ON SEQUENCES OF POLYNOMIALS AND THE DISTRIBUTION OF THEIR ZEROS

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We discuss the asymptotic behavior of the zeros of special polynomial sequences \( \{p_n\} \) such as best polynomial approximants. For example, let \( f \) be a function continuous on a compact set \( E \), analytic in the interior of \( E \) but not on \( E \) itself. Moreover, let the complement (with respect to the extended plane) of \( E \) be connected and possess a classical Green's function with pole at infinity. Then it is shown that the proportion of zeros of \( \{p_n\} \) in the neighborhood of a free boundary arc \( J \) of \( K \) is characterized asymptotically by the value at \( \infty \) of the harmonic measure of \( J \) with respect to \( K \).

1. Introduction

Let \( E \) denote a compact set in the \( z \)-plane and \( A(E) \) the collection of functions that are analytic in the interior of \( E \) and continuous on \( E \). If \( X := E \setminus E \) is connected, then the theorem of Mergelyan states that

\[
\lim_{n \to \infty} \varepsilon_n(f) = 0,
\]

where

\[
\varepsilon_n(f) := \min \{ \|f - p\|_E : p \in \Pi_n \},
\]

\( \Pi_n \) denotes the collection of polynomials of degree at most \( n \) and \( \| \cdot \|_E \) is the sup norm on \( E \). To relate the speed at which \( \varepsilon_n(f) \) tends to zero to the smoothness of \( f \) we assume furthermore that \( X \) is regular in the sense that \( K \) possesses a classical Green's function \( G(z) \) with pole at infinity.

The minimum in (1.1) will be attained by a unique polynomial

\[
p_n^*(z) = a_n^* z^n + \ldots \in \Pi_n.
\]

In [1] we proved that the coefficient \( a_n^* \) carries the information as to whether \( f \) is analytic on \( E \); namely we proved:

THEOREM 1. Let \( f \in A(E) \) and \( c = \text{cap}(E) \) be the logarithmic capacity of \( E \). Then the following assertions are equivalent:

(i) \( f \) is not analytic on \( E \).
(ii) \( \limsup_{n \to \infty} |a_n^*|^{1/n} = 1/c \).

Formula (ii) is reminiscent of the Cauchy-Hadamard formula for the
radius of convergence of a power series. Together with Walsh's theory of harmonic majorants (cf. [6]), it leads to the following divergence result.

**THEOREM 2**  Let \( f \in \mathcal{A}(E) \), but \( f \) not analytic on \( E \). Moreover, let \( S \) be a continuum (not a single point) of \( K \), then

\[
(1.3) \quad \limsup_{n \to \infty} \|p_n^*\|_S^{1/n} > 1,
\]

i.e., the sequence \( (p_n^*)_0^\infty \) diverges on \( S \).

For example, let \( f(x) = |x| \) on \( E = [-1,1] \). Then the best approximating polynomials diverge on every continuum \( S \) outside \([-1,1]\) (\( S \) not a single point). This is in contrast to the approximation of \( |x| \) by rational functions where Blatt, Isersles and Saff [3] have established the following result:

Let \( R_n^*(x) \) be the (real) rational function of degree at most \( n \) of best uniform approximation to \( f(x) = |x| \) on \([-1,1]\). Then

\[
(1.4) \quad \lim_{n \to \infty} R_n^*(z) = \begin{cases} 
  z & \text{if } \Re z > 0 \\
  -z & \text{if } \Re z < 0.
\end{cases}
\]

The following result of Blatt and Saff [1] provides an analogue of the classical theorem of Jentzsch (cf. [5, p. 238]).

**THEOREM 3**  Let \( f \in \mathcal{A}(E) \), but \( f \) not analytic on \( E \). Assume that \( f \) does not vanish identically on the interior of any component of \( E \). Then every point of the boundary of \( E \) is a limit point of zeros of the sequence of best approximating polynomials \( (p_n^*)_0^\infty \).

2. Distribution of zeros of polynomial sequences

Next, we want to obtain Szegö-type results for the distribution of the zeros of the polynomials \( p_n^* \). These polynomials \( p_n^* \) as well as maximally convergent polynomials when \( f \) is analytic on \( E \) are special cases of the following more general situation (cf. [2]):

Let \( E \) be a compact and bounded set in the extended plane \( \mathbb{C}^* \) with connected and regular complement \( K \), \( c = \text{cap}(E) \), \( (p_n) \) a polynomial sequence satisfying the following properties:

(A 1) \( n \in \mathbb{N} := \{n_1 < n_2 < n_3 < \ldots \} \),

(A 2) \( p_n \in \mathbb{P} : p_n(z) = a_n z^n + \ldots \),

(A 3) \( \lim_{n \to \infty} \|a_n\|^{1/n} = 1/c \).
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\begin{align}
(A 4) \quad \lim_{n \to \infty} \|p_n\|_E^{1/n} &= 1, \\
(A 5) \quad \text{for any compact set } C \text{ in the interior of } E \text{ the number of zeros } \\
Z_n(C) \text{ of } p_n \text{ in } C \text{ satisfies } Z_n(C) &= o(n) \text{ as } n \to \infty.
\end{align}

In (A 3) - (A 5) the limits are considered for \( n \in \mathbb{N} \).

In [1] the distribution of the zeros of polynomial sequences satisfying (A 1) - (A 4) was studied in the neighborhood of an analytic Jordan arc \( J \subset \partial E \) for the case when \( K \) is simply connected (see also [2]). In this note we announce a generalization of the results of [1] for multiply connected \( K \) and also give a new characterization in terms of harmonic measures.

**Theorem 4** Let \( \{p_n\} \) satisfy (A 1) - (A 5). Furthermore, let \( J \) be a subarc in the interior of a free boundary arc of \( K \) and let \( D \) be a neighborhood of the interior of \( J \) such that

\begin{equation}
U \cap \partial E = J.
\end{equation}

If \( Z_n(D) \) denotes the number of zeros of \( p_n \) in \( D \), then

\begin{equation}
\lim_{n \to \infty} \frac{Z_n(D)}{n} = \omega(\omega; J),
\end{equation}

where \( \omega(z; J) \) is the harmonic measure of the arc \( J \) with respect to \( K \).

**References**


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