We outline recent results on convergence of Padé approximants for a class of functions including the partial theta function.

Let \( f(z) = \sum_{j=0}^{\infty} a_j z^j \) be a formal power series. Let \( m, n, a, b \) be nonnegative integers. The \( m/n \) Padé approximant of \( f(z) \) is a rational function \( p(z)/q(z) \), where \( p(z) \) and \( q(z) \) have degree at most \( m \) and \( n \) respectively, and \( q(z) \neq 0 \).

A model function for \((1,1)\) is the partial theta function

\[
q(z) = \sum_{j=1}^{\infty} j(z^{j-1})^{2j},
\]

for which the limit in \((1.1)\) may be replaced by equality for all \( j \geq 1 \).

Let \( n = 0 \) and \( n = 1 \) and \( n = 1 \).
denominator $Q_{mn}(z)$ in $[m/n](z)$ satisfies [3]

\[(1.4) \quad Q_{mn}(z) = G_n(-zq^m), \quad m \geq n - 1 \geq 0,\]

where $G_n(z)$ is the Rogers-Szegő polynomial

\[(1.5) \quad G_n(z) = G_n(z; q) := \sum_{j=0}^{n} \binom{n}{j} z^j, \quad n = 1, 2, 3, \ldots .\]

and where

\[(1.6) \quad \binom{n}{j} := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-j})}{(1-q)(1-q^2)\cdots(1-q^j)}, \quad 1 \leq j \leq n,\]

\[= 1, \quad j = 0.\]

We also set $G_0(z) = 1$.

2. Rogers-Szegő Polynomials

The zero distribution of $(G_n(z))_1^\infty$ may be obtained from the following result [3]:

**Theorem 1.** Let $\| \cdot \|$ denote the sup norm on $|z| = 1$.

(a) If $\theta/(2\pi)$ is a rational number $\mu/\nu$, where $\mu, \nu$ are relatively prime positive integers, then

\[(2.1) \quad \lim_{n \to \infty} \| \mu \|^{1/\nu} = 2^{1/\nu}.\]

(b) If $\theta/(2\pi)$ is irrational.

\[(2.2) \quad \lim_{n \to \infty} \| G_n \|^{1/\nu} = 1.\]

It can be shown [3] that all zeros of $(G_n(z))_1^\infty$ lie in the annulus $r_0^{-1} < |z| < r_0$, where $r_0 := 2\sqrt{2} + 1 + \sqrt{1 + q}$.

Regarding the limiting behavior of $(G_n(z))_1^\infty$, we can prove

**Theorem 2.** If integer, there integers, such

\[(2.3) \quad \text{One impor} \]

3. For rows

**Theorem 3 (Row:)

radius of conver:

\[(3.1) \quad \lim_{j \to \infty} \Delta_{nq} := \text{The full } r

uniformly to } f(\)

\[(3.2) \quad \Delta_{nq} := \text{If } r_+ = r

converges local

(c) \text{If } r_+ = r

converges local

Part (c) is the conjecture by Buslaev. Con

Suppose th

$h_q(z)$ "good" if $|z| < 1$; otherw

Theorem 3 (c) s
Pade Approximants of Partial Theta Functions and the Rogers-Szegoe Polynomials

Theorem 2. If $\Theta(2x)$ is irrational and $t$ is a nonnegative integer, there exists an increasing sequence $\xi$ of positive integers, such that locally uniformly in $|z| < 1$,

$$\lim_{n \to \infty} G_n(z) = G_\xi(z).$$

One important case of (2.3) is $\xi = 0$, for which $G_\xi = 1$.

3. Convergence of Padé Approximants

For rows of the Padé table, we can prove [3]:

Theorem 3 (Rows). Let $n \geq 2$. Suppose $f(z) = \sum_{j=0}^{\infty} a_j z^j$ has radius of convergence $r$ ($0 < r < \infty$) and that

$$\lim_{j \to \infty} a_j / a_{j+1} = q = e^{i \theta}. \quad \Theta(2x) \text{ irrational.}$$

Let $r_- = \liminf_{j \to \infty} |a_j / a_{j+1}|$ and $r_+ = \limsup_{j \to \infty} |a_j / a_{j+1}|$. Then

(a) The full row sequence $((m/n)(z))_{m=1}^\infty$ converges locally uniformly to $f(z)$ in $|z| < \Delta_n r^-$, where

$$\Delta_n := \min\{|z| : G_n(z) = 0\} \in (0, 1).$$

(b) There exists a subsequence of $((m/n)(z))_{m=1}^\infty$ that converges locally uniformly to $f(z)$ in $|z| < \Delta_n r^-$.

(c) If $r_- = r < \infty$, then no subsequence of $((m/n)(z))_{m=1}^\infty$ converges locally uniformly in $|z| < \Delta_n r + \varepsilon$, for any $\varepsilon > 0$.

Part (c) above furnishes a class of counterexamples to the conjecture of Baker and Graves-Morris, recently resolved by Buslaev, G.čar and Suetin [1].

Suppose that we call a sequence of Padé approximants to $h_n(z)$ "good" if it converges locally uniformly throughout $|z| < 1$; otherwise it is "bad". When $\Theta(2x)$ is irrational, Theorem 3 (c) shows that every subsequence of every row
\((\{m/n\}(z))_{n=1}^{\infty}\) with \(n \geq 2\) fixed, is bad. By contrast, a subsequence of the main diagonal \((\{n/n\}(z))_{n=1}^{\infty}\) is good (so that the Baker-Gammel-Wills Conjecture is true for \(h_q(z)\)).

while some other diagonal subsequence is bad:

**Theorem 4. (Main Diagonal).** Let \(\theta/(2\pi)\) be irrational. Then

(a) \((\{n/n\}(z))_{n=1}^{\infty}\) converges in capacity to \(h_q(z)\) in \(|z| < 1\), and converges locally uniformly to \(h_q(z)\) in \(|z| < \Delta_q\), where

\[\Delta_q := \inf(\Delta_{nq} : n \geq 1) \in (0, 1).\]

(b) There exists \((n_j)_{j=1}^{\infty}\) such that \((n_j/n_j)(z)_{j=1}^{\infty}\) converges locally uniformly to \(h_q(z)\) in \(|z| < 1\).

(c) There exists \((n_j)_{j=1}^{\infty}\) such that \((n_j/n_j)(z)\) has a pole \(z_j\) with \(\lim_{j \to \infty} |z_j| = \Delta_q\).

Further details and proofs will appear in [3].

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