DISTRIBUTION OF ZEROS OF POLYNOMIAL SEQUENCES, ESPECIALLY BEST APPROXIMATIONS

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In [2] the asymptotic behavior of the zeros of polynomials of near best approximation to functions f on a compact set E was studied in the case when f is not everywhere analytic on E. For example, suppose E is a finite union of compact intervals of the real line and f is continuous, but not analytic on E; then we have shown that every point of E is a limit point of zeros of the polynomials of best uniform approximation to f on E. Moreover, if the complement K of E is simply connected and the boundary of E consists of a finite number of analytic Jordan arcs, then the distribution of the zeros of the polynomials of best uniform approximation was analyzed. The purpose of this paper is to give a new interpretation of this distribution, namely to show that these zeros are uniformly distributed with respect to the normal derivative of G(x,y) on ∂E, where G(x,y) is Green's function of K; furthermore results are obtained for the general case when K is connected.

1. Introduction

Let E be a closed bounded set in the z-plane ($z = x+iy$)

whose complement K (with respect to the extended plane) is connected and regular in the sense that K has a Green's function $G(x,y)$ with pole at infinity: $G(x,y)$ is harmonic in K except at infinity and in a neighborhood of the point of infinity we have

$$(1.1) \qquad G(x,y) = \log (x^2+y^2)^{1/2} + G_0(x,y),$$

where $G_0(x,y)$ approaches a finite value at infinity; moreover, $G(x,y)$ is continuous in the closed region $\overline{K}$ except at infinity and vanishes on the boundary of K (Walsh [8]). The function

$$(1.2) \qquad t = \Phi(z) := e^{G(x,y) + i\,H(x,y)},$$

where $H(x,y)$ is conjugate to $G(x,y)$ in K, maps K onto the exterior of the unit disk. Hence, it follows by (1.1) that

$$(1.3) \qquad |\Phi(z)/z| = 1/c + O(1/z)$$

as $z \to \infty$, where the constant $c > 0$ is called the (logarithmic) capacity of the set E.

For each $\sigma > 1$ we consider the equipotential locus

$$(1.4) \qquad \Gamma_\sigma := \{z = x+iy \in K\colon\ G(x,y) = \log \sigma\}$$

with interior

$$(1.5) \qquad E_\sigma := E \cup \{z = x+iy \in K\colon\ 0 < G(x,y) < \log \sigma\}.$$

If K is simply connected, the function $\Phi(z)$ is single-valued in K and each locus Γ_σ is an analytic Jordan curve. If K is multiply connected, the function $\Phi(z)$ cannot be single-valued in K and has critical points, i.e. points where $\Phi'(z) = 0$. Each locus Γ_σ consists of a finite number of Jordan curves which are mutually exterior except for a finite number of critical points. Moreover, the normal derivative $\frac{\partial G}{\partial n}$ exists at every point of the locus Γ_σ except for such critical points (n being the exterior normal for $\overline{E}_\sigma$).

If a function $f(z)$ is analytic on E, there exists a largest real number σ (finite or infinite), say $\sigma = \rho$, such that $\rho > 1$ and $f(z)$ is single-valued and analytic on E. Then, denoting by Π_n the collection of all complex polynomials of degree $\leq n$, there exist (cf. [8]) polynomials $p_n \in \Pi_n$, $n = 0,1,2, \ldots$, such that we have

$$(1.6) \qquad \overline{\lim_{n\to\infty}} \; \|f-p_n\|_E^{1/n} = 1/\rho,$$

where we denote by $\|\cdot\|_E$ the uniform norm on the set E. But there exist no polynomials $p_n \in \Pi_n$ such that the left-hand side of (1.6) is less than $1/\rho$. A sequence $\{p_n\}$ satisfying (1.6) is said to converge _maximally_ to f(z) on E.

In [2] the following characterization of functions f, which are not analytic on E, is given in terms of the leading coefficients of the polynomials of best uniform approximation to f(z) on E: This result is analogous to the Cauchy-Hadamard formula for the radius of convergence of a power series.

Theorem 1: Let E be a closed bounded point set whose complement K is connected and regular, and suppose that the function f is continuous on E, analytic in the interior of E. For each $n = 0,1,2, \ldots$, let $p_n^*(z) = a_n z^n + \ldots \in \Pi_n$ be the polynomial of best uniform approximation to f on E. Then f is not analytic on E if and only if

$$(1.7) \qquad \overline{\lim_{n\to\infty}} \; |a_n|^{1/n} = 1/c,$$

where c is the capacity of E.

Just the same arguments as in the proof of the above theorem in [2] lead to an analogous result for functions f analytic on E_ρ, namely

Theorem 2: Let E be a closed bounded set whose complement K is connected and regular, and suppose the function f is analytic on E_ρ, $1 < \rho < \infty$, but not on Γ_ρ. Let $\{p_n\}$, $n = 0,1, \ldots$, $p_n(z) = a_n z^n + \ldots \in \Pi_n$, be a sequence of polynomials converging maximally to f(z) on E. Then

(1.8) $\overline{\lim_{n\to\infty}} |a_n|^{1/n} = 1/c\rho,$

where c is the capacity of E.

Hence, the polynomials of best uniform approximation in Theorem 1 or the maximally convergent polynomials of Theorem 2 can be considered as special cases of the following situation: There is given a closed, bounded set E of $\mathbb{C}$ such that the complement K of E is connected and regular, and a polynomial sequence $\{p_n\}$ satisfying the following properties:

(A1) $n \in \mathfrak{N} := \{n_1 < n_2 < n_3 < \ldots\},$

(A2) $p_n(z) \in \Pi_n \setminus \Pi_{n-1}$ with leading coefficient $a_n \neq 0,$

(A3) $\lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{c},$

(A4) $\lim_{n\to\infty} \|p_n\|_E^{1/n} = 1.$

In (A3) and (A4) the limits are considered for $n = n_1, n_2, n_3, \ldots$.

2. Distribution of zeros: K simply connected

In [2] the distribution of the zeros of polynomial sequences satisfying (A1) - (A4) was studied in the neighborhood of an analytic Jordan arc $J \subset \partial K$, when K is simply connected. The equation of an analytic Jordan arc J in $\mathbb{C}$ is given for $z \in J$ in parametric form $z = \gamma(t)$, where t runs through a real compact interval $[a,b]$, $a < b$, $\gamma(t)$ is continuous and $\gamma(t_1) = \gamma(t_2)$ only if $t_1 = t_2$; in addition, $\gamma(t)$ is analytic in the open interval (a,b) and $\gamma'(t) \neq 0$ for all $t \in (a,b)$. Hence, there exists a region Δ, symmetric to the interval (a,b), with the property that $\gamma(t)$ is analytic for all $t \in \Delta$. If, moreover, $J \subset \partial K$ and the region Δ can be chosen in such a way that $\gamma(t) \in K$ when t lies in the upper half Δ^+ of Δ,

(2.1) $\Delta^+ := \{t \in \Delta : \operatorname{Im}(t) > 0\},$

and that $\gamma(t) \notin K$ for $t \in \Delta^-$, where

(2.2) $$\Delta^- := \{t \in \Delta : \operatorname{Im}(t) < 0\},$$

then J is a _free one-sided boundary arc of_ K; if, for an appropriate Δ, $\gamma(t) \in K$ for all $t \in \Delta^+ \cup \Delta^-$, then J is a _free two-sided boundary arc of_ K.

A point $z \in \partial K$ is an _accessible boundary point of_ K if there exists a Jordan arc J with endpoint z such that all other points of J lie in K. If K is simply connected and all points of ∂K are accessible boundary points of K, then, for the inverse mapping $\psi(t)$ of $\Phi(z)$, there exists a continuous extension to $\{t : |t| \leq 1\}$. Therefore, suppose J is a free one-sided boundary arc of K, then there are two arguments α and β, $\alpha < \beta < \alpha + 2\pi$, such that

(2.3) $$\psi^{-1}(J) = \{t = e^{i\varphi} : \alpha \leq \varphi \leq \beta\};$$

if J is a free two-sided boundary arc of K, then

(2.4) $$\psi^{-1}(J) = \{t = e^{i\varphi} : \alpha \leq \varphi \leq \beta \text{ or } \tilde{\alpha} \leq \varphi \leq \tilde{\beta}\},$$

where $\alpha < \beta \leq \tilde{\alpha} < \tilde{\beta} \leq \alpha + 2\pi$.

In stating the next theorem it is convenient to introduce the following notation: For any set C in $\mathbb{C}$ let $Z_n(C)$ be the number of zeros of the polynomial p_n in C, counted with their multiplicities, where $\{p_n\}_{n \in \mathcal{N}}$ is a given sequence of polynomials satisfying (A1) - (A4). Then, in [2], the following result was proved.

Theorem 3: Let E _be a closed bounded set whose complement_ K _is simply connected and suppose that all boundary points of_ K _are accessible boundary points. Furthermore, let_ J _be a subarc in the interior of a free one-sided boundary arc of_ K _such that the connected component_ B _of_ $\mathring{E}$, _where_ $J \subset \overline{B}$, _is a Jordan region, and assume_

(2.5) $$Z_n(C) = o(n) \text{ as } n \to \infty$$

for any compact set C in B. If D is a neighborhood of the interior of J such that $\overline{D} \cap \partial E = J$, then for the distribution of the zeros of the polynomials p_n in D holds

$$(2.6) \qquad \lim_{n\to\infty} \frac{Z_n(D)}{n} = \frac{\beta-\alpha}{2\pi},$$

where α and β are defined by (2.3).

Theorem 4: Let E be a closed bounded point set whose complement K is simply connected and suppose that all boundary points of K are accessible. If J is a subarc in the interior of a free two-sided boundary arc of K and D is a neighborhood of the interior of J such that $\overline{D} \cap \partial E = J$, then

$$(2.7) \qquad \lim_{n\to\infty} \frac{Z_n(D)}{n} = \frac{\beta-\alpha+\tilde{\beta}-\tilde{\alpha}}{2\pi},$$

where $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ are defined by (2.4).

For obtaining results in the next section when K is connected it is useful to give a new interpretation for the right-hand side of (2.6) and (2.7): Suppose the conditions of Theorem 3 are satisfied and J is a free one-sided boundary arc of K, then $\Phi(z)$ can be analytically extended to J and

$$\beta - \alpha = \int_\alpha^\beta dt = \frac{1}{i} \int_J (\log \Phi(z))' dz$$

$$= \frac{1}{i} \int_J \frac{\Phi'(z)}{\Phi(z)} dz = \frac{1}{i} \int_J \left(\frac{\partial G(x,y)}{\partial x} - i \frac{\partial G(x,y)}{\partial y} \right) dz$$

where J is oriented in such a way that K lies to the right. If J has the equation $z = \gamma(t)$, $a \leq t \leq b$, the direction of the tangent is determined by the angle $\alpha = \arg \gamma'(t)$ and we can write

$$\left(\frac{\partial G}{\partial x} - i \frac{\partial G}{\partial y}\right) dz = \left(\frac{\partial G}{\partial x} - i \frac{\partial G}{\partial y}\right) \gamma'(t)\, dt$$

$$= |\gamma'(t)| \left(\frac{\partial G}{\partial x} - i \frac{\partial G}{\partial y}\right) (\cos \alpha + i \sin \alpha)\, dt$$

$$= |\gamma'(t)| \left(\frac{\partial G}{\partial s} + i \frac{\partial G}{\partial n}\right) dt.$$

Here, $\frac{\partial G}{\partial s}$ is the tangential derivative which is identically zero in the interior of J, since $G(x,y) \equiv 0$ on the boundary of K. The expression $\frac{\partial G}{\partial n}$ is the right-hand normal derivative with respect to the curve J. Or, with other words, $\frac{\partial G}{\partial n}$ is the normal derivative where n is the normal directed into K.

Summarizing we have obtained

$$(2.8) \qquad \beta - \alpha = \int_a^b \frac{\partial G}{\partial n} |\gamma'(t)| \, dt = \int_J \frac{\partial G}{\partial n} |dz|.$$

If J is a free two-sided boundary arc of K, we consider the region Δ of section 2 such that $\gamma(t) \in K$ for all $t \in \Delta^+ \cup \Delta^-$. Then, there exists a harmonic extension $G_1(x,y)$ of the function $G(x,y)$, defined in $\gamma(\Delta^+)$, across J into some neighborhood of int(J), where

$$(2.9) \qquad \mathrm{int}(J) = \{\gamma(t): a < t < b\}.$$

This follows from Schwarz's principle of reflection. Moreover, let us denote by n_1 the normal of the curve J directed into $\gamma(\Delta^+)$. Analogously, let $G_2(x,y)$ be the extension of $G(x,y)$, defined in $\gamma(\Delta^-)$, to some neighborhood of int(J) and let n_2 be the normal of the curve J directed into $\gamma(\Delta^-)$. Then we obtain

$$(2.10) \qquad \beta - \alpha + \tilde{\beta} - \tilde{\alpha} = \int_J \left(\frac{\partial G_1}{\partial n_1} + \frac{\partial G_2}{\partial n_2}\right) |dz|.$$

Hence, we may replace in Theorem 3 equation (2.6) by

$$(2.11) \qquad \lim_{n\to\infty} \frac{Z_n(D)}{n} = \frac{1}{2\pi} \int_J \frac{\partial G}{\partial n} |dz|$$

and equation (2.7) in Theorem 4 by

$$(2.12) \qquad \lim_{n\to\infty} \frac{Z_n(D)}{n} = \frac{1}{2\pi} \int_J \left(\frac{\partial G_1}{\partial n_1} + \frac{\partial G_2}{\partial n_2}\right) |dz|.$$

3. Distribution of zeros: K connected

The first main result for the more general case, K connected, can be stated as follows.

Theorem 5: *Let E be a closed bounded set whose complement K is connected and regular, $\{p_n\}_{n\in\mathbb{N}}$ a sequence of polynomials satisfying (A1) - (A4).*
Then, for any $\sigma > 1$,

$$(3.1) \qquad \lim_{n\to\infty} \frac{Z_n(K\setminus\overline{E}_\sigma)}{n} = 0.$$

Moreover, the convergence relation

$$(3.2) \qquad \lim_{n\to\infty} \sum_{z_{n,k}\in\overline{E}_\sigma} \frac{1}{z-z_{n,k}} = \frac{\Phi'(z)}{\Phi(z)}$$

holds locally uniformly in $K\setminus\overline{E}_\sigma$.

Now we are in position to formulate results about the distribution of the zeros of p_n where K does not have to be simply connected any more.

Theorem 6: *Let E be a closed bounded point set whose complement K is connected and regular, $\{p_n\}_{n\in\mathbb{N}}$ a sequence of polynomials satisfying (A1) - (A4), $\sigma > 1$. If J is a Jordan curve contained in Γ_σ, then*

$$(3.3) \qquad \lim_{n\to\infty} \frac{Z_n(S)}{n} = \frac{1}{2\pi} \int_J \frac{\partial G}{\partial n} \, |dz|,$$

where S is the region interior to J.

As an application let us consider an example due to Walsh [10]: Let E be the set $|z(z-1)| \leq 1/16$ bounded by the lemniscate $|z(z-1)| = 1/16$, so $G(x,y) = \frac{1}{2}\log|z(z-1)| + \log 4$. We choose f(z) identically zero in the right-hand oval of the lemniscate bounding E and $1/(1-4z)$ in the left hand oval. Then $f(z)$ is analytic in E_ρ, where $\rho = \sqrt{3}$ is maximal, and E_ρ is the lemniscate $|z(z-1)| = 3/16$ passing through $z = 1/4$. Let $p_{2n-1}(z)$ be the polynomial of degree $2n-1$ which is determined by interpolation to $f(z)$ in the points $z = 0$ and $z = 1$, each considered of multiplicity n. Define $p_{2n}(z) := p_{2n-1}(z)$, then the sequence converges maximally to $f(z)$ on E. Since $G(x,y)$ has a critical point at $z = x+iy = 1/2$ on E_2, by Theorem 5, for any $\sqrt{3} < \sigma \leq 2$ the right-hand and the left-hand oval contain $n + o(n)$ zeros of $p_{2n}(z)$, at least for a subsequence of $\{p_{2n}\}$. But this result holds for the whole sequence $\{p_{2n}\}$, since $\lim\limits_{n\to\infty} |a_{2n}|^{1/2n} = 4$ and the capacity of E is 1/4. For the right-hand oval this can be easily verified, since n zeros of $p_{2n}(z)$ lie at the point $z = 1$.

In the above-mentioned paper of Walsh [10] it was shown that every point z_o of the boundary of Γ_ρ which is a limit of points of E_ρ on which $f(z) \neq 0$, is again a limit point of zeros of the polynomials p_n; therefore nothing was said about the right-hand oval in the example above.

4. Proofs

Proof of Theorem 5: For any $\sigma > 1$, the equation (3.1) was already proved in [2].

Let $Z_{n,\sigma} := Z_n(K \setminus \overline{E}_\sigma)$ and

(4.1) $$p_n(z) = \frac{a_n}{|a_n|} \cdot \tilde{p}_n(z) \cdot q_n(z),$$

where $q_n \in \Pi_{Z_{n,\sigma}}$ is the monic polynomial whose zeros are the zeros of $p_n(z)$ in $K \setminus \overline{E}_\sigma$. Let K* be a simply connected subregion of K such that the point at infinity lies in K*. We define a fixed branch of $\Phi(z)$ in K* by

(4.2) $$\Phi(z) = \frac{1}{c} z + \alpha_o + O\left(\frac{1}{z}\right) \quad \text{for } z \to \infty$$

and set for $z \in K^*$

$$(4.3) \qquad h_n(z) := \frac{\{\tilde{p}_n(z)\}^{1/(n-Z_{n,\sigma})}}{\Phi(z)},$$

where the branch of the numerator is chosen such that $h_n(\infty) > 0$. For $z \in E$ we have

$$|\tilde{p}_n(z)| \leq \frac{|p_n(z)|}{(d_1)^{Z_{n,\sigma}}},$$

where d_1 is the minimal distance of Γ_σ to the set E. Since

$$\log \frac{|\tilde{p}_n(z)|}{|\Phi(z)|^{n-Z_{n-\sigma}}}$$

is harmonic in K and continuous in $\overline{K}$, except at the zeros of $\tilde{p}_n(z)$, we conclude from the maximum principle for harmonic functions that

$$\frac{|\tilde{p}_n(z)|}{|\Phi(z)|^{n-Z_{n,\sigma}}} \leq \frac{\|p_n\|_E}{(d_1)^{Z_{n,\sigma}}}.$$

Hence, we obtain from (3.1) and (A4) that the functions $h_n(z)$ are uniformly bounded in $K^* \setminus \overline{E}_\sigma$ and satisfy

$$(4.4) \qquad \overline{\lim_{n\to\infty}} |h_n(z)| \leq \overline{\lim_{n\to\infty}} \|p_n\|_E^{1/(n-Z_{n,\sigma})} = 1.$$

Moreover, because of the normalization in (4.2) and the condition (A3), it follows that

$$(4.5) \qquad \lim_{n\to\infty} h_n(\infty) = 1.$$

Since each function $h_n(z)$ is analytic in $K^* \setminus \overline{E}_\sigma$, we conclude from (4.4), (4.5) and the maximum principle that the functions $h_n(z)$ converge uniformly to the constant function 1 in any compact subset of $K^* \setminus \overline{E}_\sigma$. Consequently the functions $\log h_n(z)$ converge uniformly to zero in any compact set of $K^* \setminus \overline{E}_\sigma$, if we take for the logarithm the branch with $\log h_n(\infty) = 0$. Then, on differentiating $\log h_n(z)$ we get

$$\lim_{n\to\infty} \frac{1}{n-Z_{n,\sigma}} \frac{\tilde{p}_n'(z)}{p_n(z)} = \frac{\Phi'(z)}{\Phi(z)}$$

or, using (3.1),

$$\lim_{n\to\infty} \frac{1}{n} \sum_{z_{n,k}\in\overline{E}_\sigma} \frac{1}{z-z_{n,k}} = \frac{\Phi'(z)}{\Phi(z)} \tag{4.6}$$

locally uniformly in $K^* \setminus \overline{E}_\sigma$. Now, we observe that the function

$$\frac{\Phi'(z)}{\Phi(z)} = \frac{\partial G(x,y)}{\partial x} - i\,\frac{\partial G(x,y)}{\partial y}$$

is independent of the branch of $\Phi(z)$. Since (4.6) holds locally uniformly in $K^* \setminus \overline{E}$, where $K^* \subset K$ is any simply connected region with $\infty \in K^*$, it follows, that (3.2) is true. □

<u>Proof of Theorem 6</u>: For any $\sigma > 1$, the locus Γ_σ consists of a finite number of Jordan curves which are mutually exterior except for a finite number of critical points of $\Phi(z)$. Let us fix a function $f(z)$ analytic in E_σ and continuous in the interior and the boundary of each Jordan curve of Γ_σ, except at the critical points. Then, by Cauchy's integral formula, we obtain for any ρ, $1 < \rho < \sigma$, from (3.2)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{z_{n,k}\in\overline{E}_\rho} f(z_{n,k}) = \frac{1}{2\pi i} \int_{\Gamma_\sigma} f(z) \frac{\Phi'(z)}{\Phi(z)}\, dz,$$

where Γ_σ is oriented in such a way that E_σ lies to the left. Since $Z_n(E_\sigma \setminus \overline{E}_\rho) = o(n)$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{z_{n,k}\in E_\sigma} f(z_{n,k}) = \frac{1}{2\pi i} \int_{\Gamma_\sigma} f(z) \frac{\Phi'(z)}{\Phi(z)}\, dz. \tag{4.7}$$

Now, let us consider the function f(z) defined by f(z) = 1 for $z \in \overline{S}$ and f(z) = 0 for $z \in \overline{E}_\sigma \setminus \overline{S}$. Then we obtain from (4.7) and (3.1):

$$\lim_{n\to\infty} \frac{Z_n(\overline{S})}{n} = \lim_{n\to\infty} \frac{Z_n(S)}{n}$$

$$= \frac{1}{2\pi i} \int_J \frac{\Phi'(z)}{\Phi(z)} dz$$

$$= \frac{1}{2\pi} \int_J \frac{\partial G(x,y)}{\partial n} |dz| . \quad \square$$

5. References

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