REMARKS ON THE BEHAVIOUR OF ZEROS OF BEST APPROXIMATING POLYNOMIALS AND RATIONAL FUNCTIONS

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ABSTRACT

We discuss some qualitative differences between best polynomial approximants and best rational approximants with respect to the locations of their zeros and poles and their analytic continuation properties. In particular, we prove that the rational functions of degree n of the best uniform approximation to $f(x) = |x|$ on $[-1,1]$ have all their zeros and poles on the imaginary axis and converge in the right-half and left-half planes. Further, we consider the sharpness of a result of Blatt and Saff concerning the asymptotic behaviour of zeros of polynomials of "near best" approximation.

1. INTRODUCTION

Let E denote a compact set in the z-plane and A(E) the collection of functions that are analytic in the interior of E and continuous on E. The famous theorem of Mergelyan asserts that if E does not separate the plane and $f \in A(E)$, then there exists a sequence of polynomials that converges uniformly to f on E. To relate the error in approximation to the degree of the polynomial, we set

$$E_n(f) := \min \{||f - p||_E : p \in \Pi_n\}, \tag{1.1}$$

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where Π_n denotes the collection of polynomials of degree at most n and $||\cdot||_E$ is the sup norm on E. Assuming E contains infinitely many points, the minimum in (1.1) will be attained for a unique polynomial $p_n^* \in \Pi_n$; that is,

$$E_n(f) = ||f - p_n^*||_E. \tag{1.2}$$

Roughly speaking, the speed at which $E_n(f)$ tends to zero is related to the smoothness of f (the "nicer" the function, the faster the convergence). For example, when f is analytic on E (i.e., f is analytic on an open set $G \supset E$), the convergence rate is geometric. To be precise we state

<u>Theorem</u> 1.1 (Walsh [9, §4.7]). <u>Suppose</u> E <u>is a compact set whose complement</u> K <u>with respect to the extended plane is connected and regular in the sense that</u> K <u>possesses a classical Green's function with pole at infinity</u>. <u>Let</u> $f \in A(E)$. <u>Then</u> f <u>is analytic on</u> E <u>if and only if</u>

$$\limsup_{n\to\infty} E_n(f)^{1/n} < 1. \tag{1.3}$$

In a recent paper of Blatt and Saff [4], it is shown that the leading coefficients of the polynomials of best uniform approximation to f on E "carry the information" as to whether f is analytic on E. They proved

<u>Theorem</u> 1.2. <u>Let</u> E <u>be as in Theorem</u> 1.1, $f \in A(E)$, <u>and</u> $p_n^* = a_n^* z^n + \cdots \in \Pi_n$ <u>satisfy</u> (1.2). <u>Then</u> f <u>is analytic on</u> E <u>if and only if</u>

$$\limsup_{n\to\infty} |a_n^*|^{1/n} < 1/c, \tag{1.4}$$

<u>where</u> $c = \mathrm{cap}(E)$ <u>is the logarithmic capacity of</u> E.

The constant c is also called the <u>transfinite diameter</u> of E (cf. [9, §4.4]) and is given by

$$\ln c = \lim_{z\to\infty} [\ln|z| - G(z)], \tag{1.5}$$

where $G(z)$ is the Green's function with pole at infinity for the complement of E. When E is a disk $|z - z_0| \le r$, then c is its radius; if E is the real interval $[a,b]$, then $c = (b - a)/4$.

It is an easy consequence of the Bernstein-Walsh lemma (cf. [9, §4.6]) that, for any $f \in A(E)$,

$$\limsup_{n\to\infty} |a_n^*|^{1/n} \leq 1/c. \tag{1.6}$$

Hence we can state Theorem 1.2 in the following equivalent form.

Theorem 1.2'. *Let E be as in Theorem 1.1, $f \in A(E)$, and $p_n^* = a_n^* z^n + \ldots \in \Pi_n$ satisfy (1.2). Then f is not analytic on E (f has at least one singularity on the boundary of E) if and only if*

$$\limsup_{n\to\infty} |a_n^*|^{1/n} = 1/c, \quad c := \operatorname{cap}(E). \tag{1.7}$$

Formula (1.7) is reminiscent of the Cauchy-Hadamard formula for the radius of convergence of a power series. Together with Walsh's theory of *exact harmonic majorants* (cf. [8]), it leads to the following result concerning the divergence of best approximating polynomials.

Theorem 1.3 ([4]). *Let E be as in Theorem 1.1, $f \in A(E)$, but f not analytic on E. Then the sequence of polynomials $\{p_n^*\}_0^\infty$ of best uniform approximation to f on E diverges on every continuum exterior to E. More precisely, for any continuum $S \subset \mathbb{C}\setminus E$ (S not a single point)*

$$\limsup_{n\to\infty} \|p_n^*\|_S^{1/n} > 1. \tag{1.8}$$

This theorem asserts that if f has a singularity at some point of the boundary of E, then the sequence of best approximating polynomials cannot directly be used to obtain an analytic continuation of f. For example, this is true for $f(x) = |x|$ when approximation occurs on the interval $E := [-1,1]$.

One goal of the present note is to contrast this behaviour of best approximating polynomials with that of best approximating rational functions. Namely, we shall establish in Section 3 the following result:

Proposition 1.4. Let $R_n^*(x)$ be the rational function of degree at most n (with real coefficients) of best uniform approximation to $f(x) = |x|$ on $[-1,1]$. Then

$$\lim_{n\to\infty} R_n^*(z) = \begin{cases} z \text{ if } \operatorname{Re} z > 0, \\ -z \text{ if } \operatorname{Re} z < 0. \end{cases} \tag{1.9}$$

The reader should bear in mind the celebrated example of D.J. Newman [6] which shows that the error in best rational approximation to $f(x) = |x|$ on $[-1,1]$ is substantially smaller than the error in best polynomial approximation. The result of (1.9) demonstrates an important qualitative difference; namely, the best approximating rationals yield an analytic continuation of f while the best approximating polynomials diverge outside $[-1,1]$.

There is another qualitative difference that concerns the asymptotic behaviour of the zeros and poles of the best approximants. The following general result of Blatt and Saff for best approximating polynomials provided an analogue of the classical theorem of Jentzsch [5, p. 352] for power series.

Theorem 1.5 ([4]). With the hypotheses of Theorem 1.3, assume that f does not vanish identically on the interior of any component of E. Then every point of the boundary of E is a limit point of zeros of the sequence of best approximating polynomials $\{p_n^*\}_1^\infty$.

We remark that Theorem 1.5 applies in the case when E is any finite union of real intervals and f is continuous on E, but not analytic at every point of E. In such a case, every point of E will attract zeros of the polynomials of best uniform approximation to f on E. This is true, in particular, for the best polynomial approximants to $f(x) = |x|$ on $E := [-1,1]$. In contrast, we prove in Section 3 the following property for best rational approximants:

Proposition 1.6. For each $n \geq 1$, the rational function R_n^* of best uniform approximation to $f(x) = |x|$ on $[-1,1]$ has all its zeros and poles on the imaginary axis. Moreover, the poles and zeros of R_n^* are interlaced along the upper-half (lower-half) of the imaginary axis.

It was shown by Blatt and Saff [4] that the asymptotic behaviour described in Theorem 1.5 for the zeros of the polynomials p_n^* also holds for polynomials p_n of "near best" approximation to f on E in the sense that

$$\| f - p_n \|_E = E_n(f) + \varepsilon_n, \quad n = 0,1,2,\ldots, \tag{1.10}$$

where $\limsup_{n\to\infty} \varepsilon_n^{1/2} < 1$. In Section 2 we give an example to show that the conclusion of Theorem 1.5 fails to hold if condition (1.10) is weakened to be

$$\| f - p_n \|_E \leq (1 + \delta) E_n(f), \quad n = 0,1,\ldots, \tag{1.11}$$

for any constant $\delta > 0$.

2. SHARPNESS OF THEOREM 1.5.

Let E denote the finite circular sector

$$E := \{z = re^{i\theta} : |\theta| \leq \theta_0 < \pi, \ 0 \leq r \leq 1\}, \tag{2.1}$$

and, for fixed $\delta > 0$, set

$$E(\delta) := (1 + \delta)E = \{(1 + \delta)z : z \in E\}. \tag{2.2}$$

Let p_n^* denote the polynomial of best uniform approximation to $f(z) = \sqrt{z}$ on E, where the square root is the principal branch. According to Theorem 1.5, every boundary point of E will be a limit point of zeros of the sequence $\{p_n^*\}_1^\infty$.

Now observe that the polynomials

$$p_n(z) := \sqrt{1+\delta}\, p_n^*(z/(1 + \delta)), \quad n = 0,1,\ldots, \tag{2.3}$$

satisfy

$$\| \sqrt{z} - p_n \|_E \leq \| \sqrt{z} - p_n \|_{E(\delta)} = \sqrt{1+\delta}\, E_n(\sqrt{z}), \tag{2.4}$$

when $E_n(\sqrt{z}) = \| \sqrt{z} - p_n^* \|_E$ is the minimum error on E. (In fact fact, p_n is the polynomial in Π_n of best uniform approximation to $f(z) = \sqrt{z}$ on $E(\delta)$.) However, not every boundary point of E is a limit point of zeros of the sequence $\{p_n\}_1^\infty$; the p_n's are bounded away from zero in a neighbourhood of each point of the open arc $z = e^{i\theta}$, $|\theta| < \theta_0$. Hence the asymptotic behaviour of zeros described in Theorem 1.5 does not, in general, hold for sequences of polynomials satisfying (1.11).

3. BEST RATIONAL APPROXIMANTS TO $|x|$

As in Section 1, we let R_n^* denote the unique rational function (with real coefficients) with numerator and denominator of degree at most n of best uniform approximation to $f(x) = |x|$ on $[-1,1]$. Because of uniqueness and the eveness of f we can write for each $m = 0,1,\ldots,$

$$R^*_{2m+1}(x) = R^*_{2m}(x), \quad R^*_{2m}(x) = r^*_m(x^2), \tag{3.1}$$

where $r^*_m(x)$ is the rational function of degree at most m that is of best uniform approximation to $g(x) := \sqrt{x}$ on $[0,1]$.

<u>Lemma</u> 3.1. <u>The rational</u> $r^*_m(x)$ <u>interpolates</u> $g(x) = \sqrt{x}$ <u>in</u> $2m + 1$ <u>distinct points of</u> $(0,1)$.

<u>Proof</u>. For each pair (k,ℓ) of nonnegative integers, the functions $1,x,\ldots,x^k,g(x),xg(x),\ldots,x^\ell g(x)$ form a Descartes system on $(0,+\infty)$. Hence $g(x)$ is hypernormal (cf. [7, §5.1]) which implies that $g(x) - r^*_m(x)$ has on $[0,1]$ an alternation set of $2m + 2$ points and so (at least) $2m + 1$ distinct zeros in $(0,1)$. □

As a consequence of Lemma 3.1 we observe that the rational function

$$s_m(z) := 1/r^*_m(z+1) \tag{3.2}$$

interpolates the <u>Stieltjes function</u> $1/\sqrt{z+1}$ in $2m + 1$ distinct real points. Hence we can apply the following result.

<u>Lemma</u> 3.2. <u>Let</u> $h(z)$ <u>be a Stieltjes function of the form</u>

$$h(z) = \int_0^1 \frac{d\mu(t)}{1 + zt}, \tag{3.3}$$

where $d\mu$ is a finite positive measure on $[0,1]$. Suppose that the rational $s_m(z) = P_m(z)/Q_m(z)$, $P_m, Q_m \in \Pi_m$, interpolates $h(z)$ in the zeros of a real polynomial of precise degree $2m+1$ that does not vanish on the cut $(-\infty,-1]$. Then the zeros $\{\xi_k\}_1^m$ of P_m and zeros $\{\eta_k\}_1^m$ of Q_m are all real and satisfy

$$\xi_1 < \eta_1 < \dots < \eta_{m-1} < \xi_m < \eta_m < -1. \qquad (3.4)$$

Proof. For the case when s_m interpolates h in a single real point $x_0 > -1$ of multiplicity $2m+1$, that is, s_m is the (m,m) Padé approximant of h at x_0, then the conclusion (3.4) is well-known (cf. Baker [2, p. 213]). In fact, for such interpolants, we have

$$s_m(z) = A_0^{(m)} + \sum_{k=1}^{m} \frac{A_k^{(m)}}{1+zt_k^{(m)}}, \quad t_k^{(m)} \in (0,1), \qquad (3.5)$$

where $A_j^{(m)} > 0$ for all $j = 0,1,\dots,m$. The assertion of Lemma 3.2 for interpolation in the zeros of an arbitrary real polynomial $D(z)$ of degree $2m+1$ that does not vanish on $(-\infty,-1]$ can be deduced via a continuity argument from the Padé case (see also Baker [1] and Barnsley [3]). The representation (3.5) with $A_j^{(m)} > 0$ for all j persists in the general case. Moreover, the denominator $Q_m(z)$ when transformed to $z^m Q_m(-1/z)$ is orthogonal with respect to $t\,d\mu(t)/|t^{2m+1}D(-1/t)|$ on $[0,1]$. □

Proof of Proposition 1.6. Since the rational $1/r_m^*(z+1)$ interpolates $h(z) = 1/\sqrt{z+1}$ (which is of the form (3.3)) in $2m+1$ real points, Lemma 3.2 implies that the zeros and poles of this rational satisfy (3.4). Hence, from the relations (3.1), we deduce Proposition 1.6. □

Proof of Proposition 1.4. Because of the relations (3.2) and (3.1), it suffices to prove that the rationals $\{s_m\}_0^\infty$ defined in (3.2) converge to $1/\sqrt{z+1}$ uniformly on compact subsets of $\mathbb{C} \setminus (-\infty,-1]$. Since $r_m^*(x)$ is the best uniform approximation to $\sqrt{x}$ on $[0,1]$, it follows that $s_m(x) \to 1/\sqrt{x+1}$ on $(-1,0]$.

Moreover, thanks to (3.5), we can write

$$s_m(z) = A_0^{(m)} + \int_0^1 \frac{d\mu_m(t)}{1 + zt}, \tag{3.6}$$

where $d\mu_m$ is a discrete positive measure on $[0,1]$. Putting $z = 0$ in (3.6) we see that

$$s_m(0) = A_0^{(m)} + \int_0^1 d\mu_m \to 1 \text{ as } m\to\infty. \tag{3.7}$$

Consequently, since $A_0^{(m)} > 0$ for all m, the sequences $\{A_0^{(m)}\}$ and $\{\mu_m([0,1])\}$ are bounded. From (3.6) we now deduce that the rational functions $\{s_m\}$ are uniformly bounded on compact subsets of $\mathbb{C} \setminus (-\infty, -1]$, and so they form a normal family in $\mathbb{C} \setminus (-\infty,-1]$. Since any limit function of this family must equal $1/\sqrt{x+1}$ on $(-1,0]$, it follows that $s_m(z) \to 1/\sqrt{z+1}$ uniformly on compact subsets of $\mathbb{C} \setminus (-\infty,-1]$. □

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