A de Montessus-type theorem for CF approximation

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Abstract: The following de Montessus-type theorem for Carathéodory–Fejér (CF) approximants is proven: Let f be meromorphic in $|z| < \rho$ ($\rho > 1$), analytic in $|z| \leqslant 1$, and with a total of n poles $\zeta_1, \ldots, \zeta_n$ (multiplicity included) in $1 < |z| < \rho$. Then, as $m \to \infty$, the CF approximants r_{mn}^{cf} of f from R_{mn} converge on $\{z \in \mathbb{C}; \; |z| < \rho\} \setminus \{\zeta_1, \ldots, \zeta_n\}$ to f, uniformly on every compact subset. Here, r_{mn}^{cf} may be either the type 1 or the type 2 CF approximant, and a similar result holds for the untruncated CF approximant.

1. Introduction

The basic convergence result of the Padé theory, the de Montessus de Ballore theorem, describes the convergence of Padé fractions of meromorphic functions when the numerator degree m tends to infinity while the denominator degree n remains fixed. An analogous result, due to Walsh [8,9] holds for best uniform rational approximations on closed bounded sets whose complement is connected and has a Green's function with pole at infinity. While de Montessus' proof of his theorem was based on Hadamard's theory for locating polar singularities and thus made this result appear to be restricted to the Padé setting, Saff's proof [5] and its abbreviated version due to Shapiro [6] applies also to more general interpolation schemes and resembles Walsh's proof for best uniform approximants. Here the same basic ideas are modified to establish such a convergence result for rational Carathéodory–Fejér (CF) approximation introduced by Trefethen [7]. In the setting considered here, CF approximation is based on the generalization, due to Clark [2,3] and Adamjan, Arov, and Krein [1] of classical minimal extension results of Carathéodory and Fejér, Schur, and Takagi. (For these and further related references see [4].)

2. Notation and definitions

Let $D(\rho) := \{z \in \mathbb{C}; |z| < \rho\}$, $S(\rho) := \{z \in \mathbb{C}; |z| = \rho\}$, $D := D(1)$, and $S := S(1)$. The L^∞-norm on a compact set A is denoted by $\| \cdot \|_A$.

P_m is the set of complex polynomials of degree at most m. R_{mn} is the set of rational functions of numerator degree at most m and denominator degree at most n, and $R_{mn}^0 \subset R_{mn}$ is the subset of functions analytic on the closed unit disk $\overline{D}$. Finally, $\tilde{R}_{mn}$ is the set of functions representable in the form $\tilde{r} = \tilde{p}/q$ where $q \in P_n$ has no zeros in $\overline{D}$ and $z \to z^{-m}\tilde{p}(z)$ is bounded analytic in $1 < |z| \leqslant \infty$. (Thus, $R_{mn}^0 \subset \tilde{R}_{mn}$.)

The *untruncated CF approximation* $\tilde{r}_{mn}$ of $f \in L^\infty(S)$ is a best approximation of f from $\tilde{R}_{mn}$ in the $\| \cdot \|_S$ norm. If f is continuous (as in our theorems below) this best approximation is unique. According to the theory of Adamjan, Arov, and Krein [1], $\tilde{r}_{mn} = \tilde{p}_{mn}/q_{mn}$ can be computed by determining the Schmidt series (i.e., the singular value decomposition) of an infinite Hankel matrix whose entries are Fourier coefficients of f. The rational CF approximant is then obtained by projecting $\tilde{r}_{mn}$ suitably into R_{mn}^0. We distinguish two types [4,7]: If

$$\tilde{p}_{mn}(z) = \sum_{k=-\infty}^{m} a_k z^k, \quad 1 < |z| < \infty, \tag{1}$$

$$\tilde{r}_{mn}(z) \sim \sum_{k=-\infty}^{\infty} c_k z^k \quad z \in S \tag{2}$$

(the latter being a Fourier series), the *type 1* and the *type 2 CF approximations* $r_{mn}^{(1)}$, $r_{mn}^{(2)} \in R_{mn}^0$ are defined by

$$r_{mn}^{(1)} := p_{mn}^{(1)}/q_{mn}, \qquad p_{mn}^{(1)}(z) := \sum_{k=0}^{m} a_k z^k, \tag{3}$$

and, if $m \geqslant n - 1$,

$$r_{mn}^{(2)} := p_{mn}^{(2)}/q_{mn} \sim \sum_{k=0}^{\infty} c_k z^k, \tag{4}$$

respectively [4,7]. Note that $r_{mn}^{(2)}(z) \sim \tilde{r}_{mn}(z) - \sum_{k<0} c_k z^k$, where the series is analytic in $|z| > 1$ and zero at ∞ (i.e., the series is in $(H^2)^\perp$); hence $r_{mn}^{(2)} = O(z^{m-n})$ as $z \to \infty$ and the denominator is indeed q_{mn}.

We prove first the de Montessus-type theorem for the untruncated CF approximants of a meromorphic function, then we show that truncation does not effect the convergence in an asymptotically essential way.

3. Convergence of the untruncated CF approximants

Theorem 1. *Let f be meromorphic in $|z| < \rho$ ($\rho > 1$), analytic in $|z| \leqslant 1$, and with a total of n poles $\zeta_1, \ldots, \zeta_n$ (multiplicity included) in $1 < |z| < \rho$. Then, as $m \to \infty$, the untruncated CF approximations $\tilde{r}_{mn}$ of f from $\tilde{R}_{mn}$ ($m \in \mathbb{Z}$) converge on*

$$S_f := \{1 \leqslant |z| < \rho\} \setminus \{\zeta_1, \ldots, \zeta_n\}$$

to f, uniformly on every compact subset K of S_f. In particular,

$$q_{mn}(z) \to q(z) := \prod_1^n (z - \zeta_k), \quad \forall z \in \mathbb{C},$$

and $\tilde{p}_{mn} \to fq$ on $\{1 \leqslant |z| < \rho\}$ if q_{mn} is monic.

Proof. The function $s(z) := f(z)q(z)$ is analytic in $D(\rho)$ and the mth partial sum s_m of its Taylor series satisfies for each $1 \leqslant \tau < \rho$

$$\limsup_{m \to \infty} \| s - s_m \|_{S(\tau)}^{1/m} \leqslant \tau/\rho < 1. \tag{5}$$

Consequently, $t_{mn} := s_m/q \in R_{mn}^0$ converges to f on $S_f^0 := S_f \cup D = D(\rho) \setminus \{\zeta_1, \ldots, \zeta_n\}$, uniformly on compact subsets. In particular,

$$\limsup_{m \to \infty} \| f - t_{mn} \|_S^{1/m} \leqslant 1/\rho. \tag{6}$$

By the optimality of $\tilde{r}_{mn}$, $\| f - \tilde{r}_{mn} \|_S \leqslant \| f - t_{mn} \|_S$, thus also

$$\limsup_{m \to \infty} \| t_{mn} - \tilde{r}_{mn} \|_S^{1/m} \leqslant 1/\rho. \tag{7}$$

Let us assume for the moment that the denominators q_{mn} are renormalized such that $\| q_{mn} \|_S = 1$. Then (7) implies

$$\limsup_{m \to \infty} \| s_m q_{mn} - \tilde{p}_{mn} q \|_S^{1/m} \leqslant 1/\rho. \tag{8}$$

Therefore, the function $[s_m(z)q_{mn}(z) - \tilde{p}_{mn}(z)q(z)]/z^{m+n}$, which is analytic in $1 < |z| \leqslant \infty$ and has a nontangential limit almost everywhere on S, is bounded in $1 < |z| \leqslant \infty$ by ρ_1^{-m} (for arbitrary $\rho_1 < \rho$) times a constant γ independent of m. Consequently, on $S(\tau)$ with $1 \leqslant \tau < \rho$,

$$\| s_m q_{mn} - \tilde{p}_{mn} q \|_{S(\tau)} \leqslant \gamma \tau^{m+n}/\rho_1^m, \tag{9}$$

so that

$$\limsup_{m \to \infty} \| s_m q_{mn} - \tilde{p}_{mn} q \|_{S(\tau)}^{1/m} \leqslant \tau/\rho < 1. \tag{10}$$

Now extract from $\{ q_{mn} \}_{m=0}^\infty \subset P_n$ any convergent subsequence, $\{ q_{mn} \}_{m \in M}$, and let q^* be its limit. Consider a disk $\Delta \subset D(\rho)$ around a zero ζ_k of multiplicity ν_k such that $\Delta \cap D = \emptyset$, Δ contains neither other zeros of q nor zeros of s, and q^* does not vanish on the boundary $\partial \Delta$. Clearly, such a disk Δ can be found by choosing its radius small enough. For sufficiently large $m \in M$ both $|s_m|$ and $|q_{mn}|$ are bounded away from zero on Δ and $\partial \Delta$, respectively. By Rouché's theorem it follows then from (10) that $|q_{mn}|$ has at least ν_k zeros in Δ. But since this construction can be repeated at every zero of q, q_{mn} can have at most ν_k zeros in Δ since its degree is at most n. Finally, since we extracted an arbitrary convergent subsequence, the zeros of the elements of the full sequence $\{ q_{mn} \}_{m=0}^\infty$ converge to the zeros ζ_k of q, so that $q_{mn} \to q$ if q_{mn} is chosen monic.

On any compact set $K \subset S_f$, we now deduce from (5) and (10) that $\tilde{r}_{mn}$ converges uniformly to $f = s/q = \lim s_m/q$. More precisely, if $K \subset \overline{D(\tau)}$, $1 \leqslant \tau < \rho$, then

$$\limsup_{m \to \infty} \| f - \tilde{r}_{mn} \|_K^{1/m} \leqslant \tau/\rho < 1.$$

Of course, the numerators $\tilde{p}_{mn}$ converge to s in S_f and, since $\tilde{p}_{mn}$ and s are analytic at the poles ζ_k, the maximum principle implies convergence of $\tilde{p}_{mn}$ to s throughout $\{1 \leqslant |z| < \rho\}$. $\square$

Remark. In the chordal metric, $\tilde{r}_{mn}$ converges to f on $1 < |z| < \rho$.

4. Convergence of (truncated) CF approximants

Theorem 2. *Under the assumptions of Theorem 1, $r_{mn}^{(1)} \to f$ and $r_{mn}^{(2)} \to f$ on $D(\rho)\setminus\{\zeta_1,\ldots,\zeta_n\}$ as $m \to \infty$, uniformly on every compact subset.*

Proof. Denote by P^+ the orthogonal projection of $L^2(S)$ onto H^2. (P^+f can be written as the Cauchy integral of f on S.) By definition,

$$p_{mn}^{(1)} = P^+(\tilde{p}_{mn}) = P^+(\tilde{r}_{mn}q_{mn}), \qquad r_{mn}^{(2)} = P^+(\tilde{r}_{mn}) \quad \text{if } m \geqslant n - 1.$$

Because $\tilde{p}_{mn}$ and fq are analytic in $1 < |z| < \rho$, we have for $|z| < \tau < \rho$

$$f(z)q(z) - p_{mn}^{(1)}(z) = \int_{S(\tau)} \frac{f(u)q(u) - \tilde{p}_{mn}(u)}{z - u}\,du.$$

By estimating the integral it follows from Theorem 1 that $p_{mn}^{(1)} \to fq$ uniformly on compact subsets of $D(\tau)$, and since $q_{mn} \to q$ our claim on the convergence of $r_{mn}^{(1)}$ is established.

For $r_{mn}^{(2)}$ we get similarly for $z \in K$, K any compact subset of $D(\rho)\setminus\{\zeta_1,\ldots,\zeta_n\}$, and m large

$$f(z) - r_{mn}^{(2)}(z) = \int_\Gamma \frac{f(u) - \tilde{r}_{mn}(u)}{z - u}\,du,$$

where Γ consists of $S(\tau)$ ($\tau < \rho$ such that $K \subset D(\tau)$, $\zeta_k \in D(\tau)(\forall k)$) and of small circles around the poles ζ_k. (Γ is obtained by distorting S in $D(\rho)\setminus\{\zeta_1,\ldots,\zeta_n\}$; sections between $S(\tau)$ and the small circles cancel.) Estimating the integral and applying Theorem 1 establishes the claim concerning $r_{mn}^{(2)}$. $\square$

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