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PADÉ APPROXIMANTS OF PARTIAL THETA FUNCTIONS AND THE ROGERS-SZEGÖ POLYNOMIALS

D. S. Lubinsky and E. B. Saff

We outline recent results on convergence of Padé approximants for a class of functions including the partial theta function.

1. Introduction

Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be a formal power series. Let m,n be nonnegative integers. The m,n Padé approximant to f(z) is a rational function $[m/n](z) = P(z)/Q(z)$, where P and Q have degree at most m and n respectively, $Q \not\equiv 0$, and $(fQ - P)(z) = O(z^{m+n+1})$ formally.

In [2], one of the authors investigated convergence of Padé approximants for functions f(z) that have "smooth" coefficients, in the sense that

$$(1.1) \qquad \lim_{j\to\infty} a_{j-1}a_{j+1}/a_j^2 = q \in \mathbb{C}.$$

A model function for (1.1) is the partial theta function

$$(1.2) \qquad h_q(z) := \sum_{j=0}^{\infty} q^{j(j-1)/2} z^j,$$

for which the limit in (1.1) may be replaced by equality for all $j \geq 1$.

We consider the Padé approximants of $h_q(z)$ for the delicate case when $|q| = 1$, say

$$(1.3) \qquad q := e^{i\theta}, \ \theta \in [0,2\pi).$$

When $\theta/(2\pi)$ is rational, $h_q(z)$ is a rational function, but when $\theta/(2\pi)$ is irrational, $h_q(z)$ has the unit circle as its natural boundary. In the latter case, the normalized Padé

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denominator $Q_{mn}(z)$ in $[m/n](z)$ satisfies [3]

$$(1.4) \qquad Q_{mn}(z) = G_n(-zq^m), \qquad m \geq n-1 \geq 0,$$

where $G_n(z)$ is the <u>Rogers-Szegö polynomial</u>

$$(1.5) \qquad G_n(z) = G_n(z;q) := \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} z^j, \quad n = 1,2,3,\ldots,$$

and where

$$(1.6) \qquad \begin{bmatrix} n \\ j \end{bmatrix} := \begin{cases} \dfrac{(1-q^n)(1-q^{n-1})\ldots(1-q^{n+1-j})}{(1-q)(1-q^2)\ldots(1-q^j)}, & 1 \leq j \leq n, \\ 1 & , j = 0. \end{cases}$$

We also set $G_0(z) \equiv 1$.

2. <u>Rogers-Szegö Polynomials</u>

The zero distribution of $\{G_n(z)\}_1^\infty$ may be obtained from the following result [3]:

THEOREM 1. <u>Let</u> $\|\cdot\|$ <u>denote the sup norm on</u> $|z| = 1$.
(a) <u>If</u> $\theta/(2\pi)$ <u>is a rational number</u> μ/ν, <u>where</u> μ, ν <u>are relatively prime positive integers, then</u>

$$(2.1) \qquad \lim_{n\to\infty} \|G_n\|^{1/n} = 2^{1/\nu}.$$

(b) <u>If</u> $\theta/(2\pi)$ <u>is irrational</u>,

$$(2.2) \qquad \lim_{n\to\infty} \|G_n\|^{1/n} = 1.$$

It can be shown [3] that all zeros of $\{G_n(z)\}_1^\infty$ lie in the annulus $r_0^{-1} < |z| < r_0$, where $r_0 := 2\sqrt{2} + 1 + |1 + q|$. Regarding the limiting behavior of $\{G_n(z)\}_1^\infty$, we can prove

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of $(G_n(z))_1^\infty$ lie in

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$z))_1^\infty$, we can prove

THEOREM 2. If $\theta/(2\pi)$ is irrational and ℓ is a nonnegative integer, there exists an increasing sequence $\mathcal{J}$ of positive integers, such that locally uniformly in $|z| < 1$,

$$\lim_{n\in\mathcal{J}} G_n(z) = G_\ell(z). \tag{2.3}$$

One important case of (2.3) is $\ell = 0$, for which $G_\ell \equiv 1$.

3. Convergence of Padé Approximants

For rows of the Padé table, we can prove [3]:

THEOREM 3 (Rows). Let $n \geq 2$. Suppose $f(z) = \sum_{j=0}^{\infty} a_j z^j$ has radius of convergence r $(0 < r \leq \infty)$ and that

$$\lim_{j\to\infty} a_{j-1}a_{j+1}/a_j^2 = q = e^{i\theta}, \qquad \theta/(2\pi) \text{ irrational}. \tag{3.1}$$

Let $r_- := \liminf_{j\to\infty} |a_j/a_{j+1}|$ and $r_+ := \limsup_{j\to\infty} |a_j/a_{j+1}|$.

Then

(a) The full row sequence $\{[m/n](z)\}_{m=1}^{\infty}$ converges locally uniformly to $f(z)$ in $|z| < \Delta_{nq} r_-$, where

$$\Delta_{nq} := \min\{|z| : G_n(z) = 0\} \in (0,1). \tag{3.2}$$

(b) There exists a subsequence of $\{[m/n](z)\}_{m=1}^{\infty}$ that converges locally uniformly to $f(z)$ in $|z| < \Delta_{nq} r$.

(c) If $r_+ = r < \infty$, then no subsequence of $\{[m/n](z)\}_{m=1}^{\infty}$ converges locally uniformly in $|z| < \Delta_{nq} r + \varepsilon$, for any $\varepsilon > 0$.

Part (c) above furnishes a class of counterexamples to the conjecture of Baker and Graves-Morris, recently resolved by Buslaev, Gončar and Suetin [1].

Suppose that we call a sequence of Padé approximants to $h_q(z)$ "good" if it converges locally uniformly throughout $|z| < 1$; otherwise it is "bad". When $\theta/(2\pi)$ is irrational, Theorem 3 (c) shows that every subsequence of every row

$([m/n](z))_{m=1}^{\infty}$ with $n \geq 2$ fixed, is bad. By contrast, a subsequence of the main diagonal $([n/n](z))_{n=1}^{\infty}$ is good (so that the Baker-Gammel-Wills Conjecture is true for $h_q(z)$), while some other diagonal subsequence is bad:

THEOREM 4. <u>(Main Diagonal). Let $\theta/(2\pi)$ be irrational. Then</u>
(a) $([n/n](z))_{n=1}^{\infty}$ <u>converges in capacity to</u> $h_q(z)$ <u>in</u> $|z| < 1$, <u>and converges locally uniformly to</u> $h_q(z)$ <u>in</u> $|z| < \Delta_q$, <u>where</u> $\Delta_q := \inf\{\Delta_{nq} : n \geq 1\} \in (0,1)$.
(b) <u>There exists</u> $(n_j)_1^{\infty}$ <u>such that</u> $([n_j/n_j](z))_1^{\infty}$ <u>converges locally uniformly to</u> $h_q(z)$ in $|z| < 1$.
(c) <u>There exists</u> $(n_j)_1^{\infty}$ <u>such that</u> $[n_j/n_j](z)$ <u>has a pole</u> z_j <u>with</u> $\lim_{j\to\infty} |z_j| = \Delta_q$.

Further details, and proofs, will appear in [3].

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