ON SEQUENCES OF POLYNOMIALS AND THE DISTRIBUTION OF THEIR ZEROS

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We discuss the asymptotic behavior of the zeros of special polynomial sequences $\{p_n\}$ such as best polynomial approximants. For example, let f be a function continuous on a compact set E, analytic in the interior of E but not on E itself. Moreover, let the complement (with respect to the extended plane) of E be connected and possess a classical Green's function with pole at infinity. Then it is shown that the proportion of zeros of $\{p_n\}$ in the neighborhood of a free boundary arc J of K is characterized asymptotically by the value at ∞ of the harmonic measure of J with respect to K.

1. Introduction

Let E denote a compact set in the z-plane and A(E) the collection of functions that are analytic in the interior of E and continuous on E. If $K := \mathbb{C}^* \setminus E$ is connected, then the theorem of Mergelyan states that $\lim_{n\to\infty} E_n(f) = 0$, where

$$(1.1) \qquad E_n(f) := \min \{\|f-p\|_E : p \in \Pi_n\},$$

Π_n denotes the collection of polynomials of degree at most n and $\|\cdot\|_E$ is the sup norm on E. To relate the speed at which $E_n(f)$ tends to zero to the smoothness of f we assume furthermore that K is regular in the sense that K possesses a classical Green's function G(z) with pole at infinity.

The minimum in (1.1) will be attained by a unique polynomial

$$(1.2) \qquad p_n^*(z) = a_n^* z^n + \dots \in \Pi_n.$$

In [1] we proved that the coefficient a_n^* carries the information as to whether f is analytic on E; namely we proved:

THEOREM 1. *Let $f \in A(E)$ and $c = \operatorname{cap}(E)$ be the logarithmic capacity of E. Then the following assertions are equivalent:*

(i) *f is not analytic on E.*

(ii) $\limsup_{n\to\infty} |a_n^*|^{1/n} = 1/c.$

Formula (ii) is reminiscent of the Cauchy-Hadamard formula for the

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radius of convergence of a power series. Together with Walsh's theory of harmonic majorants (cf. [6]), it leads to the following divergence result.

THEOREM 2 Let $f \in A(E)$, but f not analytic on E. Moreover, let S be a continuum (not a single point) of K, then

$$(1.3) \qquad \limsup_{n\to\infty} \|p_n^*\|_S^{1/n} > 1,$$

i.e. the sequence $\{p_n^*\}_0^\infty$ diverges on S.

For example, let $f(x) = |x|$ on $E = [-1,1]$. Then the best approximating polynomials diverge on every continuum S outside $[-1,1]$ (S not a single point). This is in contrast to the approximation of $|x|$ by rational functions where Blatt, Iserles and Saff [3] have established the following result:

Let $R_n^*(x)$ be the (real) rational function of degree at most n of best uniform approximation to $f(x) = |x|$ on $[-1,1]$. Then

$$(1.4) \qquad \lim_{n\to\infty} R_n^*(z) = \begin{cases} z & \text{if} \quad \mathrm{Re}\, z > 0 \\ -z & \text{if} \quad \mathrm{Re}\, z < 0. \end{cases}$$

The following result of Blatt and Saff [1] provides an analogue of the classical theorem of Jentzsch (cf. [5, p. 238]).

THEOREM 3 Let $f \in A(E)$, but f not analytic on E. Assume that f does not vanish identically on the interior of any component of E. Then every point of the boundary of E is a limit point of zeros of the sequence of best approximating polynomials $\{p_n^*\}_0^\infty$.

2. Distribution of zeros of polynomial sequences

Next, we want to obtain Szegö-type results for the distribution of the zeros of the polynomials p_n^*. These polynomials p_n^* as well as maximally convergent polynomials when f is analytic on E are special cases of the following more general situation (cf. [2]):

Let E be a compact and bounded set in the extended plane $\mathbb{C}^*$ with connected and regular complement K, $c = \mathrm{cap}(E)$, $\{p_n\}$ a polynomial sequence satisfying the following properties:

(A 1) $\quad n \in \mathfrak{N} := \{n_1 < n_2 < n_3 < \dots\}$,

(A 2) $\quad p_n \in \Pi_n \setminus \Pi_{n-1} : p_n(z) = a_n z^n + \dots$,

(A 3) $\quad \lim_{n\to\infty} |a_n|^{1/n} = 1/c$, (A 4) $\lim_{n\to\infty} \|p_n\|_E^{1/n} = 1$,

(A 5) for any compact set C in the interior of E the number of zeros $Z_n(C)$ of p_n in C satisfies $Z_n(C) = o(n)$ as $n\to\infty$.

In (A 3) - (A 5) the limits are considered for $n \in \mathcal{N}$.

In [1] the distribution of the zeros of polynomial sequences satisfying (A 1) - (A 4) was studied in the neighborhood of an analytic Jordan arc $J \subset \partial E$ for the case when K is simply connected (see also [2]). In this note we announce a generalization of the results of [1] for multiply connected K and also give a new characterization in terms of harmonic measures.

THEOREM 4 Let $\{p_n\}$ satisfy (A 1) - (A 5). Furthermore, let J be a subarc in the interior of a free boundary arc of K and let D be a neighborhood of the interior of J such that

(2.1) $\overline{D} \cap \partial E = J$.

If $Z_n(D)$ denotes the number of zeros of p_n in D, then

(2.2) $$\lim_{n\to\infty} \frac{Z_n(D)}{n} = \omega(\infty;J),$$

where $\omega(z;J)$ is the harmonic measure of the arc J with respect to K.

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