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A Weierstrass-type theorem for certain weighted polynomials

After the notion of incomplete polynomials was introduced by G. G. Lorentz in 1976 [4], there has been extensive research concerning these polynomials; especially to the theory of orthogonal polynomials. The essential question which serves as the starting point for these investigations is the following:

Suppose $w(x)$ is a nonnegative weight function continuous on its support $\Sigma \subset \mathbb{R} = (-\infty, \infty)$. (By the support of w we mean the closure of the set where w is positive.) Assume that $w(x)$ vanishes at points of Σ; that is,

$$Z := \{x \in \Sigma : w(x) = 0\} \neq \phi$$

(or in the case Σ is unbounded, then $|x|\ w(x) \to 0$ as $|x| \to \infty$). If Π_n denotes the collection of all polynomials of degree at most n, then the sup norm over Σ of the weighted polynomial $[w(x)]^n P_n(x)$, $P_n \in \Pi_n$, actually "lives" (is attained) on some compact set $\mathcal{S} \subseteq \Sigma \setminus Z$ which is independent of n and P_n. The problem is to determine the smallest such set $\mathcal{S}$.

Using potential theoretic methods, we solved this problem for a rather general setting in [7]. As a further consequence of our results, it follows that if w satisfies certain technical conditions and if a bounded function $f : \Sigma \to \mathbb{R}$ is the uniform limit on Σ of a sequence of weighted polynomials of the form $\{[w(x)]^n P_n(x)\}$, $P_n \in \Pi_n$, $n \to \infty$, then f is continuous on Σ and $f(x) = 0$ for all $x \in \Sigma \setminus \mathcal{S}$ (see [8]). In the converse direction, no general Weierstrass-type theorem for such weighted polynomials has yet been proved. However, the following conjecture, which is stated in [9] for exponential weights, seems likely to be true.

Conjecture. Suppose that w is an admissible weight in the sense of [7] and that $f : \Sigma \to \mathbb{R}$ is a bounded continuous function. Then f is a uniform limit on Σ of a sequence of weighted polynomials of the form $\{[w(x)]^n P_n(x)\}$, $P_n \in \Pi_n$, $n \to \infty$, if and only if $f(x) = 0$ for all $x \in \Sigma \setminus S$.

In this conjecture, "$n \to \infty$" means that n tends to infinity through some subsequence of the positive integers. Using the terminology of "weighted polynomial" to denote a function of the form $[w(x)]^n P_n(x)$, $P_n \in \Pi_n$, n a nonnegative integer, it is important to notice that the sum of weighted polynomials is not necessarily a weighted polynomial; however, their product is a weighted polynomial. Moreover, there is no apparent monotonicity in n for the error in best approximation by the weighted polynomials $[w(x)]^n P_n(x)$. These facts tend to complicate the analysis.

For the incomplete polynomials of Lorentz, where $\Sigma = [0, 1]$ and $w(x) = x^{\theta/(1-\theta)}$ $(0 < \theta < 1)$, it is known that $S = [\theta^2, 1]$. In this case, the conjecture was proved by Saff and Varga [10], and also by M. v. Golitschek [3].

In this note, we prove the conjecture for the Laguerre weight $\exp(-x)$ on $\Sigma = [0, \infty)$ for which it is known [11] that $S = [0, 2]$. As a consequence, we shall also prove the conjecture for the Hermite weight, $w(x) = \exp(-x^2)$, $\Sigma = \mathbb{R}$, for which we have $S = [-1, 1]$ (see [5]). More precisely, we establish the following theorems.

Theorem 1. Let $f \in C[0, 2]$ with $f(2) = 0$, and let $\epsilon > 0$. Then there exists an integer N and polynomial P of degree at most $N - 90\sqrt{N}$ such that

$$\max_{x\in[0,2]} |f(x) - e^{-Nx}P(x)| < \epsilon, \tag{1}$$

$$\max_{x\in[2,\infty)} |e^{-Nx}P(x)| < \epsilon. \tag{2}$$

Theorem 2. Let $f \in C[-1, 1]$ with $f(1) = f(-1) = 0$, and let $\epsilon > 0$. Then there exists an integer M and a polynomial of degree at most $M - 90\sqrt{M}$ such that

$$\max_{x\in[-1,1]} |f(x) - \exp(-Mx^2)P(x)| < \epsilon, \tag{3}$$

$$\max_{|x|\geq 1} |\exp(-Mx^2)P(x)| < \epsilon. \tag{4}$$

The proof of Theorem 1 is similar to the one given in [10] for the case of the incomplete polynomials of Lorentz; namely, a Muntz theorem type argument [1] is utilized. Thus, we shall first approximate a suitably chosen function by weighted polynomials of the form $\exp(-nt)P_n(t)$, $P_n \in \Pi_n$, in $L^2[0, 2]$. Next, we use this result to approximate a continuously differentiable function on $[0, 2]$. Since such functions are dense in $C[0, 2]$, this will essentially prove Theorem 1. In contrast to the proof in [10] we shall use the generating formula for Laguerre polynomials instead of Gram determinants.

We let $\{L_k\}$ denote the orthonormal Laguerre polynomials of respective degree k:

$$\int_0^\infty L_k(x)L_j(x)\exp(-x)\,dx = \delta_{kj} . \tag{5}$$

For integers $n \geq 0$, $N \geq 0$, set

$$P^*_{n,N}(x) := \frac{2N}{N+1} \sum_{k=0}^{n} (-1)^k L_k(2Nx)\left(\frac{N-1}{N+1}\right)^k \in \Pi_n . \tag{6}$$

<u>Lemma 3.</u> <u>With</u>

$$J_N := \int_0^\infty |e^{-t} - e^{-Nt}P^*_{N-100[\sqrt{N}],N}(t)|^2 dt \tag{7}$$

<u>we have</u>

$$\lim_{N\to\infty} J_N = \frac{1}{2} e^{-4} . \tag{8}$$

<u>Proof</u>. For the sake of brevity of notation, we write, in this proof,

n instead of $N - 100[\sqrt{N}]$, P instead of $P^*_{n,N}$ and let $x := 2Nt$. Then

$$J_N = \int_0^\infty \left|Q(x) - \frac{1}{\sqrt{2N}} \exp[\tfrac{x}{2}(1 - \tfrac{1}{N})]\right|^2 \exp(-x)\,dx \tag{9}$$

where

$$\begin{aligned} Q(x) &:= \frac{\sqrt{2N}}{N + 1} \sum_{k=0}^{n} L_k(x) \left(\frac{1 - N}{N + 1}\right)^k \\ &= \frac{1 - w}{\sqrt{2N}} \sum_{k=0}^{n} L_k(x)\, w^k \end{aligned} \tag{10}$$

with

$$w := \frac{1 - N}{1 + N} \ . \tag{11}$$

From the generating formula for Laguerre polynomials (cf. [12, p. 101]), we have

$$\frac{1}{\sqrt{2N}} \exp(-x \frac{w}{1 - w}) = \frac{1}{\sqrt{2N}} \exp[\tfrac{x}{2}(1 - \tfrac{1}{N})] = \frac{1 - w}{\sqrt{2N}} \sum_{k=0}^{\infty} L_k(x) w^k \tag{12}$$

and so

$$\begin{aligned} J_N &= \sum_{k=n+1}^{\infty} w^{2k} \frac{(1 - w)^2}{2N} = \frac{(1 - w)^2}{2N} \cdot \frac{w^{2n} \cdot w^2}{1 - w^2} \\ &= \frac{w^2}{2} \left[(1 - \tfrac{1}{N})^{2N} \cdot (1 + \tfrac{1}{N})^{-2N}\right]^{n/N} . \end{aligned} \tag{13}$$

Since $n/N \to 1$ and $w \to -1$ as $N \to \infty$, assertion (8) follows from (13).□

<u>Lemma 4</u>. <u>Put</u>

$$S_N(x) := e^{-Nx} S^*_N(x) := \int_x^\infty e^{-Nt}\, P^*_{N-100[\sqrt{N}],N}(t)\,dt \ . \tag{14}$$

<u>Then</u>, $S^*_N \in \Pi_{N-100[\sqrt{N}]}$ <u>and</u>

$$\lim_{N\to\infty} \max_{x\in[0,2]} |e^{-x} - e^{-2} - S_N(x)| = 0. \tag{15}$$

Proof. The fact that $S_N^* \in \Pi_{N-100[\sqrt{N}]}$ follows easily by induction. In view of Lemma 3, the sequence

$$\left\{ \int_0^\infty |e^{-Nt} P^*_{N-100[\sqrt{N}],N}(t)|^2 dt \right\}$$

is bounded. Let $\delta > 0$. Using arguments similar to those in [2], [5], [8], it follows that, for each $p > 0$,

$$\lim_{N\to\infty} \int_{2+\delta}^\infty |e^{-Nt} P^*_{N-100[\sqrt{N}],N}(t)|^p dt = 0. \tag{16}$$

Let $x \in [0, 2]$, $p^*(t) := P^*_{N-100[\sqrt{N}]}(t)$. Then

$$\begin{aligned}|e^{-x} - e^{-2} - S_N(x)| &= \left|\int_x^\infty [e^{-t} - e^{-Nt}p^*(t)]^2 dt - e^{-2}\right| \\ &= \left|\int_x^{2+\delta} [e^{-t} - e^{-Nt}p^*(t)]dt + (e^{-2-\delta} - e^{-2}) - S_N(2+\delta)\right|.\end{aligned} \tag{17}$$

So,

$$\begin{aligned}&\max_{x\in[0,2]} |e^{-x} - e^{-2} - S_N(x)| \\ &\le \int_0^{2+\delta} |e^{-t} - e^{-Nt} p^*(t)|dt + |e^{-2-\delta} - e^{-2}| + |S_N(2+\delta)| \\ &\le \sqrt{2+\delta}\left\{\int_0^{2+\delta} |e^{-t} - e^{-Nt} p^*((t)|^2 dt\right\}^{\frac{1}{2}} + |e^{-2-\delta} - e^{-2}| \\ &\quad + |S_N(2+\delta)|.\end{aligned} \tag{18}$$

In view of Lemma 3 and (16),

$$\int_0^{2+\delta} |e^{-t} - e^{-Nt}p^*(t)|^2 dt = J_N - \int_{2+\delta}^{\infty} |e^{-t} - e^{-Nt}p^*(t)|^2 dt$$
$$= \frac{1}{2} e^{-4} - \frac{1}{2} e^{-4-2\delta} + o(1), \quad N \to \infty . \tag{19}$$

Substituting (19) into (18) and noting that (16) also implies that $S_N(2 + \delta) = o(1)$ as $N \to \infty$, we get

$$\limsup_{N \to \infty} \max_{x \in [0,2]} |e^{-x} - e^{-2} - S_N(x)|$$
$$\leq \sqrt{2 + \delta}\, \{\frac{1}{2} e^{-4} - \frac{1}{2} e^{-4-2\delta}\}^{1/2} + |e^{-2} - e^{-2-\delta}| . \tag{20}$$

Finally, (15) follows on letting $\delta \to 0$. □

__Corollary 5__. __With__ S_N^* __defined in__ (14),

$$\lim_{N\to\infty} \max_{|t|\leq 1} |e^{-2t^2} - e^{-2} - e^{-2Nt^2} S_N^*(2t^2)| = 0. \tag{21}$$

__Proof of Theorem 1__. In view of the classical Weierstrass theorem, we assume without loss of generality that f is a polynomial. Since $f(2) = 0$, the function g defined by

$$g(x) := \frac{f(x)}{e^{-x} - e^{-2}} , \tag{22}$$

is continuous for all x. Let $e > 0$ and set

$$G := \max_{x\in [0,3]} |g(x)|, \quad \lambda := e/(G+2), \quad \eta := \min(1, \lambda). \tag{23}$$

By the Weierstrass theorem and Lemma 4, there exist an integer N and $R \in \Pi_{9[\sqrt{N}]}$ such that

$$|g(x) - R(x)| < \eta, \quad x \in [0, 3] \tag{24a}$$

and

$$|e^{-x} - e^{-2} - e^{-Nx} S_N^*(x)| < \eta, \quad x \in [0, 2]. \tag{24b}$$

Then, with $P := RS_N^* \in \Pi_{N-90[\sqrt{N}]}$, we have

$$|f(x) - e^{-Nx} P(x)| \le G \cdot \eta + (1 - e^{-2} + \eta) \cdot \eta < (G + 2)\eta \le \epsilon \tag{25}$$

for $x \in [0, 2]$.

It remains to show that the integer N can be selected so that

$$|e^{-Nx}P(x)| < \epsilon, \quad x \in [2, \infty) \tag{26}$$

where $P = RS_N^*$ and $R \in \Pi_{9[\sqrt{N}]}$ depends on N. Let $\delta < 1$ be a fixed positive number that will be specified later. Since the sequence $\{\exp(-Nx)R\, S_N^*\}$ is uniformly bounded on $[0, 2]$, it follows (as with (16)) that

$$\max_{[2+\delta,\infty)} |e^{-Nx}P(x)| = \max_{[2+\delta,\infty)} |e^{-Nx}R(x)S_N^*(x)| \to 0 \quad \text{as} \quad N \to \infty. \tag{27}$$

For $x \in [2, 2 + \delta]$ we have from (24a) and (14)

$$|e^{-Nx}R(x)S_N^*(x)| \le (G + 1) \int_x^\infty |e^{-Nt}P^*(t)|dt, \tag{28}$$

where $P^*(t) = P^*_{N-100[\sqrt{N}],N}(t)$. Furthermore, for $x \in [2, 2 + \delta]$,

$$\begin{aligned}\int_x^\infty |e^{-Nt}P^*(t)|dt &\le \int_2^{2+\delta} |e^{-Nt}P^*(t)|dt + \int_{2+\delta}^\infty |e^{-Nt}P^*(t)|dt \\ &\le \sqrt{\delta} \cdot \left\{\int_2^{2+\delta} |e^{-Nt}P^*(t)|^2 dt\right\}^{\frac{1}{2}} \\ &\qquad + \int_{2+\delta}^\infty |e^{-Nt}P^*(t)|dt .\end{aligned} \tag{29}$$

Now from (16), the last integral in (29) tends to zero as $N \to \infty$, and

and also, from Lemma 3, the sequence

$$\left\{ \int_2^{2+\delta} |e^{-Nt}p^*(t)|^2 dt \right\}^{\frac{1}{2}}$$

is bounded, say by the constant A. Thus

$$\limsup_{N \to \infty} \max_{x \in [2,2+\delta]} \int_x^{\infty} |e^{-Nt}p^*(t)| dt \le A \cdot \sqrt{\delta} . \tag{30}$$

Finally, using (30), (28) and (27), we see that it is possible to select δ and N so that (26) is satisfied. □

Proof of Theorem 2. This is similar to the proof of Theorem 1. We use Corollary 5 instead of Lemma 4. □

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