may be approximated by rationals geometrically fast, so that

$$\liminf_{n\to\infty}\Big(\min_{1\le j\le n}|\chi-j/n|\Big)^{1/n}<1.$$

The set $\mathcal{H}$ is dense in $(0,1)$ and its elements are all transcendental. Further, $\mathcal{H}$ has linear Lebesgue measure zero and even Hausdorff logarithmic dimension 2 (see Lubinsky [23]). It follows that $\mathcal{G}$ is correspondingly thin on the unit circle, but still dense on this circle.

Theorem 2.5. *Let $\theta/(2\pi)$ be irrational. If $q \notin \mathcal{G}$, let $\mathcal{J}$ be any increasing sequence of positive integers for which*

$$\lim_{n\in\mathcal{J}} q^n = 1. \tag{2.21}$$

If $q \in \mathcal{G}$, let $R(q) < \eta < 1$ and let $\mathcal{J}$ be any increasing sequence of positive integers for which

$$\eta^{-n}|1-q^n| < \eta^{-j}|1-q^j|, \quad 1 \le j < n; \qquad n \in \mathcal{J}. \tag{2.22}$$

Then, locally uniformly in $|z|<1$,

$$\lim_{n\in\mathcal{J}} G_n(z) = 1. \tag{2.23}$$

Furthermore, for each fixed positive integer l,

$$\lim_{n\in\mathcal{J}} G_{n+l}(z) = G_l(z), \tag{2.24}$$

locally uniformly in $|z|<1$.

Note that if $q \in \mathcal{G}$, and $R(q) < \eta < 1$, then

$$\liminf_{n\to\infty} \eta^{-n}|1-q^n| = 0.$$

It is thus possible to choose a sequence $\mathcal{J}$ for which (2.22) is valid.

While the above theorem yields quite precise information about the limit points of zeros of $\{G_n(z)\}_1^\infty$, it does not yield regions that contain all zeros, or contain no zeros of the full sequence $\{G_n(z)\}_1^\infty$. Using the three-term recurrence relation

$$G_{n+1}(z) = (1+z)G_n(z) - (1-q^n)zG_{n-1}(z), \qquad n = 1, 2, 3, \ldots, \tag{2.25}$$

and other identities for $\{G_n(z)\}_1^\infty$ that are essentially the recurrence relations for the denominator polynomials in the continued fraction (1.4), we shall prove:

Theorem 2.6.

(i) *Let $s>0$ and let $\mathcal{C}_s$ denote the set of all $z \in \mathbf{C}$ satisfying*

$$\min_{|w|=1}\{|z^{-1}+w[1-w(1+q)]| - |1-w|s - s^{-1}\} \ge 0. \tag{2.26}$$

Then $\mathcal{C}_s$ contains no zeros of $\{G_n(-zq^n)\}_{n=1}^\infty$.

(ii) *Let*

$$r_0 := 2\sqrt{2}+1+|1+q|. \tag{2.27}$$

Then the annulus $\{z\colon r_0^{-1} < |z| < r_0\}$ contains all zeros of $\{G_n(z)\}_1^\infty$.

At present, it is not known whether (i) or (ii) is sharp. In any event, it will become clear in Section 3 that (ii) is sufficient to guarantee locally uniform convergence in $|z| < r_0^{-1}$ of sequences of Padé approximants to $h_q(z)$. Since zero free disks for $G_n(z)$ play an important role in our convergence result for the

Table 2.1

n	1	2	3	5	8	11	14	17
$\inf\{\Delta_{nq}: \|q\|=1\}$	1.0	0.58	0.45	0.39	0.35	0.29	0.27	0.24

Padé approximants of $h_q(z)$, we shall state one more result concerning the zero of $G_n(z)$ of smallest modulus. Set

$$\Delta_{nq} := \min\{|z|: G_n(z; q) = 0\}, \qquad n = 1, 2, 3, \ldots, \tag{2.28}$$

$$\Delta_q := \inf\{\Delta_{nq}: n \geq 1\}, \tag{2.29}$$

and

$$\Delta := \inf\{\Delta_q : |q| = 1\}. \tag{2.30}$$

Theorem 2.7. *Let $\theta/(2\pi)$ be irrational. Then*

$$1 > \Delta_{nq} \geq (2\sqrt{2} + 1 + |1+q|)^{-1} \geq 3 - 2\sqrt{2}, \qquad n = 2, 3, \ldots, \tag{2.31}$$

and

$$\Delta_q = \liminf_{n\to\infty} \Delta_{nq} < \limsup_{n\to\infty} \Delta_{nq} = 1. \tag{2.32}$$

Except for the leftmost inequality in (2.31), the assertions of Theorem 2.7 follow from Theorems 2.5 and 2.6. Moreover, Theorem 2.5 implies that for any positive integer l, each zero of $G_l(z)$ is a limit point of zeros of $\{G_n(z)\}_1^\infty$, which is considerably stronger than (2.32).

One question left unanswered by the above results, is the size of Δ. We have the trivial bounds $1 > \Delta \geq 3 - 2\sqrt{2} = 0.1715\ldots$. Some numerical upper bounds for Δ appear in Table 2.1.

3. Convergence Theorems for Padé Approximants

Let

$$f(z) = \sum_{j=0}^{\infty} a_j z^j,$$

and m, n be nonnegative integers. Recall that the m, n Padé approximant to $f(z)$ is $[m/n](z) = P(z)/Q(z)$, where P and Q have degree at most m and n, respectively, $Q \not\equiv 0$, and

$$f(z)Q(z) - P(z) = O(z^{m+n+1}).$$

Let $a_j = 0$, $j < 0$, and

$$D(m/n) := \det \begin{bmatrix} a_m & a_{m+1} & \cdots & a_{m+n-1} \\ a_{m-1} & a_m & \cdots & a_{m+n-2} \\ \vdots & \vdots & & \vdots \\ a_{m-n+1} & a_{m-n+2} & \cdots & a_m \end{bmatrix}, \qquad m, n = 0, 1, 2, \ldots. \tag{3.1}$$

When $D(m/n)\neq 0$, it can be shown (see [3] and [4]) that

$$[m/n](z)=P_{mn}(z)/Q_{mn}(z), \tag{3.2}$$

where

$$Q_{mn}(0)=1 \quad\text{and}\quad P_{mn}(0)=a_0, \tag{3.3}$$

$$Q_{mn}(z)=\frac{1}{D(m/n)}\det\begin{bmatrix} a_m & a_{m+1} & \cdots & a_{m+n} \\ a_{m-1} & a_m & \cdots & a_{m+n-1} \\ \vdots & \vdots & & \vdots \\ a_{m-n+1} & a_{m-n+2} & \cdots & a_{m+1} \\ z^n & z^{n-1} & \cdots & 1 \end{bmatrix}, \tag{3.4}$$

and

$$P_{mn}(z)=\frac{1}{D(m/n)}\det\begin{bmatrix} a_m & a_{m+1} & \cdots & a_{m+n} \\ a_{m-1} & a_m & \cdots & a_{m+n-1} \\ \vdots & \vdots & & \vdots \\ a_{m-n+1} & a_{m-n+2} & \cdots & a_{m+1} \\ \sum_{j=n}^{m} a_{j-n}z^j & \sum_{j=n-1}^{m} a_{j-n+1}z^j & \cdots & \sum_{j=0}^{m} a_j z^j \end{bmatrix}. \tag{3.5}$$

From the results of Section 2, and those in [24] and [25], we obtain the following theorem concerning the rows of the Padé table.

Theorem 3.1. *Let $f(z)=\sum_{j=0}^{\infty} a_j z^j$, where $a_j\neq 0$ for j large enough, and assume that*

$$\lim_{j\to\infty} a_{j-1}a_{j+1}/a_j^2=q, \tag{3.6}$$

where $q=e^{i\theta}$ and $\theta/(2\pi)$ is irrational. Let the radius of convergence of f be $r\in(0,\infty]$, and let

$$r_-:=\liminf_{j\to\infty}|a_j/a_{j+1}| \quad\text{and}\quad r_+:=\limsup_{j\to\infty}|a_j/a_{j+1}|. \tag{3.7}$$

Fix $n\geq 2$.

(i) *If $r_->0$, then*

$$\lim_{m\to\infty}[m/n](z)=f(z), \tag{3.8}$$

locally uniformly in $|z|<\Delta_{nq}r_-$, where Δ_{nq} is defined in (2.28).

(ii) *There is an increasing sequence $\mathcal{J}$ of the positive integers for which*

$$\lim_{m\in\mathcal{J}}[m/n](z)=f(z), \tag{3.9}$$

locally uniformly in $|z|<\Delta_{nq}r$.

(iii) *If $r_+=r<\infty$, then* (ii) *is sharp in the sense that for m large enough, $[m/n](z)$ has a pole z_m with*

$$\limsup_{m\to\infty}|z_m|=\Delta_{nq}r<r. \tag{3.10}$$

Thus no subsequence of $\{[m/n]\}_{m=1}^{\infty}$ *can converge locally uniformly in the disk* $|z| \le \Delta_{nq} r + \varepsilon$, *for any* $\varepsilon > 0$.

Theorem 3.1(iii) shows that even when the power series coefficients of f are reasonably behaved, nevertheless not even a subsequence of any row (with denominator degree ≥ 2) need converge pointwise throughout the largest disk of analyticity of f. This provides a class of counterexamples to the conjecture of Baker and Graves-Morris, recently resolved by Buslaev, Gončar, and Suetin [8]. See Section 3 of [25] for a brief review of convergence theorems for rows of the Padé table.

For the special case $f = h_q$, one can prove more than Theorem 3.1(ii); namely, for $m \ge n-1$, $[m/n](z)$ has a pole on the circle $|z| = \Delta_{nq} < 1$. This is an immediate consequence of part (ii) of the following theorem, which is essentially contained in the papers of Balk [5], Gragg [16], and Wynn [45].

Theorem 3.2. *Let* $[m/n](z)$ *and* $D(m/n)$ *denote, respectively, the m, n Padé approximant and m, n Toeplitz determinant associated with* $h_q(z)$.

(i) If $m \ge n-1 > 0$,

$$D(m/n) = \{q^{m(m-1)/2}\}^n \prod_{j=1}^{n-1} (1-q^j)^{n-j}. \tag{3.11}$$

(ii) *If* $m \ge n-1 > 0$, *and q is not a* jth *root of unity for some* $1 \le j \le n$, *then the normalized Padé denominator* $Q_{mn}(z)$ *to* $h_q(z)$ *satisfies*

$$Q_{mn}(-zq^{-m}) = G_n(z). \tag{3.12}$$

Using this last result, we can obtain convergence results for the upper half of the Padé table of $h_q(z)$.

Theorem 3.3. *Let* $\theta/(2\pi)$ *be irrational.*

(i) *Let* $\{m_j\}_{j=1}^{\infty}$ *and* $\{n_j\}_{j=1}^{\infty}$ *be sequences of positive integers satisfying*

$$m_j \ge n_j - 1, \qquad j = 1, 2, 3, \ldots, \tag{3.13}$$

and

$$\lim_{j\to\infty} m_j = \infty. \tag{3.14}$$

(a) *Then* $\{[m_j/n_j]\}_{j=1}^{\infty}$ *converges in capacity to* h_q *in* $|z| < 1$. *More precisely, given* $0 < \varepsilon < r < 1$,

$$\operatorname{cap}\{z\colon |z| \le r \text{ and } |h_q(z) - [m_j/n_j](z)| \ge (|z|(1+\varepsilon))^{m_j+n_j}\} \to 0 \qquad (as\ j \to \infty). \tag{3.15}$$

(b) *Further* $\{[m_j/n_j]\}_{j=1}^{\infty}$ *converges locally uniformly to* h_q *in* $|z| < \Delta_q$, *where* Δ_q *is defined in* (2.29). *More precisely, given* $0 < \varepsilon < r < \Delta_q$,

$$\max\{|h_q(z) - [m_j/n_j](z)|/(|z|(1+\varepsilon))^{m_j+n_j}\colon |z| \le r\} \to 0 \qquad as \quad j \to \infty. \tag{3.16}$$

(ii) *The result of* (i)(b) *is sharp in the following sense: there exists a sequence of positive integers* $\{n_j\}_1^\infty$ *such that*

$$\lim_{j\to\infty} n_j = \infty, \tag{3.17}$$

and such that whenever $\{m_j\}_1^\infty$ *satisfies* (3.13), *then* $[m_j/n_j](z)$ *has a pole* z_j, $j = 1, 2, 3, \ldots$, *with* $\lim_{j\to\infty}|z_j| = \Delta_q$.

Note first that the convergence in capacity above is not a consequence of the Nuttall-Pommerenke theorem [29], [31] or Stahl's theorem [40], since h_q has the unit circle as its natural boundary. It is well known (Lubinsky [22], Rakhmanov [32], Stahl [38]) that diagonal sequences of Padé approximants need not converge in capacity in any open set within the domain of analyticity of f, when f has singularities of positive capacity in **C**. Perhaps the only published convergence result for Padé sequences formed from functions with natural boundaries is due to Gammel and Nuttall [14], but the class of functions to which it applies does not include h_q.

It is an easy consequence of Theorems 2.4 and 3.2 that given sequences of positive integers $\{m_j\}_1^\infty$ and $\{n_j\}_1^\infty$ satisfying (3.13) and (3.14), $\{Q_{m_jn_j}(z)\}_1^\infty$ is uniformly bounded in each compact subset K of $|z| < R(q)$. Consequently, in each such K, the number of zeros of $Q_{m_jn_j}(z)$ in K is bounded above by a number independent of j. These observations, and the fact that $\{G_n(z)\}_1^\infty$ has property U, imply that, at least in $|z| < R(q)$, the convergence in capacity above can be improved to convergence almost everywhere. Moreover, as in convergence of rows of the Padé table, $\{[m_j/n_j]\}_1^\infty$ even converges outside a set of Hausdorff logarithmic dimension one (see [20]).

The following result proves the Baker-Gammel-Wills conjecture [3, p. 188] for the class of functions h_q, $\theta/(2\pi)$ irrational. See [22] for a brief review of progress on the Baker-Gammel-Wills conjecture.

Theorem 3.4. *Let* $\theta/(2\pi)$ *be irrational. If* $q \notin \mathcal{G}$, *let* $\mathcal{J} = \{n_j\}_1^\infty$ *be any increasing sequence of positive integers with*

$$\lim_{n\in\mathcal{J}} q^n = 1. \tag{3.18}$$

If $q \in \mathcal{G}$, *let* $R(q) < \eta < 1$ *and let* $\mathcal{J} = \{n_j\}_1^\infty$ *be any increasing sequence of positive integers with*

$$\eta^{-n}|1-q^n| < \eta^{-j}|1-q^j|, \qquad 1 \le j < n; \qquad n \in \mathcal{J}. \tag{3.19}$$

Then for any sequence of positive integers $\{m_j\}_1^\infty$ *satisfying* (3.13),

$$\lim_{j\to\infty}[m_j/n_j](z) = h_q(z)$$

locally uniformly in $|z| < 1$. *Furthermore, for each fixed positive integer* $l \ge 2$ *and* *for each fixed*

$$r \in (0,1) \cap \{|z|: G_l(z) = 0\},$$

$[(m_j+l)/(n_j+l)](z)$ *has a pole* z_j, $j = 1, 2, \ldots$, *with*

$$\lim_{j\to\infty} |z_j| = r. \tag{3.20}$$

Our final result concerns the zero distribution of sequences of normalized Padé numerators and denominators.

Theorem 3.5. *Let* $\theta/(2\pi)$ *be irrational.*

(i) *Let* $\{m_j\}_1^\infty$ *and* $\{n_j\}_1^\infty$ *be sequences of positive integers satisfying* (3.13) *and* (3.17). *Then the normalized Padé denominators* $\{Q_{m_j n_j}(z)\}_1^\infty$ *have property W. Furthermore, if* $q \notin \mathcal{G}$, *the normalized Padé numerators* $\{P_{m_j n_j}(z)\}_1^\infty$ *have property U.*

(ii) *If* $q \in \mathcal{G}$ *and* $\mathcal{J}$ *is any increasing sequence of positive integers satisfying* (3.19), *then* $\{P_{nn}\}_{n\in\mathcal{J}}$ *does not have property U. In particular, the sequence of leading coefficients* $\{A_{nn}\}_{n\in\mathcal{J}}$ *satisfies*

$$\limsup_{n\in\mathcal{J}} |A_{nn}|^{1/n} < 1.$$

We remark that if $q \in \mathcal{G}$ but $R(q) > 0$, one can show that for some increasing sequence of integers $\hat{\mathcal{J}}$, $\{P_{nn}\}_{n\in\hat{\mathcal{J}}}$ has property U.

There are very few results on the distribution of zeros and poles of diagonal Padé approximants in the non-Stieltjes case. Perhaps the only function for which a complete analysis has been undertaken is the exponential function (Saff and Varga [35], [36], [37]). For a general class of analytic functions, but under implicit conditions on the poles, Gončar [15] proved that the distribution of the poles must be a certain equilibrium distribution. Stahl's theorem [40] is based on analysis of the distribution of poles of diagonal Padé sequences.

The zero distribution of rows of the Padé table has been studied by Edrei [9], [10] in the case when the function has finite radius of meromorphy, or is entire of finite order. The zeros of partial sums of entire functions of finite order have been analyzed by Edrei [11], [12] and Edrei, Saff and Varga [13]. For a large class of Stieltjes functions with complex weights, Al. Magnus [26] has recently investigated the distribution and location of the poles of diagonal Padé sequences.

4. Zero Distribution of the Rogers–Szegö Polynomials

In this section we shall prove Theorems 2.1, 2.3, 2.4, and 2.5. We begin with some basic properties (valid for all q) of the Rogers–Szegö polynomials defined in (1.5).

Lemma 4.1.

(i) *The following three-term recurrence relations are valid:*

(4.1) $$G_{n+1}(z)=(1+z)G_n(z)-(1-q^n)zG_{n-1}(z), \qquad n=1,2,3,\ldots,$$

(4.2) $$G_n(z)=G_{n-1}(z)+q^{n-1}zG_{n-1}(z/q), \qquad n=1,2,3,\ldots,$$

(4.3) $$G_n(qz)=G_n(z)-(1-q^n)zG_{n-1}(z), \qquad n=1,2,3,\ldots,$$

where

(4.4) $$G_0(z)\equiv 1; \qquad G_1(z)=1+z.$$

(ii) $G_n(z)$ *is monic and inversive, in the sense that*

(4.5) $$z^nG_n(1/z)=G_n(z), \qquad n=1,2,3,\ldots.$$

(iii) *For fixed* n, $G_n(z;q)$ *and its coefficients* (1.6) *are polynomials in the parameter* q.

Proof. The identities (4.1)-(4.5) follow by comparing coefficients of like powers of z on both sides of each identity. For example, (4.1) is equivalent to

$$\begin{bmatrix} n+1 \\ j \end{bmatrix}=\begin{bmatrix} n \\ j \end{bmatrix}+\begin{bmatrix} n \\ j-1 \end{bmatrix}-(1-q^n)\begin{bmatrix} n-1 \\ j-1 \end{bmatrix},$$

$j=1,2,\ldots,n-1$. This last relation follows by easy manipulations from (1.6). Part (iii) of the lemma follows by induction on n from the recurrence relation (4.1). ■

We next turn to the proof of Theorem 2.1 for the case where $\theta/(2\pi)$ is irrational. To this end we need a special case of the q-binomial theorem; namely, if

(4.6) $$H_n(z)=H_n(z;q):=\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}q^{j(j+1)/2}z^j,$$

then

(4.7) $$H_n(z)=\prod_{j=1}^{n}(1+q^jz).$$

See Szegö [41, p. 245].

Lemma 4.2. *Set*

(4.8) $$F(x):=\log|1+e^{2\pi ix}|, \qquad x\in\mathbf{R},$$

and

(4.9) $$\beta:=\theta/(2\pi).$$

Then for $n=1,2,3,\ldots$

(4.10) $$1\le\|G_n\|\le(n+1)^{1/2}\|H_n\|,$$

and

$$1 \le \|H_n\|^{1/n} \le \max_{s\in[0,1]} \exp\left\{ n^{-1} \sum_{j=1}^{n} F(\langle j\beta\rangle + s) \right\}, \tag{4.11}$$

where $\langle x\rangle$ denotes the fractional part of a real number x.

Proof. Since $G_n(0)=1$, the maximum modulus principle implies that $\|G_n\| \ge 1$, which yields the lower estimate in (4.10). To obtain the right-hand inequality in (4.10), we shall use the L_2 norm $\|\cdot\|_2$ of a polynomial on the unit circle. Since the absolute values of the coefficients of G_n and H_n are the same (compare (1.5) and (4.6)), we see that

$$\begin{aligned} \|G_n\|_2 &= \left(\sum_{j=0}^{n} \left| \begin{bmatrix} n \\ j \end{bmatrix} \right|^2 \right)^{1/2} = \|H_n\|_2 \\ &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} |H_n(e^{i\phi})|^2 \, d\phi \right\}^{1/2} \\ &\le \|H_n\|. \end{aligned} \tag{4.12}$$

Next, by the Cauchy-Schwarz inequality,

$$\|G_n\| \le \sum_{j=0}^{n} \left| \begin{bmatrix} n \\ j \end{bmatrix} \right| \le (n+1)^{1/2} \left\{ \sum_{j=0}^{n} \left| \begin{bmatrix} n \\ j \end{bmatrix} \right|^2 \right\}^{1/2} \le (n+1)^{1/2} \|H_n\|,$$

by (4.12). This yields the upper bound in (4.10). The lower bound in (4.11) follows as before. To obtain the upper bound, we use (4.7):

$$\begin{aligned} \|H_n\|^{1/n} &= \max_{s\in[0,1]} \exp\left(n^{-1} \sum_{j=1}^{n} \log|1 + q^j e^{2\pi i s}| \right) \\ &= \max_{s\in[0,1]} \exp\left(n^{-1} \sum_{j=1}^{n} \log|1 + e^{2\pi i (j\beta + s)}| \right). \end{aligned}$$

Using the periodicity of $e^{2\pi i x}$ and using (4.8), we obtain the upper bound in (4.11). ∎

To estimate the sum in (4.11), we use the theory of uniform distribution, and estimates involving discrepancy, as applied by Niederreiter [28] in his analysis of certain rules of numerical integration.

Definition 4.3. Let $\mathcal{S} = \{x_j\}_1^\infty$ be a sequence of real numbers in $[0, 1]$. For $n = 1, 2, 3, \ldots$, the nth *discrepancy* $D_n(\mathcal{S})$ is defined by

$$D_n(\mathcal{S}) := \sup_{t\in[0,1]} |Z_n[0, t]/n - t|,$$

where, for $t \in [0, 1]$, $Z_n[0, t]$ denotes the number of elements of $\{x_j\}_1^n$ lying in $[0, t]$.

Lemma 4.4. *Let $g(x)$ be continuous on $[0, 1]$ with ordinary modulus of continuity $\omega(g; \cdot)$. With the notation of Definition 4.3,*

$$\left| n^{-1} \sum_{j=1}^{n} g(x_j) - \int_0^1 g(t)\, dt \right| \le \omega(g; D_n(\mathcal{S})), \qquad n = 1, 2, 3, \ldots.$$

Proof. See Niederreiter [28, p. 140]. ∎

Proof of Theorem 2.1 when $\theta/(2\pi)$ is irrational. Let $\beta=\theta/(2\pi)$ and $\mathcal{S}=\{\langle j\beta\rangle\}_{j=1}^{\infty}$, and

$$S_n := \max_{s\in[0,1]} \exp\left\{n^{-1}\sum_{j=1}^{n} F(\langle j\beta\rangle+s)\right\},$$

$n=1,2,3,\ldots$. Further, for $M>0$, let

$$F_M(x):=\max\{-M, F(x)\}, \qquad x\in\mathbf{R}.$$

Then F_M is continuous in $\mathbf{R}$ and periodic with period 1. Let $\omega(F_M;\cdot)$ denote the modulus of continuity of F_M on $\mathbf{R}$. Applying Lemma 4.4 to $F_M(t+s)$, $t\in[0,1]$, we see that

$$\begin{aligned}(4.13)\qquad S_n &\le \max_{s\in[0,1]} \exp\left\{n^{-1}\sum_{j=1}^{n} F_M(\langle j\beta\rangle+s)\right\}\\ &\le \max_{s\in[0,1]} \exp\left\{\int_0^1 F_M(t+s)\,dt+\omega(F_M\,;D_n(\mathcal{S}))\right\}\\ &= \exp\left(\int_0^1 F_M(t)\,dt+\omega(F_M\,;D_n(\mathcal{S}))\right\},\end{aligned}$$

by the periodicity of F_M. Since β is irrational, it is a classical result that $\mathcal{S}$ is uniformly distributed modulo 1 (see, for example, [18, p. 8]) and hence [18, p. 89],

$$\lim_{n\to\infty} D_n(\mathcal{S})=0.$$

Thus, letting $n\to\infty$ in (4.13),

$$\limsup_{n\to\infty} S_n \le \exp\left(\int_0^1 F_M(t)\,dt\right),$$

for each $M>0$. By the monotone convergence theorem, we may let $M\to\infty$ to deduce that

$$\begin{aligned}\limsup_{n\to\infty} S_n &\le \exp\left(\int_0^1 F(t)\,dt\right)\\ &= \exp\left((2\pi)^{-1}\int_0^{2\pi} \log|1+e^{i\phi}|\,d\phi\right).\end{aligned}$$

Since $\log|1+z|$ is harmonic in $|z|<1$,

$$(2\pi)^{-1}\int_0^{2\pi} \log|1+r\,e^{i\phi}|\,d\phi=\log 1=0, \qquad r\in[0,1).$$

We may use dominated convergence to let $r\to 1^-$ in this last integral, so that

$$\limsup_{n\to\infty} S_n \le 1.$$

Together with (4.11), this shows that

$$(4.14)\qquad \lim_{n\to\infty} \|H_n\|^{1/n}=1.$$

Then (4.10) yields the desired limit

$$\lim_{n\to\infty} \|G_n\|^{1/n} = 1. \qquad \blacksquare$$

We turn to the proof of Theorem 2.1 for rational $\theta/(2\pi)$, and first prove:

Lemma 4.5. *Let $\theta/(2\pi) = \mu/\nu$, where μ, ν are positive integers whose greatest common divisor is 1. Then, if*

$$n = k\nu + l,$$

where k, l are nonnegative integers with $0 \le l < \nu$,

$$G_n(z) = (1+z^\nu)^k G_l(z). \tag{4.15}$$

Proof. We prove (4.15) by a double induction on k and l (the outer induction on k, the inner on l). First, for $k=0$, (4.15) is trivial for all $0 \le l < \nu$. Assume now, as an induction hypothesis on k, that we have proved (4.15) for a given k and $0 \le l < \nu$. We shall prove it for $k+1$ and $0 \le l < \nu$. When $l=0$, (4.2) yields

$$\begin{aligned} G_{(k+1)\nu}(z) &= G_{(k+1)\nu-1}(z) + q^{(k+1)\nu-1} z G_{(k+1)\nu-1}(z/q) \\ &= (1+z^\nu)^k [G_{\nu-1}(z) + q^{\nu-1} z G_{\nu-1}(z/q)], \end{aligned}$$

by our induction hypothesis and as $q^\nu = 1$. Applying (4.2), we obtain

$$G_{(k+1)\nu}(z) = (1+z^\nu)^k G_\nu(z).$$

Next, since μ/ν is in its lowest form, $q^j \ne 1$ for $1 \le j < \nu$. Hence, for $1 \le j < \nu$,

$$\begin{bmatrix} \nu \\ j \end{bmatrix} = \frac{(1-q^\nu)(1-q^{\nu-1}) \cdots (1-q^{\nu+1-j})}{(1-q)(1-q^2) \cdots (1-q^j)} = 0.$$

Thus, $G_\nu(z) = 1 + z^\nu$, and

$$G_{(k+1)\nu}(z) = (1+z^\nu)^{k+1} = (1+z^\nu)^{k+1} G_0(z),$$

which is (4.15) with k replaced by $k+1$ and $l=0$.

Assume now, as an induction hypothesis on l, that we have proved

$$G_{(k+1)\nu+j}(z) = (1+z^\nu)^{k+1} G_j(z), \tag{4.16}$$

for $j = 0, 1, 2, \ldots, l$. We prove (4.16) for $j = l+1$. By (4.2),

$$\begin{aligned} G_{(k+1)\nu+l+1}(z) &= G_{(k+1)\nu+l}(z) + q^{(k+1)\nu+l} z G_{(k+1)\nu+l}(z/q) \\ &= (1+z^\nu)^{k+1} [G_l(z) + q^l z G_l(z/q)] \\ &= (1+z^\nu)^{k+1} G_{l+1}(z), \end{aligned}$$

by (4.16) and (4.2). This proves (4.16) for $j = l+1$. Thus, (4.15) is true with k replaced by $k+1$ and $0 \le l < \nu$, so our induction step on k is complete. $\blacksquare$

Proof of Theorem 2.1 when $\theta/(2\pi)$ is rational. From (4.15), for $n = k\nu + l$,

$$\|G_n\| \le 2^k \|G_l\| \le 2^{n/\nu} \max_{0 \le j \le \nu} \|G_j\|. \tag{4.17}$$

Also,

$$\|G_n\| \geq 2^k \max\{|G_l(z)|: z^\nu = 1\} =: 2^k C_l,$$

where $C_l > 0$ since $G_l(z)$ has degree less than ν. Thus

$$\|G_n\| \geq 2^{(n-\nu)/\nu} \min_{0 \leq j < \nu} C_j. \tag{4.18}$$

The desired limit of (2.3) follows from (4.17) and (4.18). ∎

We shall use results of Blatt and Saff [7] in the

Proof of Theorem 2.3. Let M_n denote the number of zeros of $p_n(z)$ in $|z| \geq 1+\varepsilon$. By Theorem 3.1 in [7],

$$\lim_{n \in \mathscr{J}} m_n/n = 0.$$

Also the number N_n of zeros of $p_n(z)$ outside $\mathscr{A}_\varepsilon$ satisfies $N_n = 2m_n$, in view of (2.8). Hence (2.9) holds. Let us write

$$p_n(z) = A_n \tilde{p}_n(z) \hat{q}_n(z), \qquad n \in \mathscr{J},$$

where $\tilde{p}_n(z)$ and $\hat{q}_n(z)$ are monic, and $\hat{q}_n(z)$ is of degree N_n, and its zeros are those of $p_n(z)$ outside $\mathscr{A}_\varepsilon$. Then as all zeros of $\hat{q}_n(z)$ are at least a distance of ε from the unit circle,

$$\|\tilde{p}_n\| = |A_n|^{-1} \|p_n/\hat{q}_n\| \leq |A_n|^{-1} \|p_n\| \varepsilon^{-N_n}.$$

Since $\tilde{p}_n(z)$ is monic, and since $N_n = o(n)$, we see that

$$\lim_{n \in \mathscr{J}} \|\tilde{p}_n\|^{1/n} = \lim_{n \in \mathscr{J}} \|p_n\|^{1/n} = 1.$$

It is then an easy consequence of the Bernstein–Walsh lemma (Walsh [44]) that, for each $r > 1$,

$$\lim_{n \in \mathscr{J}} \|\tilde{p}_n(z)/z^{n-N_n}\|_{L_\infty(|z|=r)}^{1/(n-N_n)} = 1.$$

Further, $\tilde{p}_n(z)/z^{n-N_n}$ is analytic and nonzero for $|z| > 1+\varepsilon$, and takes the value 1 at ∞. Hence, with a suitable choice of branches, $\{\tilde{p}_n(z)^{1/(n-N_n)}/z\}_{n \in \mathscr{J}}$ is a family of functions analytic and locally single-valued in $|z| > 1+\varepsilon$, with value 1 at ∞, and satisfying

$$\lim_{n \in \mathscr{J}} \|\tilde{p}_n(z)^{1/(n-N_n)}/z\|_{L_\infty(|z|=r)} = 1, \qquad |z| > 1+\varepsilon.$$

Using normality and the maximum modulus theorem, we obtain (2.11). Next, since w_n/z is a zero of $\tilde{p}_n(z)$ whenever z is, we see that

$$z^{n-N_n} \tilde{p}_n(w_n/z) = \hat{v}_n \tilde{p}_n(z), \qquad n \in \mathscr{J},$$

where $\hat{v}_n$ has unit modulus. Then locally uniformly in $|z| < 1-\varepsilon$,

$$\{\hat{v}_n w_n^{-n+N_n} \tilde{p}_n(z)\}^{1/(n-N_n)} = \{(z/w_n)^{n-N_n} \tilde{p}_n(w_n/z)\}^{1/(n-N_n)} \to 1,$$

as $n \to \infty$, $n \in \mathcal{J}$, by (2.11). Thus (2.12) is valid. Finally, (2.13) is an immediate consequence of Theorem 3.3 in [7]. ■

In the proof of Theorem 2.4 we shall need the following surprising lemma, which follows from results of Hardy and Littlewood [17]:

Lemma 4.6. *Let $\theta/(2\pi)$ be irrational. Then*

$$\liminf_{n\to\infty}\left\{\prod_{j=1}^{n}|1-q^j|\right\}^{1/n}=\liminf_{n\to\infty}|1-q^n|^{1/n}=:R(q). \tag{4.19}$$

Proof. Let us define formal power series

$$\phi(z;q):=\sum_{n=1}^{\infty} z^n/(n(1-q^n))$$

and

$$\Phi(z;q):=\sum_{n=0}^{\infty} z^n\Big/\prod_{j=1}^{n}(1-q^j).$$

With the aid of the remarkable identity

$$\Phi(z;q)=\exp(\phi(z;q)),$$

Hardy and Littlewood [17, p. 86] showed that even for $|q|=1$, $\theta/(2\pi)$ irrational, $\Phi(z;q)$ and $\phi(z;q)$ have the same radius of convergence. The Cauchy-Hadamard formula for the radius of convergence of a power series, as applied to $\Phi(z;q)$ and $\phi(z;q)$, then yields (4.19). ■

In the proof of Theorem 2.4 we shall find it convenient to consider

$$A_j:=\sup_{n\ge j}\left|\begin{bmatrix} n \\ j \end{bmatrix}\right|, \qquad j=0,1,2,\ldots, \tag{4.20}$$

and the *majorant series* $A(z)$ associated with $\{G_n(z)\}_1^{\infty}$,

$$A(z):=\sum_{j=0}^{\infty} A_j z^j. \tag{4.21}$$

The relationship between general sequences of polynomials and their associated majorant series was first investigated by Rosenbloom [34], and the following proof makes use of his ideas.

Proof of Theorem 2.4(i). We first show that

$$A_j=\|H_j(z;q^{-1})\|\Big/\prod_{k=1}^{j}|1-q^k|, \qquad j=0,1,2,\ldots, \tag{4.22}$$

where H_j is defined in (4.6). Setting $w_n := -q^{n+1}$, we see from (1.6) that

$$\begin{bmatrix} n \\ j \end{bmatrix} = \prod_{k=1}^{j} \{(1+q^{-k}w_n)/(1-q^k)\}$$

$$= H_j(w_n; q^{-1}) \Big/ \prod_{k=1}^{j} (1-q^k),$$

by (4.6) and (4.7). Since $\{w_n\}_1^\infty = \{-q^{n+1}\}_1^\infty$ is dense on the unit circle, and j is fixed, we obtain (4.22) for each $j = 0, 1, 2, \ldots$. It follows from (4.14) and Lemma 4.6 that

$$\limsup_{j\to\infty} A_j^{1/j} = R(q)^{-1}. \tag{4.23}$$

Thus $A(z)$ has radius of convergence $R(q)$. Further, from (4.20) it follows that, if $R(q) > 0$,

$$|G_n(z)| \le A(|z|), \qquad n = 0, 1, 2, \ldots, \quad |z| < R(q),$$

and so $\{G_n(z)\}_1^\infty$ is uniformly bounded in each compact subset of $|z| < R(q)$. Next, let $r > 0$. It follows from Cauchy's estimates that

$$\left| \begin{bmatrix} n \\ j \end{bmatrix} \right| \le \|G_n\|_{L_\infty(|z|=r)} r^{-j}, \qquad n, j = 0, 1, 2, \ldots. \tag{4.24}$$

If r is such that

$$C := \sup_{n\ge 0} \|G_n\|_{L_\infty(|z|=r)} < \infty,$$

then, taking the supremum over n in (4.24), we obtain

$$A_j \le Cr^{-j}, \qquad j = 0, 1, 2, \ldots.$$

Taking jth roots and using (4.23), we deduce that

$$R(q)^{-1} \le r^{-1},$$

or $r \le R(q)$. Thus $\{G_n(z)\}_1^\infty$ cannot be uniformly bounded in $\{z\colon R(q) < |z| < R(q) + \varepsilon\}$, for any $\varepsilon > 0$. ∎

Proof of Theorem 2.4(ii). Suppose $\mathscr{S}$ is an increasing sequence of positive integers satisfying (2.17). Then for each fixed nonnegative integer j,

$$\lim_{n\in\mathscr{S}} \begin{bmatrix} n \\ j \end{bmatrix} = \frac{(1-w)(1-wq^{-1})\ldots(1-wq^{-j+1})}{(1-q)(1-q^2)\ldots(1-q^j)} =: I_{wj}.$$

In particular, this last limit relation and (4.20) imply that

$$|I_{wj}| \le A_j, \qquad j = 0, 1, 2, \ldots,$$

so that

$$I_w(z) = \sum_{j=0}^{\infty} I_{wj} z^j$$

has radius of convergence $\geq R(q)$. Let $0<r<R(q)$ and let l be a nonnegative integer. Then, for $|z|\leq r$,

$$|G_n(z)-I_w(z)|\leq \sum_{j=0}^{l}\left|\begin{bmatrix} n\\ j\end{bmatrix}-I_{wj}\right| r^j+2\sum_{j=l+1}^{\infty}A_jr^j\to 2\sum_{j=l+1}^{\infty}A_jr^j$$

as $n\to\infty$, $n\in\mathcal{J}$. Since l is arbitrary and $A(z)$ has radius of convergence $R(q)$, we may let $l\to\infty$ to deduce that

$$\lim_{n\in\mathcal{J}} G_n(z)=I_w(z),$$

uniformly in $|z|\leq r$. ■

Proof of Theorem 2.5. We first prove (2.23). If $q\notin\mathcal{G}$, then $R(q)=1$ and (2.23) follows from Theorem 2.4(ii) and (2.21). Suppose now $q\in\mathcal{G}$, and let $R(q)<\eta<1$ and let $\mathcal{J}$ be an increasing sequence of positive integers satisfying (2.22). Since

$$\begin{bmatrix} n\\ j\end{bmatrix}=\begin{bmatrix} n\\ n-j\end{bmatrix},\qquad j=0,1,2,\ldots,n,$$

we have, using (1.6),

$$\begin{aligned}\left|\sum_{j=1}^{n-1}\begin{bmatrix} n\\ j\end{bmatrix}z^j\right| &\leq \sum_{1\leq j\leq n/2}\left|\begin{bmatrix} n\\ j\end{bmatrix}\right|(|z|^j+|z|^{n-j})\\ &\leq 2\sum_{1\leq j\leq n/2}\left|\frac{1-q^n}{1-q^j}\right|\left|\begin{bmatrix} n-1\\ j-1\end{bmatrix}\right|(\max\{1,|z|\})^{n-j}\\ &\leq 2\sum_{1\leq j\leq n/2}\|G_{j-1}\|(\eta\max\{1,|z|\})^{n-j},\end{aligned}$$

by Cauchy's estimates and (2.22). Note that $n-j\geq n/2$ in this last sum. It is then an easy consequence of Theorem 2.1 that this last sum tends to 0 as $n\to\infty$, $n\in\mathcal{J}$, uniformly for $|z|\leq r<\eta^{-1}$. Hence

$$\lim_{n\in\mathcal{J}}\{G_n(z)-(1+z^n)\}=0,$$

uniformly in $|z|\leq 1<\eta^{-1}$. Thus (2.23) holds regardless of whether $q\in\mathcal{G}$ or $q\notin\mathcal{G}$. Note also that if $g\in\mathcal{G}$, then (2.22) implies (2.21). Next, (2.21), (2.23), and (4.1) show that locally uniformly in $|z|<1$,

$$\lim_{n\in\mathcal{J}} G_{n+1}(z)=1+z=G_1(z).$$

Thus (2.24) holds for $l=1$. Using the recurrence relation (4.1) and a simple induction argument, we then obtain (2.24) for $l=1,2,3,\ldots$. Indeed, (2.24) holds for $l=0$ and $l=1$. Assume now it is true for $1,2,3,\ldots,l$. Then, by (4.1), we have locally uniformly in $|z|<1$,

$$\begin{aligned}\lim_{n\in\mathcal{J}} G_{n+l+1}(z)&=(1+z)G_l(z)-(1-q^l)zG_{l-1}(z)\\ &=G_{l+1}(z).\end{aligned}$$

■

Finally, we prove that there is no ratio asymptotic of the form $|G_{n+1}(z)/G_n(z)|\to|z|$:

Lemma 4.7. *Suppose* $|z|>1$ *and* $|1+z^{-1}|\neq 1$. *Then* $\lim_{n\to\infty}|G_{n+1}(z)/G_n(z)|$ *does not exist.*

Proof. Let $\mathcal{J}$ be as in Theorem 2.5, and consider $\{G_n\}_{n\in\mathcal{J}}$. This sequence of polynomials has no limit points of zeros in $|z|<1$, and hence none in $|z|>1$. Then Theorem 2.3 shows that for $|z|>1$,

$$\lim_{n\in\mathcal{J}}|G_n(z)|^{1/n}=|z|.$$

Hence, if $|z|>1$ and

$$r=\lim_{n\to\infty}|G_{n+1}(z)/G_n(z)|$$

exists, then necessarily $r=|z|$. Dividing (4.1) by $G_n(z)$ and letting $n\to\infty$ through $\mathcal{J}$, we obtain

$$\lim_{n\to\infty}|G_{n+1}(z)/G_n(z)|=\lim_{n\in\mathcal{J}}|1+z-(1-q^n)z/\{G_n(z)/G_{n-1}(z)\}|$$

or $|z|=|1+z|$, so that $|1+z^{-1}|=1$. ∎

5. Location of the Zeros of the Rogers–Szegö Polynomials

In this section we prove Theorems 2.6 and 2.7. In the proof of Theorem 2.6 we use continued fraction methods; more specifically, the method of the "fundamental inequalities" as applied by Wall [43, pp. 40–41]. Rather than derive the continued fraction for $h_q(z)$, we proceed directly from (4.2) and (4.3), which are essentially the recurrence relations for the denominators in that continued fraction. Let

(5.1) $$B_{2n}^*(z):=G_n(-zq^n),\qquad n=0,1,2,\ldots,$$

(5.2) $$B_{2n+1}^*(z):=G_n(-zq^{n+1}),\qquad n=0,1,2,\ldots,$$

(5.3) $$a_{2n}^*:=-q^{2n-1},\qquad n=0,1,2,3,\ldots,$$

and

(5.4) $$a_{2n+1}^*:=(1-q^n)q^n,\qquad n=1,2,3,\ldots.$$

Lemma 5.1. *The following identities are valid:*

(5.5) $$B_{k+1}^*(z)=B_k^*(z)+a_{k+1}^*zB_{k-1}^*(z),\qquad k=1,2,3,\ldots,$$

and

(5.6) $$B_{k+2}^*(z)=\{1+(a_{k+1}^*+a_{k+2}^*)z\}B_k^*(z)-a_k^*a_{k+1}^*z^2B_{k-2}^*(z),\qquad k=2,3,4,\ldots.$$

Proof. For $k=2n-1$, (5.5) follows from (4.2) by replacing z in (4.2) by $-zq^n$. For $k=2n$, (5.5) follows from (4.3) by replacing z in (4.3) by $-zq^n$.

Next, applying (5.5) twice,

(5.7) $$\begin{aligned}B_{k+2}^*(z)&=B_{k+1}^*(z)+a_{k+2}^*zB_k^*(z)\\&=B_k^*(z)+a_{k+1}^*zB_{k-1}^*(z)+a_{k+2}^*zB_k^*(z).\end{aligned}$$

Furthermore, from (5.5),

$$B_k^*(z)=B_{k-1}^*(z)+a_k^*zB_{k-2}^*(z).$$

Solving for $B^*_{k-1}(z)$ and substituting into (5.7), we obtain

$$B^*_{k+2}(z) = B^*_k(z) + a^*_{k+1} z(B^*_k(z) - a^*_k z B^*_{k-2}(z)) + a^*_{k+2} z B^*_k(z),$$

and (5.6) follows. ■

It is noteworthy that (5.6) is essentially the recurrence relation for the denominator polynomials in the even and odd parts of the continued fraction (1.4) for $h_q(z)$.

Lemma 5.2. *Let $s>0$. Assume that $0<|z|<1$ and for all $|w|=1$,*

$$|z^{-1} + w[1 - w(1+q)]| - |1-w|s - s^{-1} \geq 0. \tag{5.8}$$

Then, for $k = 2, 4, 6, \ldots,$

$$B^*_k(z) \neq 0. \tag{5.9}$$

Proof. We shall show by induction that for positive even integers k, (5.9) holds and

$$\rho_k(z) := z B^*_{k-2}(z) / B^*_k(z) \tag{5.10}$$

satisfies

$$|\rho_k(z)| \leq s. \tag{5.11}$$

First, for $k=2$, $B^*_2(z) = G_1(-zq) = 1 - zq \neq 0$, since $|z|<1$. Further,

$$|\rho_2(z)| = |zG_0(-z)/G_1(-zq)| = |z^{-1} - q|^{-1} \leq s,$$

by (5.8) with $w=1$. Thus (5.9) and (5.11) hold for $k=2$.

Now, let us assume (5.9) and (5.11) for a given positive even integer k. We prove (5.9) and (5.11) for $k+2$. We may write $k = 2l$ for some positive integer l. By (5.3) and (5.4),

$$a^*_{k+1} + a^*_{k+2} = (1-q^l)q^l - q^{2l+1}$$

while

$$a^*_k a^*_{k+1} = -q^{3l-1}(1-q^l).$$

Letting $w = q^l$, and dividing by $zB^*_k(z)$ in (5.6), we obtain

$$B^*_{k+2}(z)/(zB^*_k(z)) = (z^{-1} + (1-w)w - qw^2) + w^3(1-w)q^{-1}\rho_k(z),$$

by (5.10). Then by our induction hypothesis (5.11),

$$|B^*_{k+2}(z)/(zB^*_k(z))| \geq |z^{-1} + w[1-w(1+q)]| - |1-w|s \geq s^{-1},$$

by (5.8). Thus $B^*_{k+2}(z) \neq 0$, and also

$$|\rho_{k+2}(z)|^{-1} \geq s^{-1},$$

which yields (5.11) for $k+2$. ■

Proof of Theorem 2.6. First, taking account of (5.1), we see that assertion (i) of Theorem 2.6 is a restatement of Lemma 5.2. To prove assertion (ii), we note that (2.26) is satisfied if, for all $|w|=1$,

$$|z|^{-1} \geq |1 - w(1+q)| + |1-w|s + s^{-1},$$

and this in turn is satisfied if

$$|z|^{-1} \geq 1 + |1+q| + 2s + s^{-1}.$$

The value of s that minimizes $2s + s^{-1}$ is $s = 1/\sqrt{2}$. Thus $G_n(z) \neq 0$, $n = 0, 1, 2, \ldots$, if

$$0 < |z| \leq (2\sqrt{2} + 1 + |1+q|)^{-1} = r_0^{-1}.$$

As $G_n(0) = 1$ and $G_n(z)$ is inversive (cf. (4.5)), all zeros of $G_n(z)$ lie in the annulus $\{z: r_0^{-1} < |z| < r_0\}$, $n = 0, 1, 2, \ldots$. ■

Proof of Theorem 2.7. First, with the notation (2.28), Theorem 2.6(ii) implies that

$$\Delta_{nq} \geq (2\sqrt{2} + 1 + |1+q|)^{-1} \geq (3 + 2\sqrt{2})^{-1} = 3 - 2\sqrt{2}.$$

To prove $\Delta_{nq} < 1$, for $n \geq 2$, let us assume to the contrary that $\Delta_{nq} = 1$, so that $G_n(z)$ has no zeros in $|z| < 1$. In view of (4.5), this implies that all zeros of $G_n(z)$ lie on the unit circle. Now, for $|z| = 1$, (4.5) may be rewritten in the form

$$z^n G_n(\bar{z}) = G_n(z),$$

so all nonreal zeros of $G_n(z)$ occur in conjugate pairs. As $G_n(z)$ is monic, it follows that $G_n(z)$ is a polynomial with real coefficients. Then

$$\begin{aligned} 0 = \operatorname{Im}\begin{bmatrix} n \\ 1 \end{bmatrix} &= \operatorname{Im}\left\{\frac{1-q^n}{1-q}\right\} \\ &= \sin(n\theta/2)\sin((n-1)\theta/2)/\sin(\theta/2) \\ &\neq 0, \end{aligned}$$

as $n \geq 2$ and $\theta/(2\pi)$ is irrational. Thus $\Delta_{nq} < 1$ and (2.31) is valid. Next, it follows from Theorem 2.5 and Hurwitz' theorem, that for each positive integer l, each zero of $G_l(z)$ in $|z| < 1$ is a limit point of zeros of $\{G_n(z)\}_1^\infty$. Further, for any sequence of positive integers $\mathcal{J}$ satisfying the hypotheses of Theorem 2.5, $\{G_n(z)\}_{n \in \mathcal{J}}$ has no limit points of zeros in $|z| < 1$. Thus (2.32) follows. ■

6. Convergence of Padé Approximants

We begin this section with a simple proof that $h_q(z)$ has the unit circle as its natural boundary, when $\theta/(2\pi)$ is irrational:

Theorem 6.1. *Let $\theta/(2\pi)$ be irrational. Then $h_q(z)$ has the unit circle as its natural boundary.*

Proof. Firstly, for $0 < r < 1$,

$$\begin{aligned} \|h_q\|_{L_\infty(|z|=r)} &\geq \|h_q\|_{L_2(|z|=r)} \\ &= \left\{\sum_{j=0}^{\infty} r^{2j}\right\}^{1/2} \\ &= (1-r^2)^{-1/2} \to \infty, \end{aligned}$$

as $r \to 1^-$. Thus $h_q(z)$ cannot be bounded on the open disk $|z| < 1$, and there is at least one point z_0 on the unit circle such that $h_q(z)$ is unbounded in a neighborhood

$$\mathcal{N}_\varepsilon(z_0) := \{z : |z| < 1 \text{ and } |z - z_0| < \varepsilon\}, \qquad \varepsilon > 0,$$

of z_0 inside the unit circle.

Next it follows from (1.1) (compare (1.3)) that

$$h_q(qz) = (h_q(z) - 1)/z, \qquad |z| < 1.$$

Hence $h_q(z)$ is unbounded in the neighborhood $\mathcal{N}_\varepsilon(qz_0)$ of qz_0 and so also in $\mathcal{N}_\varepsilon(q^j z_0)$, $j = 0, 1, 2, 3, \ldots$. Since $\{q^j z_0\}_{j=1}^{\infty}$ is dense on the unit circle, the desired result follows. ■

We delay the proof of Theorem 3.1 until the end of this section and proceed with the

Proof of Theorem 3.2(i). Let

$$a_m := q^{m(m-1)/2}, \qquad m = 0, 1, 2, \ldots. \tag{6.1}$$

We show by induction on n that

$$D(m/n) = (a_m)^n \prod_{j=1}^{n-1} (1 - q^j)^{n-j}, \qquad m \geq n - 1 \geq -1. \tag{6.2}$$

Firstly, $D(m/0) = 1$ and $D(m/1) = a_m$, $m \geq 0$, so (6.2) is true for $n = 0, 1$. Assume now that we have established (6.2) for $m \geq n - 1 \geq 0$ and $n = 0, 1, 2, \ldots, l$. We prove it for $n = l + 1$, with the aid of the familiar "star" identity [3], [4]

$$D(m/n+1)D(m/n-1) = D(m/n)^2 - D(m-1/n)D(m+1/n), \qquad m, n = 0, 1, 2, \ldots. \tag{6.3}$$

Then from (6.2) for $n = l - 1, l$, and $m \geq l$,

$$\begin{aligned} D(m/l+1) &= \frac{D(m/l)^2}{D(m/l-1)}\left\{1 - \frac{D(m-1/l)D(m+1/l)}{D(m/l)^2}\right\} \\ &= a_m^{l+1} \prod_{j=1}^{l-1} (1 - q^j)^{2(l-j)-(l-1-j)}\{1 - (a_{m-1}a_{m+1}/a_m^2)^l\} \\ &= a_m^{l+1} \prod_{j=1}^{l-1} (1 - q^j)^{l+1-j}\{1 - q^l\} \\ &= a_m^{l+1} \prod_{j=1}^{l} (1 - q^j)^{l+1-j} \end{aligned}$$

so that (6.2) is true for $n = l + 1$. ■

Proof of Theorem 3.2(ii). In the proof of (3.12) we need one of the Frobenius triangle identities,

$$Q_{mn}(z) = Q_{m,n-1}(z) - zQ_{m-1,n-1}(z)\frac{D(m-1/n-1)D(m+1/n)}{D(m/n-1)D(m/n)}, \tag{6.4}$$

which is valid whenever $D(m-1/n-1)$, $D(m/n-1)$, and $D(m/n)$ are nonzero ([3, p. 31]). Using (6.2), and letting

$$z=-ua_m/a_{m+1}=-uq^{-m}$$

in (6.4), we obtain, for $m\geq n-1\geq 0$,

$$Q_{mn}(-uq^{-m})=Q_{m,n-1}(-uq^{-m})+uq^{n-1}Q_{m-1,n-1}(-uq^{-m+1}/q). \tag{6.5}$$

We use this to prove by induction on n, that

$$Q_{mn}(-uq^{-m})=G_n(u), \qquad m\geq n-1\geq 0. \tag{6.6}$$

Firstly, for $n=0$ and $m\geq 0$,

$$Q_{m0}(-uq^{-m})=1=G_0(u), \qquad m\geq 0,$$

while, for $n=1$ and $m>0$, (3.4) shows that

$$Q_{m1}(-uq^{-m})=1+uq^{-m}a_{m+1}/a_m=1+u=G_1(u).$$

Thus (6.6) is true for $n=0, 1$ and $m\geq n-1$. Now let us assume that (6.6) is true with n replaced by $n-1$ and $m\geq(n-1)-1$. Then (6.5) yields, if $m\geq n-1$,

$$Q_{mn}(-uq^{-m})=G_{n-1}(u)+uq^{n-1}G_{n-1}(u/q)=G_n(u),$$

by (4.2). Thus by induction on n, (6.6) is true for all $n\geq 0$. ■

In the proof of Theorem 3.3 we shall use:

Lemma 6.2. *Let $\theta/(2\pi)$ be irrational. Let $0<\delta<1$. Then there exists a positive constant C independent of m, n, and z, such that for $m\geq n-1\geq 0$,*

$$\|Q_{mn}\|/|Q_{mn}(z)|\leq C(1+\delta)^{m+n}, \tag{6.7}$$

for $|z|\leq 1-\delta$, $z\notin E_{mn}$, where

$$\operatorname{cap}(E_{mn})\to 0 \quad \text{as} \quad m+n\to\infty. \tag{6.8}$$

Proof. Let $0<\eta<\delta$. By Theorem 2.1, there exists $C_1>0$ such that

$$\|G_n\|\leq C_1(1+\eta)^n, \qquad n=0, 1, 2, \ldots.$$

Then, by Theorem 3.2, for $m\geq n-1\geq 0$,

$$\|Q_{mn}\|/|Q_{mn}(z)|\leq C_1(1+\eta)^n/|G_n(-zq^m)|. \tag{6.9}$$

Let us write

$$G_n(z)=\tilde{G}_n(z)\tilde{Q}_n(z), \tag{6.10}$$

where $\tilde{G}_n(z)$ and $\tilde{Q}_n(z)$ are monic, and the zeros of $\tilde{Q}_n(z)$ are those of $G_n(z)$ outside the annulus $\{z\colon 1-\eta<|z|<1+\eta\}$. By Theorem 2.3, the degree N_n of $\tilde{Q}_n(z)$ satisfies

$$\lim_{n\to\infty} N_n/n=0.$$

Further, since $\delta > \eta$, Theorem 2.3 shows that

$$\lim_{n\to\infty} |\tilde{G}_n(z)|^{1/n} = 1 \qquad \text{uniformly for} \quad |z| \le 1-\delta.$$

Hence, for some C_1 and n_0 independent of η,

$$(6.11) \qquad |\tilde{G}_n(z)| \ge C_2(1-\eta)^n, \qquad |z| \le 1-\delta, \quad n \ge n_0.$$

We shall define E_{mn} differently for $n < n_0$ and $n \ge n_0$: for $m \ge n-1 \ge 0$, let

$$(6.12) \qquad E_{mn} := \{z : |G_n(-zq^m)| \le (1-\eta)^{m+n}\}, \qquad n < n_0,$$

and

$$(6.13) \qquad E_{mn} := \{z : |\tilde{Q}_n(-zq^m)| \le (1-\eta)^{m+n}\}, \qquad n \ge n_0.$$

Since $G_n(z)$ and $\tilde{Q}_n(z)$ are monic, we see that

$$\operatorname{cap}(E_{mn}) \le (1-\eta)^{(m+n)/n} \le (1-\eta)^{(m+n)/n_0}, \qquad n < n_0,$$

and

$$\operatorname{cap}(E_{mn}) \le (1-\eta)^{(m+n)/N_n}, \qquad n \ge n_0.$$

Then (6.8) follows. Finally, suppose $m \ge n-1 \ge 0$, $|z| \le 1-\delta$, and $z \notin E_{mn}$. If $n > n_0$, (6.9) and (6.12) show that

$$\|Q_{mn}\|/|Q_{mn}(z)| \le C_1(1+\eta)^n(1-\eta)^{-(m+n)},$$

while if $n \ge n_0$, (6.9), (6.10), (6.11), and (6.13) show that

$$\|Q_{mn}\|/|Q_{mn}(z)| \le C_1(1+\eta)^n C_2^{-1}(1-\eta)^{-n}(1-\eta)^{-(m+n)}.$$

Then (6.7) follows if η is small enough. ■

Proof of Theorem 3.3(i)(a). Let $0 < r < 1$, and let $\delta > 0$ satisfy $r < 1-\delta < 1$. We use the contour integral error formula (see p. 195 of [3] and p. 250 of [4]):

$$h_q(z) - [m/n](z) = (2\pi i)^{-1} \int_{|t|=1-\delta} \frac{h_q(t) Q_{mn}(t)}{(t-z) Q_{mn}(z)} \left(\frac{z}{t}\right)^{m+n+1} dt,$$

for $|z| < 1-\delta$. From Lemma 6.2 it follows that, for $|z| \le r$, $z \notin E_{mn}$,

$$|h_q(z) - [m/n](z)| \le C_3(1+\delta)^{m+n}(|z|/(1-\delta))^{m+n},$$

where C_3 is independent of m, n, and z. Together with (6.8), this yields (3.15). ■

Proof of Theorem 3.3(i)(b) and (ii). The proof of (i)(b) is similar to that of (i)(a), without the exceptional set E_{mn}, since $\{G_n(z)\}_1^\infty$ has no zeros, and so $\{[m_j/n_j](z)\}_1^\infty$ has no poles in $|z| < \Delta_q$. To prove (ii), it suffices to note from Theorem 2.7, that there is a sequence $\{n_j\}_1^\infty$ satisfying (3.17) for which $G_{n_j}(z)$ has a zero z_j with

$$\lim_{j\to\infty} |z_j| = \Delta_q.$$

Since $D(m/n) \ne 0$ for $m \ge n-1$, any zero of $G_n(-zq^m)$ is a pole of $[m/n](z)$. ■

Proof of Theorem 3.4. Let $\mathcal{J}=\{n_j\}_1^\infty$ be an increasing sequence of positive integers satisfying (3.18) if $q\notin\mathcal{G}$, and (3.19) if $q\in\mathcal{G}$. By Theorem 2.5,

$$\lim_{j\to\infty} G_{n_j}(z)=1,$$

locally uniformly in $|z|<1$. Then, by Theorem 3.2,

$$\lim_{j\to\infty} Q_{m_jn_j}(z)=1$$

locally uniformly in $|z|<1$, for any sequence $\{m_j\}_1^\infty$ of positive integers satisfying (3.13). Hence $\{[m_j/n_j](z)\}_1^\infty$ converges locally uniformly in $|z|<1$. The assertion concerning $\{[(m_j+l)/(n_j+l)](z)\}_1^\infty$ follows similarly from Theorems 2.5 and 3.2. ■

For *full* paradiagonal sequences of Padé approximants, we can prove pointwise convergence in a region containing the disk $|z|<\Delta_q$ considered in Theorem 3.3:

Theorem 6.3. *Let $J\geq -1$ be an integer. Let $\mathcal{D}$ be the set of all z such that $|z|<1$, and such that for some positive number s,*

$$\min_{|w|=1}\{|z^{-1}+q^Jw[1-w(1+q)]|-|1-w|s-s^{-1}\}\geq 0. \tag{6.14}$$

Then, locally uniformly in the interior of $\mathcal{D}$,

$$\lim_{n\to\infty}[n+J/n]=h_q(z).$$

Proof. If z satisfies (6.14), then zq^J satisfies (2.26), and hence $Q_{n+J,n}(z)=G_n(-zq^{n+J})\neq 0$, $n=1,2,3,\ldots$. Thus $\{[n+J/n](z)\}_1^\infty$ has no limit points of poles in the interior of $\mathcal{D}$. ■

In the proof of Theorem 3.5 we shall need:

Lemma 6.4. *Let $m\geq n-1\geq 0$. Then the coefficient of z^m in $P_{mn}(z)$ is*

$$A_{mn}:=q^{m(m-1)/2}\prod_{j=1}^{n}(1-q^j) \tag{6.15}$$

and

$$1\leq \|P_{mn}\|\leq (m+1)(n+1)\|Q_{mn}\|. \tag{6.16}$$

Proof. From (3.5) we see that the coefficient of z^m in $P_{mn}(z)$ is

$$D(m/n+1)/D(m/n).$$

Then (3.11) yields (6.15). Next, let us write

$$Q_{mn}(z)=\sum_{k=0}^{n} q_{mnk}z^k.$$

Expanding the determinants in (3.4) and (3.5) by their last row, and with the notation (6.1), we see that

$$P_{mn}(z)=\sum_{k=0}^{n} q_{mn,n-k}\sum_{j=n-k}^{m} a_{j-n+k}z^j.$$

Hence, using Cauchy's estimates,

$$\|P_{mn}\| \leq \sum_{k=0}^{n} \|Q_{mn}\|(m-n+k+1) \leq (m+1)(n+1)\|Q_{mn}\|.$$

The lower bound in (6.16) follows as $P_{mn}(0) = h_q(0) = 1$. ■

Proof of Theorem 3.5. Let $\{m_j\}_1^\infty$ and $\{n_j\}_1^\infty$ be sequences of positive integers satisfying (3.13) and (3.17). It is an immediate consequence of Theorems 2.1 and 3.2 that $\{Q_{m_jn_j}(z)\}_1^\infty$ has property *W*. Further, it follows from (6.16) that

$$\lim_{j\to\infty} \|P_{m_jn_j}\|^{1/m_j} = 1.$$

Cauchy's inequalities show that

$$\limsup_{j\to\infty} |A_{m_jn_j}|^{1/m_j} \leq 1.$$

If $q \notin \mathcal{G}$, it follows from Lemma 4.6 and (6.15) that

$$\liminf_{j\to\infty} |A_{m_jn_j}|^{1/m_j} \geq 1.$$

Hence, if $q \notin \mathcal{G}$, $\{P_{m_jn_j}(z)\}_1^\infty$ has property *U*.

Now suppose $q \in \mathcal{G}$, $R(q) < \eta < 1$, and $\mathcal{J}$ is an infinite sequence of positive integers satisfying (3.19). Then

$$\limsup_{n\in\mathcal{J}} |1-q^n|^{1/n} \leq \eta < 1,$$

and so, by (6.15),

$$\begin{aligned}\limsup_{n\in\mathcal{J}} |A_{nn}|^{1/n} &\leq \limsup_{n\in\mathcal{J}} |1-q^n|^{1/n} \limsup_{n\in\mathcal{J}} |A_{n-1,n-1}|^{1/n} \\ &\leq \eta < 1.\end{aligned}$$ ■

Note that if $R(q) > 0$, then, by Lemma 4.6,

$$\begin{aligned}R(q) &= \liminf_{n\to\infty} |1-q^n|^{1/n} \left\{\prod_{j=1}^{n-1} |1-q^j|\right\}^{1/n} \\ &\leq R(q) \limsup_{n\to\infty} \left\{\prod_{j=1}^{n-1} |1-q^j|\right\}^{1/n},\end{aligned}$$

so that

$$\limsup_{n\to\infty} \left\{\prod_{j=1}^{n-1} |1-q^j|\right\}^{1/n} = 1.$$

Hence, at least a subsequence of $\{P_{nn}(z)\}_1^\infty$ has property *U*.

Finally, we turn to the

Proof of Theorem 3.1. Let us assume (3.6), with $\theta/(2\pi)$ irrational. It is shown in Lubinsky [25] that for each fixed positive integer n,

$$\lim_{m\to\infty} Q_{mn}(ua_m/a_{m+1}) = B_n(u),$$

where $B_1(u)=1-u$ and

$$B_n(u)=B_{n-1}(u)-uq^{n-1}B_{n-1}(u/q), \qquad n=2,3,4,\ldots.$$

With the aid of (4.2) and (4.4), we recognize that

$$B_n(u)=G_n(-u), \qquad n=2,3,4,\ldots.$$

Let us fix $n\geq 2$ and let $u_1, u_2, \ldots, u_n$ denote the zeros of $G_n(z)$. With a suitable ordering of the zeros $z_{m1}, z_{m2}, \ldots, z_{mn}$ of $Q_{mn}(z)$, Hurwitz' theorem shows that as $m\to\infty$,

$$z_{mj}=-u_j a_m/a_{m+1}(1+o(1)), \qquad j=1,2,\ldots,n. \tag{6.17}$$

(i) If $r_-=\liminf_{j\to\infty}|a_j/a_{j+1}|>0$, then it follows from (6.17) that $\{[m/n](z)\}_1^\infty$ has no limit points of poles in $|z|<\Delta_{nq}r_-$, and (3.8) follows.

(ii) Since $r_+=\limsup_{m\to\infty}|a_m/a_{m+1}|\geq r$, we can find an infinite sequence $\mathcal{J}$ of positive integers for which $\{[m/n]\}_{m\in\mathcal{J}}$ has no limit points of poles in $|z|<\Delta_{nq}r_+$ and hence also in $|z|<\Delta_{nq}r$.

(iii) If $r_+=r<\infty$, (6.17) implies that for m large enough, $[m/n](z)$ has a pole z_m that satisfies (3.10). ■

Acknowledgments. The authors wish to thank G. Andrews, R. Askey, W. Gragg, J. Liang, Al. Magnus, and L. Reichel for helpful comments and for providing references. This research was completed while the first author was visiting the University of South Florida. The second author was supported, in part, by National Science Foundation Grant No. DMS-8401370.

Added in Proof. Recent numerical results of J. Waldvogel (to appear in Tampa Approximation Seminar Proceedings, Lecture Notes in Mathematics, New York: Springer-Verlag) suggest that the actual value of Δ defined in (2.30) is $3-2\sqrt{2}$. Note that inequality (2.31) yields $\Delta\geq 3-2\sqrt{2}$.

References

1. G. E. Andrews (1981): *Ramanujan's "lost" notebook I. Partial θ-functions.* Adv. in Math., **41**:137–172.
2. R. Askey, M. E. H. Ismail (1984): *Recurrence relations, continued fractions and orthogonal polynomials.* Mem. Amer. Math. Soc., **49** (300).
3. G. A. Baker, Jr. (1975): Essentials of Padé Approximants. New York: Academic Press.
4. G. A. Baker Jr., P. R. Graves-Morris (1981): Padé Approximants, Part I: Basic Theory. Encyclopedia of Mathematics and Its Applications, vol. 13. Reading: Addison-Wesley.
5. M. B. Balk (1960): *The Padé interpolation process for certain analytic functions.* In: Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo. Moscow: Gosdartsv. Izdat. Fiz.-Mat. Lit., pp. 234–257. (Abstract: see Zbl, **173** (1969), 73; MR; **22** (1961), No. 9780.)
6. H. P. Blatt, E. B. Saff (1985): *Distribution of zeros of polynomial sequences, especially best approximations.* In: Delay Equations, Approximation and Application (G. Meinardus, G. Nurnberger, eds.). ISNM, vol. 74. Basel: Birkhäuser Verlag, pp. 71–82.
7. H. P. Blatt, E. B. Saff (1986): *Behavior of zeros of polynomials of near best approximation.* J. Approx. Theory, **46**:323–344.
8. V. I. Buslaev, A. A. Gončar, S. P. Suetin (1984): *On the convergence of subsequences of the mth row of the Padé table.* Math. USSR-Sb., **48**:535–540.

37. E. B. Saff, R. S. Varga (1978): *On the zeros and poles of Padé approximants to e^z, III.* Numer. Math. **30**:241–266.
38. H. Stahl (1985): *Divergence of diagonal Padé approximants and the asymptotic behavior of orthogonal polynomials associated with nonpositive measures.* Constr. Approx., **1**:249–270.
39. H. Stahl (1985): *On the divergence of certain Padé approximants and the behaviour of the associated orthogonal polynomials.* In: Polynomes Orthogonaux et Applications; Symposium Laguerre (C. Brezinski *et al.*, eds.). Lecture Notes in Mathematics, vol. 1171. Berlin: Springer-Verlag, pp. 321–330.
40. H. Stahl (unpublished): *The convergence of Padé approximants to functions with branch points.*
41. G. Szegö (1926): *Ein Beitrag zur Theorie der Thetafunktionen.* Sitzungsber. Preussischen Akad. der Wissen., Phys. Math. Kl., 242–251.
42. R. S. Varga (1982): Topics in Polynomial and Rational Interpolation and Approximation. Montreal: Les Presses de l'Université de Montreal.
43. H. S. Wall (1948): Analytic Theory of Continued Fractions. Princeton: Van-Nostrand Reinhold.
44. J. L. Walsh (1960): Interpolation and Approximation in the Complex Domain. (Colloquium Publications, vol. 20. Providence: American Mathematical Society.
45. P. Wynn (1967): *A general system of orthogonal polynomials.* Quart. J. Math. Oxford, **18**:81–96.

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