Fig. 10(b) Bézier coefficients $M_{2,2,1}$ on one quarter of support

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APPROXIMATION BY POLYNOMIALS THAT OMIT A POWER OF Z

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Abstract

Let E be a compact set in the z-plane and f a function analytic in the interior of E and continuous on E. Our goal is to study approximation to f on E by polynomials that omit a single power of z. Such questions arise, for example, in the analysis of certain numerical methods where it is required to approximate on E the function $f(z)\equiv 1$ by polynomials with zero constant term (cf. [4]).

1. Introduction

One important aspect of our investigation is the connection of the above problem with inequalities of Bernstein and Markoff type that relate the growth of the derivative of a polynomial to the norm of the polynomial on E. To be specific, let $\mathbf{P}_n$ denote the collection of all algebraic polynomials (with complex coefficients) of degree at most n and set

$$\mu_n(z_0)=\mu_n(z_0,E):=\sup\left\{\frac{|p'(z_0)|}{\|p\|_E},\ p\in\mathbf{P}_n,\ p\not\equiv 0\right\}, \tag{1.1}$$

where $\|.\|_E$ denotes the sup norm on E. We shall examine the behaviour

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of the sequence $\mu_n(z_0)$ as $n \to \infty$. In the classical settings where E is a disk or a line segment, this behaviour is well-known from the sharp inequalities of Bernstein and Markoff. But for general sets E, no mention is made in the recent monograph of Rahman ane Schmeisser [5] as to whether $\mu_n(z_0) \to \infty$. We prove as a special case of Corollary 2.7 and and Theorem 2.9, that if the complement $\mathbf{C} \setminus E$ is connected and z_0 lies either on the boundary of E or in the exterior of E, then $\mu_n(z_0) \to \infty$.

The inspiration for the present work comes from two sources. First the elegant paper of G. Szego [6] that analyzes the behaviour of $\mu_n(z_0)$ in the case when E is a Jordan region bounded by finitely many analytic arcs. Second is the work of M. Hasson [1], [2] who studied the convergence behaviour of "incomplete polynomials" in the case when E is a real interval. By working in the complex domain, the present paper lends geometric insight to the later results.

The outline of the paper is as follows. In Section 2 we introduce the necessary notation and discuss some properties of best uniform approximation by polynomials that omit the term z^k. In Section 3, we give some asymptotic results for the error in this uniform approximation and provide examples of our results.

2. Notation and Basic Properties

We assume throughout that the compact set E contains infinitely many points. As in the introduction, $\|.\|_E$ denotes the uniform norm on E and $\mathbf{P}_n$ the collection of all polynomials of degree at most n. We also define

$$B_n(E) := \{p \in \mathbf{P}_n : \|p\|_E \leq 1\}, \tag{2.1}$$

$$\mu_{n,k}(E) := \sup\{|p^{(k)}(0)| : p \in B_n(E)\}, \tag{2.2}$$

$$P_{n,k} := \{p(z) = \sum_{j=0}^{n} a_j z^j;\ a_k = 0\} = \{p \in \mathbf{P}_n;\ p^{(k)}(0) = 0\}, \tag{2.3}$$

$$C(E) := \{f;\ f \text{ continuous on } E\}, \tag{2.4}$$

$$A(E) := \{f \in C(E);\ f \text{ analytic in the interior of } E\}, \tag{2.5}$$

$$E_n(f) := \inf\{\|f - p\|_E;\ p \in \mathbf{P}_n\}, \tag{2.6}$$

$$E_{n,k}(f) := \inf\{\|f - p\|_E;\ p \in P_{n,k}\}. \tag{2,7}$$

For k a fixed nonnegative integer and $f \in A(E)$, we shall examine the asymptotic behaviour of $E_{n,k}(f)$ as $n \to \infty$, and its relationship to the behaviour of the sequence $\mu_{n,k}(E)$. We begin with some basic lemmas.

Lemma 2.1. If k is any nonnegative integer, then

$$E_{n,k}(z^k) = \frac{k!}{\mu_{n,k}(E)}, \quad n \geq k. \tag{2.8}$$

The proof of this lemma is straightforward using the definitions given above for $\mu_{n,k}(E)$ and $E_{n,k}(f)$.

Lemma 2.2. Let $f \in C(E)$ and $p_n^*(z) = \sum_{j=0}^{n} a_{n,j}^* z^j \in P_n$ be the polynomial of best uniform approximation to f on E; that is

$$E_n(f) = \| f - p_n^* \|_E.$$

Then for all $k \geq n$,

$$\frac{|a_{n,k}^*| k!}{\mu_{n,k}(E)} - E_n(f) \leq E_{n,k}(f) \leq \frac{|a_{n,k}^*| k!}{\mu_{n,k}(E)} + E_n(f). \tag{2.9}$$

Proof. Let $p_{n,k}^* \in P_{n,k}$ satisfy

$$E_{n,k}(f) = \| f - p_{n,k}^* \|_E.$$

Then

$$E_{n,k}(f) \geq \| p_n^* - p_{n,k}^* \|_E - \| f - p_{n,k}^* \|_E$$

$$= \| a_{n,k}^* z^k - q \|_E - E_n(f) \quad (q \in P_{n,k})$$

$$\geq |a_{n,k}^*| E_{n,k}(z^k) - E_n(f).$$

Using Lemma 2.1, we then get the lower bound in (2.9). The upper estimate in (2.9) is similarly established.

Lemma 2.3. Let f, p_n^* be as in Lemma 2.2. For k a fixed nonnegative integer, $\lim_{n \to \infty} E_{n,k}(f) = 0$ if and only if

$$\lim_{n \to \infty} E_n(f) = 0 \text{ and } \lim_{n \to \infty} \frac{|a_{n,k}^*|}{\mu_{n,k}(E)} = 0. \tag{2.10}$$

Proof. If $E_{n,k}(f)\to 0$, then clearly $E_n(f)\to 0$ as $n\to\infty$. From the lower estimate in (2.9) we then deduce that

$$\lim_{n\to\infty} |a^*_{n,k}|/\mu_{n,k}(E)=0.$$

The sufficiency of the conditions (2.10) follows immediately from the upper estimate in (2.9).

Lemma 2.4. If $0\in\overset{\circ}{E}$, the interior of E, then there exist positive constants $c_1=c_1(E, k)$, $c_2=c_2(E, k)$ such that

$$0<c_1\leqslant\mu_{n,k}(E)\leqslant c_2,\quad n\geq k. \tag{2.11}$$

Proof. The upper bound in (2.11) follows by applying the Cauchy estimates on a closed disk about zero contained in E (cf. Szego[6]). We get the positive lower bound by considering $p(z)=z^k/\|z^k\|_E\in B_n(E)$ for $n\geq k$.

We remark that, by Mergelyan's theorem, $E_n(f)\to 0$ for all $f\in A(E)$ if and only if the compact set E does not separate the plane, that is, $\mathbf{C}\setminus E$ is connected.

Theorem 2.5. Suppose $0\in\overset{\circ}{E}$ and $\mathbf{C}\setminus E$ is connected. Let $f\in A(E)$. Then $\lim\limits_{n\to\infty} E_{n,k}(f)=0$ if and only if $f^{(k)}(0)=0$.

Proof. Let $p^*_n(z)=\sum\limits_{j=0}^{n} a^*_{n,j}z^j\in\mathbf{P}_n$ be the polynomial of best unifrom approximation to $f(z)$ on E. By Mergelyan's theorem, $p^*_n(z)$ converges uniformly to $f(z)$ on E. Since $0\in\overset{\circ}{E}$, it follows from the convergence of the derived sequences that

$$\lim_{n\to\infty} a^*_{n,k}=\frac{f^{(k)}(0)}{k!}.$$

The assertion of the theorem is now immediate from Lemmas 2.3 and 2.4.

The case when zero lies exterior to E is also easy to handle.

Theorem 2.6. Let $f\in A(E)$ and assume $\mathbf{C}\setminus E$ is connected. If $0\notin E$, then for each $k=0, 1, \cdots,$

$$\lim_{n\to\infty} E_{n,k}(f)=0. \tag{2.12}$$

Proof. Set $F(z):=f(z)/z^{k+1}$. Then $F\in A(E)$ and, since $\mathbf{C}\setminus E$ is connected, there exists a sequence of polynomials p_n, $p_n \in \mathbf{P}_n$, $n=0, 1, \cdots$, such that $p_n(z) \to F(z)$ uniformly on E. But then $z^{k+1} p_n(z) \to f(z)$ uniformly on E and so (2.12) follows.

On taking $f(z) = z^K$ in Theorem 2.6 and applying Lemma 2.1 we obtain:

Corollary 2.7. If $0 \notin E$ and $\mathbf{C} \setminus E$ is connected, then

$$\lim_{n\to\infty} \mu_{n,k}(E) = \infty, \quad k=0, 1, \cdots.$$

It remains to consider the more interesting case when $0 \in \partial E$, the boundary of E. The situation for $k=0$ is trivial.

Lemma 2.8. Let $f \in A(E)$, with $\mathbf{C} \setminus E$ connected, and assume $0 \in \partial E$. Then $\lim_{n\to\infty} E_{n,0}(f) = 0$ if and only if $f(0) = 0$.

It was shown by Szegö[6] that if E is a closed Jordan region bounded by a finite number of analytic arcs and if the exterior angle at $0 \in \partial E$ is $\alpha\pi$, with $0 < \alpha \leqslant 2$, then there exist positive constants A, B such that

$$Bn^{\alpha} \leqslant \mu_{n,1}(E) \leqslant An^{\alpha}, \quad n=1, 2, \cdots. \tag{2.13}$$

He has also proved that if the exterior angle at $0 \leqslant \partial E$ is zero, then $\mu_{n,1}(E)$ can increase arbitrarily slowly as $n \to \infty$. Szego did not prove, however, that $\mu_{n,1}(E)$ necessarily tends to infinity in such a case. The next theorem establishes this fact in a much more general geometric setting.

Theorem 2.9. Assume $\mathbf{C} \setminus E$ is connected and $0 \in \partial E$. Then for each $k=1, 2, \cdots$,

$$\lim_{n\to\infty} \mu_{n,k}(E) = \infty. \tag{2.14}$$

Theorem 2.9 is a consequence of the following result.

Theorem 2.10. Assume $\mathbf{C} \setminus E$ is connected and $0 \in \partial E$. If $f \in A(E)$, then for each $k=1, 2, \cdots$,

$$\lim_{n\to\infty} E_{n,k}(f) = 0. \tag{2.15}$$

Theorem 2.10 is actually a special case of a result due to A.A. Nersesyan[3] which establishes uniform convergence to f on E by polynomials having prescribed derivatives at finitely many points on ∂E. We shall give a simple proof of Theorem 2.10 that follows from a key lemma used in the proof of Mergelyan's theorem, namely

Lemma 2.11. Suppose $\mathbf{C} \setminus E$ is connected and $\zeta \in \partial E$. Then for each

δ, $0<\delta<1$, there is a polynomial $p(z)$ for which

$$|p(z)|\leqslant A/\delta,\ z\in E, \tag{2.16}$$

$$\left|p(z)-\frac{1}{\zeta-z}\right|<\frac{B\delta^2}{|\zeta-z|^3},\quad z\in E\cap\{z;\ |\zeta-z|>10\delta\}, \tag{2.17}$$

where A, B are absolute constants.
(See, for example, the appendix of Walsh's book [7, p.369].)

Proof of Theorem 2.10. To prove Theorem 2.10 it is, in fact, enough to show that given $\varepsilon>0$ there is a polynomial q_1 such that

$$\|z-z^2 q_1\|_E<\varepsilon. \tag{2.18}$$

Note that by considering

$$(z-z^2q_1)^k=z^k-z^{k+1}q_k$$

for some polynomial q_k, it follows from (2.18) that given a positive integer k and $\varepsilon>0$, there is a polynomial q_k such that

$$\|z^k-z^{k+1}\ q_k\|_E<\varepsilon \tag{2.19}$$

To deduce Theorem 2.10, first of all, given $\varepsilon>0$, choose a polynomial p, using Mergelyan's theorem, so that

$$\|f-p\|_E<\varepsilon.$$

Assume $p\in P_n$ with $n>k$ and write $p(z)=\sum_0^n a_j z^j$. From (2.19) we see that there is a polynomial q such that

$$\|a_k z^k-z^{k+1}\ q\|_E<\varepsilon$$

and therefore

$$\|f-(p-a_kz^k+z^{k+1}\ q)\|_E<2\varepsilon.$$

and $p-a_kz^k+z^{k+1}\ Q\in P_{N,K}$ for all large N. Hence Theorem 2.10 follows.

It remains to show that (2.18) is true. Consider Lemma 2.11 with $\zeta=0$, $0<\delta<1$. From (2.16) and (2.17)

$$|z^2p(z)|<A|z|^2/\delta,\quad (z\in E)$$

$$|z^2p(z)+z|<\frac{B\delta^2}{|z|},\ (z\in E,\ |z|>10\delta).$$

Hence for $z\in E$, $|z|\leqslant 10\delta$,

$$|z^2p(z)+z|\leqslant|z^2p(z)|+|z|$$

$$<\frac{A}{\delta}100\delta^2+10\delta=(100A+10)\delta;$$

and for $z\in E$, $|z|>10\delta$,

$$|z^2p(z)+z|<B\delta/10.$$

Given $\varepsilon>0$, choose δ so that

$$\max\{(100A+10)\delta,\ B\delta/10\}>\varepsilon.$$

We then obtain (2.18) by setting $q_1:=-p$.

As an application of Theorem 2.10 we obtain the folloowing best possible result concerning the behaviour of coefficients for a uniformly convergent polynomial sequence.

Corollary 2.12. Suppose $\mathbf{C}\setminus E$ is connected and $0\in\partial E$. If $p_n(z)=\sum_{j=0}^{n}a_{n,j}z^j$ is a sequence of polynomials which converge uniformly on ∂E, then, for each $k=1, 2, \cdots$,

$$a_{n,k}=o(\mu_{n,k}(E))\text{ as }n\to\infty. \tag{2.20}$$

Proof. Since $\mathbf{C}\setminus E$ is connected and p_n converges uniformly on ∂E, then p_n converges uniformly on E to a function $f\in A(E)$. As in the lower bound of (2.9) we have

$$\frac{|a_{n,k}|k!}{\mu_{n,k}(E)}\leqslant e_n(f)+E_{n,k}(f), \tag{2.21}$$

where

$$e_n(f):=\|f-p_n\|_E.$$

From Theorem 2.10 we know that $E_{n,k}(f)\to o$ as $n\to\infty$ for each $k\geq 1$. Since, also, $E_n(f)\to o$ as $n\to\infty$, the assertion (2.20) follows.

If, in Corollary 2.12, zero lies outside E, then the conclusion (2.20) holds for every $k\geq 0$.

We also remark that (2.20) is best possible in the following sense. Let ε_n be any sequenec such that $\varepsilon_n\to o$ as $n\to\infty$. Set $p_n(z)=\varepsilon_n p_n(z)$, where $p_n(z)\in B_n(E)$ satisfies

$$\mu_{n,k}(E)=|p_n^{(k)}(0)|.$$

Then the sequence p_n converges uniformly on E (to the zero funciion) and

$$\frac{|a_{n,k}|}{\mu_{n,k}(E)}=\frac{|\varepsilon_n|}{k!}.$$

Example 2.13. If E is an interval, say $E=[a,b]$, then it follows from Corollary 2.12 and the inequalities of Markoff and Bernstein [5] that if $p_n(x)=\sum_{j=0}^{n} a_{n,j}x^j$ converges uniformly on $[a, b]$, then, as $n\to\infty$,

$$a_{n,k}=o(n^k),\ k=1, 2, \cdots, \text{ if } 0\in(a, b), \tag{2.22}$$

while

$$a_{n,k}=o(n^{2k}),\ k=1, 2, \cdots, \text{ if } a=0 \text{ or } b=0. \tag{2.23}$$

If E is a closed disk, say $E=\{z;\ |z|\leqslant R\}$, and $\{P_n(z)\}$ is a sequence of polynomials of respective degrees at most n which converge uniformly on $|z|=R$, then for any point ξ with $|\xi|=R$ and any $k\geq 1$, there holds

$$|p_n^{(k)}(\xi)|=o(n^k) \text{ as } n\to\infty. \tag{2.24}$$

3. Asymptotic Results and Examples.

In this section we provide some asymptotic formulas for the error $E_{n,k}(f)$.

Lemma 3.1. Let $f\in C(E)$ and suppose $f^{(k)}(0)$ exists and is nonzero. Let $p_n^*(z)=\sum_{j=0}^{n} a_{n,j}^*\ z^j\in P_n$ be the polynomial of best uniform approximation to $f(z)$ on E. If

$$\lim_{n\to\infty} a^*_{n,k}=\frac{f^{(k)}(0)}{k!} \tag{3.1}$$

and

$$\lim_{n\to\infty} \mu_{n,k}(E)\,E_n(f)=0. \tag{3.2}$$

then

$$E_{n,k}(f)\approx\frac{|f^{(k)}(0)|}{\mu_{n,k}(E)} \text{ as } n\to\infty. \tag{3.3}$$

Proof. Multiplying the inequalities (2.9) of Lemma 2.2 by $\mu_{n,k}(E)|f^{(k)}(0)|$ and using conditions (3.1) and (3.2) immediately gives (3.3).

Theorem 3.2. Suppose E is a compact set whose complement K is connected and regular in the sense that K possesses a Green's function $G(z)$ with pole at ∞. Assume that $f(z)$ is analytic on E and $0\in E$. Then, if $f^{(k)}(0)\neq 0$, the asymptotic formula (3.3) holds.

Proof. It is well-known (cf. Walsh [7, §4.7]) that since f is analytic on E,

$$\limsup_{n\to\infty}[E_n(f)]^{1/n}<1, \tag{3.4}$$

and the polynomials $p_n^*(z)$ converge uniformly to f on some open set containing E. The latter property implies (3.1). Moreover, it is a simple consequonce of the Bernstein-Walsh lemma (cf. Walsh [7, §4.6]) that

$$\limsup_{n\to\infty} [\mu_{n,k}(E)]^{1/n} \leqslant 1. \qquad (3.5)$$

The inequalities of (3.4) and (3.5) imply that (3.2) holds. Hence, by Lemma 3.1, the asymptotic formula (3.3) follows.

A similar argument gives

Theorem 3.3. let E be as in Theorem 3.2 and suppose that $0\in\Gamma_R := \{z : G(z) = \log R\}$ $(R>1)$. If $f(z)$ is analytic inside and on Γ_R and $f^{(k)}(0) \neq 0$, then the asymptotic formula 3.3 holds.

Example 3.4. Let E be the closed disk $|z-c| \leqslant |c|$ $(c\neq 0)$ so that 0 lies on ∂E. Then, as is known,

$$\mu_{n,k}(E) = \frac{k!}{|c|^k}\begin{bmatrix} n \\ k \end{bmatrix}, \quad n\geq k. \qquad (3.6)$$

Hence, by Theorem 3.2, if f is analytic on E and $f^{(k)}(0)\neq 0$, then

$$E_{n,k}(f) \approx \frac{|f^{(k)}(0)|\ \ |c|^k}{k!\begin{bmatrix} n \\ k \end{bmatrix}} \quad \text{as } n\to\infty. \qquad (3.7)$$

Example 3.5. Let Ebe the closed disk $|z-c|\leqslant\rho$, where $|c|>\rho>0$, so that 0 lies exterior to E. Then

$$\mu_{n,k}(E) = \frac{k!\rho^n}{|c|^{n+k}}\begin{bmatrix} n \\ k \end{bmatrix}. \qquad (3.8)$$

Hence, by Theorem 3.3, if f is analytic on $|z-c|\leqslant|c|$ and $f^{(k)}(0)\neq 0$, then

$$E_{n,k}(f) \approx \frac{|f^{(k)}(0)|\ \ |c|^{n+k}}{k!\rho^n\begin{bmatrix} n \\ k \end{bmatrix}} \quad \text{as } n\to\infty. \qquad (3.9)$$

Example 3.6. Let $E=[0, b]$, $b>0$. Then

$$\mu_{n,k}(E) = \left[\frac{2}{b}\right]^k T_n^{(k)}(1), \quad \text{as } n\to\infty, \tag{3.10}$$

where $T_n(x) = \cos(n \arccos x)$ is the Chebyshev polynomial of the first kind. Hence, by Theorem 3.2, if f is analytic on [0, b] and $f^{(k)}(0) \neq 0$, then

$$E_{n,k}(f) \approx \frac{|f^{(k)}(0)|}{T_n^{(k)}(1)} \left[\frac{b}{2}\right]^k \text{ as } n\to\infty. \tag{3.11}$$

Example 3.7. Let $E=[a, b]$, where $0<a<b$. Then

$$\mu_{n,k}(E) = \left[\frac{2}{b-a}\right]^k T_n^{(k)} \left[\frac{b+a}{b-a}\right], \quad \text{as } n\to\infty. \tag{3.12}$$

Hence, by Theorem 3.3, if $f(z)$ is analytic inside and on the ellipse with foci at a, b and passing through the origin, and if $f^{(k)}(0) \neq 0$, then

$$E_{n,k}(f) \approx \frac{|f^{(k)}(0)|}{T_n^{(k)}\left[\frac{b+a}{b-a}\right]} \left[\frac{b-a}{2}\right]^k \text{ as } n\to\infty \tag{3.13}$$

Example 3.8. Suppose E is a closed Jordan region bounded by a finite number of analytic Jordan arcs. Let $0\in\partial E$ and suppose that the exterior angle at 0 is $\alpha\pi$, where $0<\alpha\leqslant 2$. If f is analytic on E and $f^{(k)}(0) \neq 0$, then there exist positive constants c_1, c_2 such that

$$c_2/n^{k\alpha} \leqslant E_{n,k}(f) \leqslant c_1/n^{k\alpha}. \tag{3.14}$$

The inequalities (3.14) follow from (3.3) and a previously mentioned result of Szego [6].

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