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Behavior of Zeros of Polynomials of Near Best Approximation*

H.-P. BLATT

Katholische Universität Eichstätt, Mathematisch-Geographische Fakultät, Ostenstraße 18, D-8078 Eichstätt, West Germany

AND

E. B. SAFF

Department of Mathematics, University of South Florida, Tampa, Florida 33620, U.S.A.

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DEDICATED TO THE MEMORY OF GÉZA FREUD

The purpose of this paper is to study the asymptotic behavior of the zeros of polynomials of near best approximation to continuous functions f on a compact set E in the case when f is analytic on the interior of E but not everywhere on the boundary. For example, suppose E is a finite union of compact intervals of the real line and f is a continuous function on E, but is not analytic on E; then we show (cf. Corollary 2.2) that every point of E is a limit point of zeros of the polynomials of best uniform approximation to f on E. This fact answers a question posed by P. Borwein who showed that, for the case when E is a single interval and f is real-valued, then the above hypotheses on f imply that *at least one* point of E is the limit point of zeros of such polynomials.

1. INTRODUCTION

Let E be a closed bounded set in the z-plane, whose complement K (with respect to the extended plane) is connected and *regular* in the sense that K has a Green's function $G(z)$ with pole at infinity: $G(z)$ is harmonic in K except at infinity, and in a neighborhood of the point of infinity we have

$$G(z) = \log |z| + G_0(z), \tag{1.1}$$

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where $G_0(z)$ is harmonic in this neighborhood and approaches a finite value at infinity; moreover, $G(z)$ is continuous in the closed region $\bar{K}$ except at infinity and vanishes on the boundary of K (Walsh [13]). The function

$$t = \Phi(z) := e^{G(z) + iH(z)}, \tag{1.2}$$

where $H(z)$ is conjugate to $G(z)$ in K, maps K onto the exterior of the unit disk. Hence, for $z \to \infty$,

$$|\Phi(z)/z| = 1/c + O(1/z), \tag{1.3}$$

with $c > 0$; if K is simply connected we normalize $H(z)$ such that $\Phi'(\infty) = 1/c$. The constant c is called the *capacity* or *transfinite diameter* of the set E. For each $\sigma \geqq 1$, we consider the equipotential locus

$$\Gamma_\sigma := \{z \in \bar{K}: G(z) = \log \sigma\}, \tag{1.4}$$

with interior

$$E_\sigma := \mathring{E} \cup \{z \in \bar{K}: 0 \leqq G(z) < \log \sigma\}, \tag{1.5}$$

where $\mathring{E}$ denotes the interior of E.

If a function $f(z)$ is continuous on E and analytic on $\mathring{E}$, there exists a largest real number σ (finite or infinite), say $\sigma = \rho$, such that $f(z)$ is single-valued and analytic on E_ρ. Then, denoting by Π_n the collection of all complex polynomials of degree $\leqq n$, there exist (cf. [13]) polynomials $p_n \in \Pi_n$, $n = 0, 1, 2, \ldots$, such that

$$\overline{\lim_{n \to \infty}} \| f - p_n \|_E^{1/n} = \frac{1}{\rho}, \tag{1.6}$$

where we denote by $\|\cdot\|_E$ the Chebyshev (uniform) norm on the set E. Moreover, there exist no polynomials $p_n \in \Pi_n$ for which the left-hand side of (1.6) is less than $1/\rho$. For $\rho > 1$, a sequence $\{p_n\}$ satisfying (1.6) is said to converge *maximally* to $f(z)$ on E.

Walsh [15] proved the following result.

THEOREM 1.1. *Let E be a closed bounded point set whose complement K is connected and regular, and suppose f is single-valued and analytic on E_ρ, where $1 < \rho < \infty$ and f cannot be analytically extended as a single-valued function to Γ_ρ. Let $\{p_n\}$, $p_n \in \Pi_n$, $n = 0, 1, 2, \ldots$, be a sequence of polynomials converging maximally to $f(z)$ on E, and z_0 a point of Γ_ρ that is a limit of points of E_ρ on which $f(z)$ is not zero. Then z_0 is a limit point of zeros of the polynomials p_n.*

It is important to notice that the case when f is continuous on E, analytic on $\mathring{E}$ but not everywhere on the boundary ∂E, i.e., $\rho = 1$, is not considered in the theorem above.

Theorem 1.1 is a generalization of a result of R. Jentzsch [6] (cf. [12, p. 238–241]):

THEOREM 1.2. *Let $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ be a power series with radius of convergence equal to* 1. *Then every point on the boundary of the unit disk is a limit point of the zeros of the partial sums*

$$s_n(z) := \sum_{\nu=0}^{n} a_\nu z^\nu, \; n = 1, 2, \ldots.$$

Theorem 1.2 was generalized by Ostrowski [8], who considered a sequence of analyitc functions $f_n(z)$ converging uniformly to $f(z)$ on every compact subset of a region D; he chose for D the largest possible region, the so-called *complete region of uniform convergence.* Under various assumptions as to the growth of $f_n(z)$, the speed of convergence and the nature of $f(z)$, he discussed the zeros of $f_n(z)$ in the neighborhood of a particular boundary point of E.

In [11] Szegö considered the distribution of zeros of a polynomial sequence $\{p_n\}_{n \in \mathfrak{N}}$ converging uniformly to a function $f(z) \not\equiv 0$ on every compact subset of a simply connected region $G = \mathring{E}$ bounded by a finite number of analytic Jordan arcs and satisfying the following properties:

(A1) $n \in \mathfrak{N} := \{n_1 < n_2 < n_3 < \cdots\}$,

(A2) $p_n(z) = a_n z^n + \cdots \in \Pi_n \backslash \Pi_{n-1}$,

(A3) $\lim_{n \to \infty} |a_n|^{1/n} = 1/c$, where $c = \operatorname{cap}(E)$.

Rosenbloom [9] discussed the analogous problem for sequences $\{p_n\}$ bounded in the neighborhood of some point.

In our theorems, especially in Section 3, the condition that $\{p_n\}$ converges to a function $f(z) \not\equiv 0$ in compact subsets of a region is replaced by the assumption

(A4) $\lim_{n \to \infty} \| p_n \|_E^{1/n} = 1$.

In (A3) and (A4), the limits are considered for $n = n_1, n_2, \ldots$.

The outline of the present paper is as follows. In Section 2, a characterization of functions $f(z)$ that fail to be analytic on E, is given in terms of the leading coefficients of the polynomials of best uniform approximation to $f(z)$ on E (Theorem 2.1). This result is analogous to the Cauchy–Hadamard formula for the radius of convergence of a power series. Moreover, we extend Theorem 1.1 to the case $\rho = 1$, if $\{p_n\}$ is a sequence of near best polynomial approximations. In Section 3, we state our main results concerning the distribution of the zeros of polynomials p_n satisfying the conditions (A1)–(A4). As special cases we obtain the results of Szegö [11]. In Section 4, we discuss the notion of exact harmonic majorant and its connection with polynomials p_n satisfying (A1)–(A4). Finally, in Section 5 the proofs of the results stated in Sections 2 and 3 are given.

2. Near Best Polynomial Approximations

THEOREM 2.1. *Let E be a closed bounded point set whose complement K is connected and regular, and suppose the function f is continuous on E, analytic in $\mathring{E}$. For each $n=0, 1, 2,\ldots$, let $p_n^*(z)=a_n z^n+\cdots \in \Pi_n$ be the polynomial of best uniform approximation to f on E. Then f is not analytic on E if and only if*

$$\overline{\lim_{n\to\infty}}\, |a_n|^{1/n}=\frac{1}{c}, \tag{2.1}$$

where c is the capacity of E.

We remark that under the conditions of Theorem 2.1, Mergelyan's theorem (cf. [13]) implies that

$$E_n(f) := \| f-p_n^* \|_E \to 0 \qquad \text{as} \quad n\to\infty.$$

COROLLARY 2.1. *Let E be as in Theorem 2.1, f continuous on E, analytic in the interior of E but not on E. If $p_n(z)=b_n z^n+\cdots$, $n=0, 1,\ldots$, is a sequence of polynomials such that*

$$\| f-p_n \|_E = E_n(f)+\varepsilon_n, \qquad n=0, 1, 2,\ldots, \tag{2.2}$$

where

$$\overline{\lim_{n\to\infty}}\, |\varepsilon_n|^{1/n}<1, \tag{2.3}$$

then

$$\overline{\lim_{n\to\infty}}\, |b_n|^{1/n}=\frac{1}{c}. \tag{2.4}$$

As a generalization of Theorem 1.1 for the case $\rho=1$, we state

THEOREM 2.2. *Let E be a closed bounded point set whose complement K is connected and regular, and suppose the function f is continuous on E,* *analytic in the interior of E but not on E. For $n=0, 1,\ldots$, let $p_n^* \in \Pi_n$ be the polynomial of best uniform approximation to f on E. If z_0 is a point of the closure of the set $\partial E \setminus \partial S$, where*

$$S := \{x \in \mathring{E}: f(z) \equiv 0 \text{ in a neighborhood of } x\}, \tag{2.5}$$

then z_0 is a limit point of zeros of the p_n^*, $n=0, 1, 2,\ldots$.

More generally, such a point z_0 is a limit point of zeros of any sequence of polynomials $\{p_n\}$, $p_n \in \Pi_n$, $n=0, 1,\ldots$, that satisfy (2.2) *and* (2.3).

If the compact set E has empty interior, then clearly the set S of (2.5) is empty and we get from Theorem 2.2 the following.

COROLLARY 2.2. *Let E be a closed bounded point set with $\mathring{E} = \varnothing$ and connected and regular complement, and suppose f is continuous on E, but not (everywhere) analytic on E. For $n=0, 1,\ldots$, let $p_n \in \Pi_n$ be a sequence of polynomials satisfying* (2.2) *and* (2.3). *Then every point of E is a limit point of zeros of the polynomials p_n.*

3. DISTRIBUTION OF ZEROS

Because of Theorem 2.1 (resp., Corollary 2.1), a *subsequence* of the polynomials of best uniform approximation p_n^* (resp., the polynomials p_n of Corollary 2.1) satisfies the conditions (A1)–(A4), when the function f cannot be analytically extended to ∂E. Henceforth *we assume throughout this section that $\{p_n\}_{n \in \mathfrak{N}}$ is a sequence of polynomials satisfying the assumptions* (A1)–(A4). Moreover, all limits as $n \to \infty$ are considered for $n = n_1, n_2,\ldots$.

In stating the next theorems it is convenient to introduce the following notation: For any set C in $\mathbb{C}$ let $Z_n(C)$ be the number of zeros of p_n in C, counted with their multiplicities.

THEOREM 3.1. *For $\sigma > 1$, let $Z_{n,\sigma} := Z_n(K \setminus \bar{E}_\sigma)$. Then*

$$\lim_{n \to \infty} \frac{Z_{n,\sigma}}{n} = 0. \tag{3.1}$$

Assume, furthermore, that the complement K is simply connected and write

$$p_n(z) = \frac{a_n}{|a_n|} \cdot \tilde{p}_n(z) \cdot q_n(z), \tag{3.2}$$

where $q_n \in \Pi_{Z_{n,\sigma}}$ *is the monic polynomial whose zeros are the zeros of* $p_n(z)$ *in* $K \backslash \bar{E}_\sigma$. *Then, for a suitable choice of branches,*

$$\lim_{n \to \infty} \frac{\{\tilde{p}_n(z)\}^{1/(n - Z_{n,\sigma})}}{\Phi(z)} = 1 \tag{3.3}$$

holds locally uniformly in $K \backslash \bar{E}_\sigma$.

If the boundary ∂E of E is a closed Jordan curve, then $\Phi(z)$ is a univalent conformal mapping from K onto the exterior of the unit disk. As is well known (cf. [5, Theorem 4, p. 44]), $\Phi(z)$ can be extended to a homeomorphism from $\bar{K}$ onto $\{t: |t| \geqq 1\}$. Let $z = \psi(t)$ denote the inverse mapping of $\Phi(z)$. If $z_{n,1}, ..., z_{n,n}$ are the zeros of the polynomial p_n and if $z_{n,k} \in \bar{K}$, then

$$z_{n,k} = \psi(t_{n,k}), \tag{3.4}$$

with $t_{n,k} = \rho_{n,k} \cdot e^{i\varphi_{n,k}}$, $\rho_{n,k} \geqq 1$, $\varphi_{n,k} \in [0, 2\pi)$.

With the above notation we state

THEOREM 3.2. *Assume that* ∂E *is a closed Jordan curve. If* $Z_n(\mathring{E}) = o(n)$ *for* $n \to \infty$, *then the arguments* $\varphi_{n,k}$ *associated with the zeros* $z_{n,k} = \psi(t_{n,k})$ *of* p_n *in* $\bar{K}$ *as in* (3.4) *are uniformly distributed on the interval* $[0, 2\pi]$ *in the sense of Weyl.*

Next, we consider the case when K is simply connected and bounded by Jordan arcs. The equation of a *Jordan arc* J in $\mathbb{C}$ is given for $z \in J$ in parametric form $z = \gamma(t)$, where t runs through a real compact interval $[a, b]$, $a < b$, $\gamma(t)$ is continuous, and $\gamma(t_1) = \gamma(t_2)$ only if $t_1 = t_2$. A Jordan arc is called *analytic* if $\gamma(t)$ is an analytic function on the open interval (a, b) and $\gamma'(t) \neq 0$ for all $t \in (a, b)$. Hence, there exists a region Δ, symmetric to the interval (a, b), with the property that $\gamma(t)$ is analytic for all $t \in \Delta$. If, moreover, $J \in \partial K$ and the region Δ can be chosen in such a way that $\gamma(t) \in K$ when t lies in the upper half of Δ, and that $\gamma(t)$ lies outside of K for t in the lower half, then J is a *free one-sided boundary arc of* K; if, for an appropriate Δ, $\gamma(t) \in K$ for all $t \in \Delta \backslash (a, b)$ then J is a *free two-sided boundary arc of* K (cf. [1, p. 234]).

A point $z \in \partial K$ is an *accessible boundary point* of K if there exists a Jordan arc J with endpoint z such that all other points of J lie in K (cf. [5, p. 35]). If K is simply connected and all points of ∂K are accessible boundary points of K, then, for the inverse mapping $\psi(t)$ of $\Phi(z)$, there exists a continuous extension on $\{t: |t| \geqq 1\}$ ([5, p. 43]). If J is a free one-sided boundary arc of K then there exist two arguments α and β, $\alpha < \beta < \alpha + 2\pi$, such that

$$\psi^{-1}(J) = \{t = e^{i\varphi}: \alpha \leqq \varphi \leqq \beta\}; \tag{3.5}$$

if J is a free two-sided boundary arc of K then there exist four points α, β, $\tilde{\alpha}$, and $\tilde{\beta}$, $\alpha<\beta\leqq\tilde{\alpha}<\tilde{\beta}\leqq\alpha+2\pi$, such that

$$\psi^{-1}(J)=\{t=e^{i\varphi}: \alpha\leqq\varphi\leqq\beta \text{ or } \tilde{\alpha}\leqq\varphi\leqq\tilde{\beta}\} \tag{3.6}$$

(cf. [5, Theorem 1, p. 37]). Moreover, in either case, the function $\psi(t)$ is analytic and $\psi'(t)\neq 0$ for all interior points t of the inverse image $\psi^{-1}(J)$ (cf. [5, Theorem 5, p. 44]).

THEOREM 3.3. *Let K be simply connected with only accessible boundary points, let J be a subarc in the interior of a free one-sided boundary arc of K such that the connected component B of $\mathring{E}$, where $J\subset\bar{B}$, is a Jordan region. Furthermore, assume*

$$Z_n(C)=o(n), \qquad \text{as } n\to\infty, \tag{3.7}$$

for any compact set C in B. Then there exists a real number ε_0, $0<\varepsilon_0<1$, such that the function $\psi(t)$ can be extended analytically to the closed region $T(\varepsilon_0)$, where $\psi(T(\varepsilon_0))\subset\bar{K}\cup B$ and, for any $0<\varepsilon<1$, $T(\varepsilon)$ is the point set

$$T(\varepsilon):=\{t=\rho e^{i\varphi}: 1-\varepsilon\leqq\rho\leqq 1+\varepsilon, \alpha\leqq\varphi\leqq\beta\} \tag{3.8}$$

and α, β are defined by (3.5). *Moreover for the distribution of the zeros of the polynomials p_n in the closed region $J(\varepsilon):=\psi(T(\varepsilon))$, $0<\varepsilon\leqq\varepsilon_0$, we have*

$$\lim_{n\to\infty}\frac{Z_n(J(\varepsilon))}{n}=\frac{\beta-\alpha}{2\pi}. \tag{3.9}$$

THEOREM 3.4. *Let K be simply connected with only accessible boundary points, let J be a subarc in the interior of a free two-sided boundary arc of K. Then, for any $\varepsilon>0$, the distribution of the zeros of the polynomials p_n in the closed region*

$$\tilde{J}(\varepsilon):=\{z=\psi(t): t=\rho e^{i\varphi}, 1\leqq\rho\leqq 1+\varepsilon, \alpha\leqq\varphi\leqq\beta \text{ or } \tilde{\alpha}\leqq\varphi\leqq\tilde{\beta}\} \tag{3.10}$$

satisfies

$$\lim_{n\to\infty}\frac{Z_n(\tilde{J}(\varepsilon))}{n}=\frac{\beta-\alpha+\tilde{\beta}-\tilde{\alpha}}{2\pi}, \tag{3.11}$$

where α, β, $\tilde{\alpha}$ and $\tilde{\beta}$ are defined by (3.6).

We remark that the above theorems apply to simply connected K where the boundary ∂K is composed of a finite number of free one-sided or two-sided boundary arcs. If the sequence $\{p_n\}$ converges locally uniformly in $\mathring{E}$ to a function $f(z)\not\equiv 0$, then by Hurwitz's theorem, condition (3.7) is

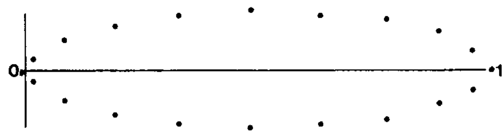

FIG. 1. Zeros of $p_{20}^*(x)$, the best approximation to $f(x)=\sqrt{x}$ on $E=[0, 1]$.

automatically satisfied. Thus Theorem 3.3 contains as special cases certain theorems of Szegö [11].

COROLLARY 3.1. *With the same hypothesis as in Theorem* 3.3 *let D be a neighborhood of the interior of J such that $\bar{D} \cap \partial E = J$. Then*

$$\lim_{n\to\infty} \frac{Z_n(D)}{n} = \frac{\beta-\alpha}{2\pi}. \tag{3.12}$$

COROLLARY 3.2. *Let K and J be as in Theorem* 3.4, *D a neighborhood of the interior of J such that $\bar{D} \cap \partial E = J$. Then*

$$\lim_{n\to\infty} \frac{Z_n(D)}{n} = \frac{\beta-\alpha+\tilde{\beta}-\tilde{\alpha}}{2\pi}. \tag{3.13}$$

To illustrate the results of Corollary 2.2 and Theorem 3.4, we include Fig. 1 and 2 which show the zeros of the best approximating polynomials $p_{20}^*(x)$ and $p_{40}^*(x)$, respectively, to $f(x)=\sqrt{x}$ on $[0, 1]$. The figures indicate the convergence of the zeros to the set $E=[0, 1]$ as well as their uniform distribution as described above. We wish to thank Professor R. S. Varga for these valuable computations.

We remark that the results of this section also apply to the polynomials $p_n^*-p_{n-1}^*$, $n=1, 2,\ldots$, for an appropriate subsequence, where $p_n^* \in \Pi_n$ is, as in Section 2, the polynomial of best uniform approximation to f on E. In the case when $E=[a, b]$ and f is real-valued, all zeros of $p_n^*-p_{n-1}^*$ lie in $[a, b]$ and must interlace the extreme points of $f-p_n^*$. Thus we get from Theorem 3.4 a Kadec-type theorem concerning the distribution of extreme points. But we must emphasize that the results of Kadec [7] and Fuchs [4] are sharper since they give estimations for the distance between the extreme points of $f-p_n^*$ and the extreme points of the Chebyshev polynomials $T_{n+1}(x)$ in the case $E=[-1, 1]$.

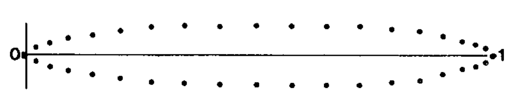

FIG. 2. Zeros of $p_{40}^*(x)$, the best approximation to $f(x)=\sqrt{x}$ on $E=[0, 1]$.

4. Exact Harmonic Majorants

An essential tool for proving the theorems of Sections 2 and 3 is the concept of exact harmonic majorants introduced by Walsh [14].

Definition [14]. Let $\{F_n\}$ be a sequence of locally single-valued analytic functions (except possibly for branch points) in a region D of the z-plane, whose modulus $|F_n(z)|$ is single-valued in D. If the function $V(z)$ is harmonic in D and if we have for every continuum S (S not a single point) in D the relation

$$\overline{\lim_{n\to\infty}} \|F_n\|_S \leq \max_{z\in S} e^{V(z)}, \tag{4.1}$$

then $V(z)$ is a *harmonic majorant* for the sequence $\{F_n\}$ in D. If in (4.1) the equality holds for every S, then $V(z)$ is an *exact harmonic majorant.*

Theorem 4.1 (Walsh [14]). *If $V(z)$ is a harmonic majorant for the sequence $\{F_n\}$ in D, and if for a single continuum S equality holds in* (4.1), *then $V(z)$ is an exact harmonic majorant of the sequence $\{F_n\}$ in D.*

There is an intimate relation between an exact harmonic majorant for the sequence $\{F_n\}$ and the zeros of the functions $F_n(z)$, namely,

Theorem 4.2 (Walsh [14]). *Let $V(z)$ be an exact harmonic majorant in D for the sequence $\{f_n^{1/n}\}$ and every subsequence of $\{f_n^{1/n}\}$, where the functions $f_n(z)$ are analytic in D. If γ is a closed disk that lies in D and if $Z_n(\gamma)$ is the number of zeros of f_n in γ, then*

$$\lim_{n\to\infty} \frac{Z_n(\gamma)}{n} = 0. \tag{4.2}$$

For the proof of our main results it is also convenient to have for reference the following lemmas.

Lemma 4.1. *Let E be a closed bounded set whose complement K in the extended z-plane is connected and regular. If $p_n(z) = a_n z^n + \cdots \in \Pi_n$, then*

$$|a_n| \leq \frac{1}{c^n} \|p_n\|_E, \tag{4.3}$$

where c is the capacity of E.

Proof. Inequality (4.3) is a consequence of the generalized Bernstein inequality due to Walsh [13, p. 78]; namely,

$$\left|\frac{p_n(z)}{\Phi(z)^n}\right| \leqq \| p_n \|_E, \qquad \text{for all} \quad z \in K, \tag{4.4}$$

where $\Phi(z)$ is the mapping of (1.2). Letting $z \to \infty$ then yields (4.3). ∎

LEMMA 4.2. *If the polynomial sequence* $\{p_n\}$ *satisfies* (A1)–(A4), *then the Green's function* $G(z)$ *is an exact harmonic majorant in* $\mathbb{C} \backslash E$ *for every subsequence of the sequence* $\{p_n^{1/n}\}_{n \in \mathfrak{N}}$.

Proof. Using (A4) and (4.4), it follows that $G(z)$ is a harmonic majorant in $\mathbb{C} \backslash E$ for the sequence $\{p_n^{1/n}\}_{n \in \mathfrak{N}}$. Let us assume that $G(z)$ is *not* an exact harmonic majorant in $\mathbb{C} \backslash E$. Then, from Theorem 4.1, we must have

$$\overline{\lim_{n \to \infty}} \; \| p_n \|_{\Gamma_\sigma}^{1/n} < \sigma, \tag{4.5}$$

for every $\sigma > 1$. Fix any such value of σ. On applying Lemma 4.1 (with E replaced by $\bar{E}_\sigma$), we get

$$|a_n| \leqq \frac{1}{(\sigma c)^n} \| p_n \|_{\Gamma_\sigma}, \tag{4.6}$$

since σc is the capacity of $\bar{E}_\sigma$. But then, from (4.5), it follows that

$$\overline{\lim_{n \to \infty}} \; |a_n|^{1/n} < \frac{1}{c},$$

which contradicts property (A3). Thus $G(z)$ is an exact harmonic majorant for $\{p_n^{1/n}\}_{n \in \mathfrak{N}}$. Clearly, the above argument applies also to any subsequence of $\{p_n^{1/n}\}_{n \in \mathfrak{N}}$. ∎

5. PROOFS

Proof of Theorem 2.1. First we note, by Lemma 4.1 and the fact that $\{\| p_n^* \|_E\}$ is a bounded sequence, that

$$\overline{\lim_{n \to \infty}} \; |a_n|^{1/n} \leqq \frac{1}{c}. \tag{5.1}$$

Now suppose that (2.1) holds. We wish to show that f cannot be analytically extended to ∂E. Assume, to the contrary, that f is analytic on E. Then f is analytic in $\bar{E}_\sigma$, for some $\sigma > 1$. But then, since p_n^* is a maximally convergent (cf. [13]) sequence, $p_n^* \to f$ uniformly on $\bar{E}_\sigma$. Hence from Lemma 4.1, we get (with E replaced by $\bar{E}_\sigma$)

$$\overline{\lim_{n\to\infty}} |a_n|^{1/n} \leqq \frac{1}{\sigma c} < \frac{1}{c}, \tag{5.2}$$

which contradicts (2.1).

Next, assume that f is not (everywhere) analytic on ∂E. If Eq. (2.1) does not hold, then from (5.1) we have

$$\overline{\lim_{n\to\infty}} |a_n|^{1/n} < \frac{1}{c}. \tag{5.3}$$

Let $T_n(z) = z^n + \cdots \in \Pi_n$, $n = 0, 1, \ldots$, be the Chebyshev polynomials for E, i.e.,

$$\| T_n \| = \min\{ \| z^n - q_{n-1} \|_E : q_{n-1} \in \Pi_{n-1}\},$$

and define

$$\tilde{p}_{n-1} := p_n^* - a_n T_n. \tag{5.4}$$

Clearly, $\tilde{p}_{n-1} \in \Pi_{n-1}$ for each $n \geqq 1$. Moreover, since

$$\lim_{n\to\infty} \| T_n \|_E^{1/n} = c \tag{5.5}$$

(cf. [5]), it follows from (5.3) that

$$\overline{\lim_{n\to\infty}} \| p_n^* - \tilde{p}_{n-1} \|_E^{1/n} < 1. \tag{5.6}$$

Also, from the extremal property of p_{n-1}^*, we have

$$\begin{aligned} E_{n-1}(f) = \| f - p_{n-1}^* \|_E &\leqq \| f - \tilde{p}_{n-1} \|_E \\ &\leqq \| f - p_n^* \|_E + \| p_n^* - \tilde{p}_{n-1} \|_E \\ &= E_n(f) + \| p_n^* - \tilde{p}_{n-1} \|_E, \end{aligned}$$

and hence

$$E_{n-1}(f) - E_n(f) \leqq \| p_n^* - \tilde{p}_{n-1} \|_E.$$

But then, from (5.6), we find

$$\overline{\lim_{n\to\infty}} \{E_{n-1}(f) - E_n(f)\}^{1/n} < 1, \tag{5.7}$$

and since, by Mergelyan's theorem, $E_n(f) \to 0$ as $n \to \infty$, we get

$$\overline{\lim_{n \to \infty}} \, [E_n(f)]^{1/n} < 1. \tag{5.8}$$

As is well known, inequality (5.8) implies that f is analytic on E (cf. [13]) which is the desired contradiction. ∎

Proof of Corollary 2.1. Clearly, from Lemma 4.1,

$$\overline{\lim_{n \to \infty}} \, |b_n|^{1/n} \leqq \frac{1}{c}. \tag{5.9}$$

Let us assume that strict inequality holds in (5.9). Since, as in the proof of Theorem 2.1, we have

$$\begin{aligned} E_{n-1}(f) &\leqq \| f - (p_n - b_n T_n) \|_E \\ &\leqq \| f - p_n \|_E + \| b_n T_n \|_E, \end{aligned}$$

it follows from (2.2) that

$$E_{n-1}(f) - E_n(f) \leqq \varepsilon_n + \| b_n T_n \|_E.$$

Thus (2.3) and the assumption of strict inequality in (5.9) yield

$$\overline{\lim_{n \to \infty}} \, \{E_{n-1}(f) - E_n(f)\}^{1/n} < 1,$$

which contradicts the fact that f is not analytic on E. ∎

We remark that the assumption of (2.3) in Corollary 2.1 can be replaced by the condition

$$\sum_{n=m}^{\infty} \varepsilon_n = o(E_{m-1}(f)), \qquad \text{as} \quad m \to \infty. \tag{2.3'}$$

Proof of Theorem 2.2. Because of Theorem 2.1 and Lemma 4.2, there exists a subset

$$\mathfrak{N} := \{n_1 < n_2 < n_3 < \cdots\}$$

of the natural numbers such that the Green's function $G(z)$ is an exact harmonic majorant in $\mathbb{C} \backslash E$ for every subsequence of $\{p_n^*(z)^{1/n}\}_{n \in \mathfrak{N}}$. Let $z_0 \in \partial E \backslash \partial S$ and assume, to the contrary, that there exists an open disk U centered at z_0 that contains no zeros of p_n^* for $n \geqq n_0$. Since $S \subset \mathring{E}$, we can choose U such that

$$U \cap S = \varnothing. \tag{5.10}$$

For $z \in U$ and each $n \geqq n_0$, we define the single-valued analytic function

$$f_n(z) := p_n^*(z)^{1/n} = \exp\left(\frac{1}{n} \log p_n^*(z)\right) \tag{5.11}$$

by taking the branch of $\log p_n^*(z)$ in U for which

$$-\pi < \operatorname{Im} \log p_n^*(z_0) \leqq \pi. \tag{5.12}$$

From the fact that $\{p_n^*\}$ is uniformly bounded on E, it follows that the functions f_n are uniformly bounded in U. Hence there exists a subsequence of $\{f_n\}_{n \in \mathfrak{N}}$, say $\{f_n\}_{n \in \hat{\mathfrak{N}}}$, $\hat{\mathfrak{N}} \subset \mathfrak{N}$, that converges locally uniformly to an analytic function $g(z)$ in U. From the boundedness of the sequence $\{p_n^*(z_0)\}$, we conclude that $|g(z_0)| \leqq 1$. Also, since $G(z)$ is an exact harmonic majorant of $\{p_n^*(z))^{1/n}\}_{n \in \hat{\mathfrak{N}}}$ in $\mathbb{C} \setminus E$, it follows that $|g(z)| > 1$ in $U \cap (\mathbb{C} \setminus E)$. Hence $g(z)$ is nonconstant in U and so the open point set

$$V := \{z \in U : |g(z)| < 1\}$$

is nonempty. Since $V \cap (\mathbb{C} \setminus E) = \varnothing$, we have $V \subset \mathring{E}$ and $f(z) = 0$ for all $z \in V$. But then $V \subset S$ and so $U \cap S \neq \varnothing$, which contradicts (5.10). The theorem is therefore proved for all $z \in \partial E \setminus \partial S$, and consequently for all points z belonging to the closure of $\partial E \setminus \partial S$.

From Corollary 2.1, it follows that the above argument also applies to any sequence $\{p_n\}$, $n = 0, 1, \ldots$, satisfying (2.2). ∎

Proof of Theorem 3.1. Let $\sigma > 1$ and choose $\sigma_0 \geqq \sigma$ such that the level curve Γ_{σ_0} consists of a single Jordan curve. Then there exists a finite covering of $K \setminus \bar{E}_\sigma$,

$$K \setminus \bar{E}_\sigma \subset (K \setminus \bar{E}_{2\sigma_0}) \cup S_1 \cup S_2 \cup \cdots \cup S_m,$$

where $S_1, S_2, \ldots, S_m$ are disks in K. Then (3.1) is proved if

$$\lim_{n \to \infty} \frac{Z_n(S_1)}{n} = \cdots = \lim_{n \to \infty} \frac{Z_n(S_m)}{n} = 0 \tag{5.13}$$

and

$$\lim_{n \to \infty} \frac{Z_n(K \setminus \bar{E}_{2\sigma_0})}{n} = 0. \tag{5.14}$$

Lemma 4.2 and Theorem 4.2 show that (5.13) is satisfied. To prove (5.14), let $z_{n,1}, z_{n,2}, \ldots, z_{n,N_n}$ be the zeros of $p_n(z)$ in $K \setminus \bar{E}_{2\sigma_0}$, $N_n := Z_n(K \setminus \bar{E}_{2\sigma_0})$. Let $z = \psi_0(t) = ct + \cdots$ denote the single-valued analytic mapping of $|t| > \sigma_0$ onto $K \setminus \bar{E}_{\sigma_0}$. Then we can write

$$z_{n,1} = \psi_0(t_{n,1}),\ z_{n,2} = \psi_0(t_{n,2}), \ldots,\ z_{n,N_n} = \psi_0(t_{n,N_n}),$$

with $|t_{n,k}| \geqq 2\sigma_0$ for $k=1, 2, \ldots, N_n$. The function

$$g(t) := \frac{p_n(\psi_0(t))}{t^n} \prod_{k=1}^{N_n} \left(\frac{\sigma_0^2 - \overline{t_{n,k}}\, t}{t - t_{n,k}} \right)$$

is analytic in $\{t\colon |t| \geqq \sigma_0\}$, even at ∞. Using the maximum principle, we get

$$\max_{|t| = \sigma_0} |g(t)| \geqq |g(\infty)|$$

or

$$\| p_n \|_{\Gamma_{\sigma_0}} \geqq |a_n|\, c^n \sigma_0^{n - N_n} \prod_{k=1}^{N_n} |t_{n,k}|.$$

Since $\| p_n \|_{\Gamma_{\sigma_0}} \leqq \sigma_0^n \| p_n \|_E$ and $|t_{n,k}| \geqq 2\sigma_0$, we obtain

$$\log \| p_n \|_E \geqq \log |a_n| + n \cdot \log c + N_n \cdot \log 2$$

or

$$N_n \leqq \frac{\log \| p_n \|_E - \log |a_n| - n \cdot \log c}{\log 2}. \tag{5.15}$$

From (A4), we know that

$$\lim_{n \to \infty} \frac{\log \| p_n \|_E}{n} = 0,$$

and together with condition (A3) we obtain from (5.15) that

$$\lim_{n \to \infty} \frac{N_n}{n} = \lim_{n \to \infty} \frac{Z_n(K \backslash \bar{E}_{2\sigma_0})}{n} = 0.$$

Consequently (3.1) is true.

Next, let $\tilde{p}_n$ and q_n as in (3.2). For $z \in K \backslash \bar{E}_\sigma$, set

$$h_n(z) := \frac{\{\tilde{p}_n(z)\}^{1/(n - Z_{n,\sigma})}}{\Phi(z)}, \tag{5.16}$$

where the branch is chosen such that $h_n(\infty) > 0$. For $z \in E$, we have

$$|\tilde{p}_n(z)| \leqq \frac{|p_n(z)|}{(d_1)^{Z_{n,\sigma}}},$$

where d_1 is the minimal distance from Γ_σ to the set E. Since

$$\frac{\tilde{p}_n(z)}{[\Phi(z)]^{n - Z_{n,\sigma}}}$$

is analytic in K, we get from the maximum principle for $z \in K$:

$$\frac{|\tilde{p}_n(z)|}{|\Phi(z)|^{n-Z_{n,\sigma}}} \leq \frac{\|p_n\|_E}{(d_1)^{Z_{n,\sigma}}}.$$

Therefore we get from (3.1) and (A4) that the functions $h_n(z)$ are uniformly bounded in $K \backslash \bar{E}_\sigma$ and satisfy

$$\overline{\lim_{n\to\infty}} |h_n(z)| \leq \overline{\lim_{n\to\infty}} \|p_n\|_E^{1/(n-Z_{n,\sigma})} = 1. \tag{5.17}$$

Moreover, because of the normalization in (5.16) and the condition (A3), it follows that

$$\lim_{n\to\infty} h_n(\infty) = 1. \tag{5.18}$$

Since each function $h_n(z)$ is analytic in $(K \backslash \bar{E}_\sigma)$, we conclude from (5.17), (5.18), and the maximum principle that the functions $h_n(z)$ converge uniformly to the constant function 1 in any compact subset of $K \backslash \bar{E}_\sigma$. ∎

Proof of Theorem 3.2. We know from the preceding proof that the functions $\log h_n(z)$ converge uniformly to zero in any compact set C of $K \backslash \bar{E}_\sigma$, if we take for the logarithm the branch with $\log h_n(\infty) = 0$. Consequently, on differentiating $\log h_n(z)$, we get

$$\lim_{n\to\infty} \frac{1}{n - Z_{n,\sigma}} \frac{\tilde{p}_n'(z)}{\tilde{p}_n(z)} = \frac{\Phi'(z)}{\Phi(z)}$$

locally uniformly in $K \backslash \bar{E}_\sigma$. Then, from the definition of $\tilde{p}_n(z)$ in (3.2), we can write

$$\lim_{n\to\infty} \frac{1}{n - Z_{n,\sigma}} \sum_{z_{n,k} \in \bar{E}_\sigma} \frac{1}{z - z_{n,k}} = \frac{\Phi'(z)}{\Phi(z)} \tag{5.19}$$

locally uniformly in $K \backslash \bar{E}_\sigma$. Now, let $f(z)$ be a polynomial. Then we get from (5.19) and the fact that $Z_{n,\sigma} = o(n)$ for $n \to \infty$:

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{z_{n,k} \in \bar{E}_\sigma} f(z_{n,k}) &= \frac{1}{2\pi i} \int_{\Gamma_{\sigma^*}} f(z) \frac{\Phi'(z)}{\Phi(z)}\, dz \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\psi(\sigma^* e^{i\varphi}))\, d\varphi, \end{aligned} \tag{5.20}$$

where $\sigma^* > \sigma$. Since the integral on the right-hand side of (5.20) is the same if we integrate over Γ_τ instead of Γ_{σ^*} for $1<\tau\leqq\sigma^*$, if follows from the uniform continuity of $f\circ\psi$ on $\{t: 1\leqq|t|\leqq\sigma^*\}$ that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{z_{n,k}\in\bar{E}_\sigma} f(z_{n,k})=\frac{1}{2\pi}\int_0^{2\pi} f(\psi(e^{i\varphi}))\,d\varphi \tag{5.21}$$

for every polynomial f. Since $Z_n(\mathring{E})=o(n)$ for $n\to\infty$, we may restrict the sum in (5.21) to be over all zeros $z_{n,k}=\rho_{n,k}e^{i\varphi_{n,k}}$ with $z_{n,k}\in\bar{K}\cap\bar{E}_\sigma$. Using now for the left-hand side of (5.21) the uniform continuity of $f\circ\psi$ on the set $\{t: 1\leqq|t|\leqq\sigma\}$ again, we obtain, together with (3.1),

$$\lim_{n\to\infty}\frac{1}{n}\sum_{z_{n,k}\in\bar{K}} f(\psi(e^{i\varphi_{n,k}}))=\frac{1}{2\pi}\int_0^{2\pi} f(\psi(e^{i\varphi}))\,d\varphi, \tag{5.22}$$

for any polynomial f, where $\varphi_{n,k}=\arg(z_{n,k})$ as in (3.4). But clearly, then (5.22) holds for any function f analytic in $\mathring{E}$ and continuous in E.

The arguments $\varphi_{n,k}$ are uniformly distributed in the sense of Weyl on $[0, 2\pi]$ iff

$$\lim_{n\to\infty}\frac{1}{n}\sum_{z_{n,k}\in\bar{K}} g(\varphi_{n,k})=\frac{1}{2\pi}\int_0^{2\pi} g(\varphi)\,d\varphi \tag{5.23}$$

holds for any continuous real-valued function $g(\varphi)$ with period 2π. Let us therefore consider a function $\xi=\lambda(z)$ mapping $\mathring{E}$ conformally onto the interior of the unit disk. Then there exists a continuous extension of $\lambda(z)$ on E, such that the inverse map $z=\mu(\xi)$ is continuous on $\{\xi: |\xi|\leqq 1\}$ (cf. [5, p. 44]). For a given continuous real-valued $g(\varphi)$ with period 2π the function

$$h(\xi):=g\left(\frac{\log\Phi(\mu(\xi))}{i}\right)$$

is continuous on $\{\xi: |\xi|=1\}$. Hence, for any $\varepsilon>0$ there exists an algebraic polynomial $q(\xi)$ such that

$$\max_{|\xi|=1}|\operatorname{Re} q(\xi)-h(\xi)|\leqq\varepsilon.$$

Using (5.22) for $f(z)=q(\lambda(z))$, we get for the real part in (5.22):

$$\left|\lim_{n\to\infty}\frac{1}{n}\sum_{z_{n,k}\in\bar{K}} g(\varphi_{n,k})-\frac{1}{2\pi}\int_0^{2\pi} g(\varphi)\,d\varphi\right|\leqq 2\varepsilon.$$

Since this is true for any $\varepsilon>0$, (5.23) is proved. ∎

Proof of Theorem 3.3. Using the first part of the proof of the preceding theorem, we know that Eq. (5.21) holds for f a polynomial, $\sigma > 1$. Now, we use the partitioning

$$\sum_{z_{n,k} \in \bar{E}_\sigma} = \sum_{z_{n,k} \in \mathring{E}} + \sum_{z_{n,k} \in \bar{E}_\sigma \setminus \mathring{E}} .$$

Since $f \circ \psi$ is uniformly continuous on $\{t: 1 \mid \leqq |t| \leqq \sigma\}$ and (3.1) holds, we conclude, as in the proof of Theorem 3.2, that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{z_{n,k} \in \bar{E}_\sigma \setminus \mathring{E}} f(z_{n,k}) = \lim_{n \to \infty} \frac{1}{n} \sum_{z_{n,k} \in \bar{K}} f(\psi(e^{i\varphi_{n,k}})),$$

where, as in (3.4), $z_{n,k} = \psi(\rho_{n,k} e^{i\varphi_{n,k}})$ for $z_{n,k} \in \bar{K}$. Hence we have

$$\begin{aligned} \lim_{n \to \infty} \frac{1}{n} &\left[\sum_{z_{n,k} \in \mathring{E}} f(z_{n,k}) + \sum_{z_{n,k} \in \bar{K}} f(\psi(e^{i\varphi_{n,k}})) \right] \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\psi(e^{i\varphi}))\, d\varphi, \end{aligned} \tag{5.24}$$

for every polynomial f. But then (5.24) is true for all functions f analytic in $\mathring{E}$ and continuous on E.

Let J be contained in the interior of the free one-sided boundary arc J_1 of K. It follows by the reflection principle that $\psi(t)$ can be extended analytically to $T(\varepsilon_0)$ for some $\varepsilon_0 > 0$ such that $\psi(T(\varepsilon_0)) \subset \bar{K} \cup B$.

Considering the Jordan region B of $\mathring{E}$, where $J \subset \bar{B}$, we define a fixed conformal one-to-one mapping $\xi = \tilde{\lambda}(z)$ from B onto the interior of the unit disk such that the continuous extension onto $\bar{B}$ satisfies

$$\tilde{\lambda}(z_1) = e^{i\alpha} \qquad \text{and} \qquad \tilde{\lambda}(z_2) = e^{i\beta},$$

where $z_1 = \psi(e^{i\alpha})$ and $z_2 = \psi(e^{i\beta})$. Let $z = \tilde{\mu}(\xi)$ denote the continuous inverse function of $\xi = \tilde{\lambda}(z)$ mapping $\{\xi: |\xi| \leqq 1\}$ onto $\bar{B}$, and let

$$J_1 = \{\psi(t): t = e^{i\varphi}, \alpha^* \leqq \varphi \leqq \beta^*\}, \tag{5.25}$$

where $\alpha^* < \alpha < \beta < \beta^*$. We fix a real number $\delta > 0$ such that

$$0 < \delta < \min\left(\alpha - \alpha^*, \beta^* - \beta, \frac{\beta - \alpha}{2}\right), \tag{5.26}$$

and construct the continuous, real-valued, 2π-periodic function $g_{\delta,1}(\varphi)$, whose graph is made up of straight-line segments connecting the points

$$(\beta + \delta - 2\pi, 0), (\alpha - \delta, 0), (\alpha, 1), (\beta, 1), \quad (\beta + \delta, 0).$$

Analogously the graph of the 2π-periodic function $g_{\delta,2}(\varphi)$ is made up of straight-line segments connecting the points

$$(\beta - 2\pi, 0), (\alpha, 0), (\alpha + \delta, 1), (\beta - \delta, 1), (\beta, 0).$$

Since Φ can be extended analytically onto the interior $\operatorname{int}(J_1)$ of J_1 [5], the functions $h_{\delta,v}(\xi)$ defined for $|\xi| = 1$ by

$$\begin{aligned} h_{\delta,v}(\xi) &:= g_{\delta,v}\left(\frac{\log \Phi(\tilde{\mu}(\xi))}{i}\right), && \text{for} \quad \xi \in \tilde{\lambda}(\operatorname{int}(J_1)), \\ &:= 0, && \text{elsewhere,} \end{aligned}$$

are continuous ($v = 1, 2$). Now, there exist algebraic polynomials $q_{\delta,1}(\xi)$ and $q_{\delta,2}(\xi)$ such that

$$\max_{|\xi|=1} |\operatorname{Re}(q_{\delta,v}(\xi)) - h_{\delta,v}(\xi)| < \delta. \tag{5.27}$$

Let S be some connected component of $E \backslash \bar{B}$: Since K is connected, the intersection $\bar{S} \cap \bar{B}$ contains exactly one point. Therefore, for $v = 1, 2$, the continuous extension $Q_{\delta,v}(z)$ to E of the function $q_{\delta,v}(\tilde{\lambda}(z))$, where $Q_{\delta,v}(z)$ is constant on each connected component of $E \backslash \bar{B}$, is well-defined. Moreover, $Q_{\delta,v}(z)$ is analytic in $\mathring{E}$ and we may apply (5.24) to $f(z) = Q_{\delta,v}(z)$. Considering the real part of (5.24), we get with (5.27) for the right-hand side

$$\operatorname{Re}\left(\frac{1}{2\pi}\int_0^{2\pi} Q_{\delta,v}(\psi(e^{i\varphi})\right) d\varphi = \frac{\beta - \alpha}{2\pi} + R_v(\delta),$$

where $\lim_{\delta \to 0} R_v(\delta) = 0$ for $v = 1, 2$. Because of (5.27) there exists a positive number $\varepsilon_1 \leqq \varepsilon_0$ such that

$$\operatorname{Re}(Q_{\delta,1}(z)) \geqq 1 - 2\delta, \qquad \text{for all} \quad z \in J(\varepsilon_1) \cap \bar{B}, \tag{5.29}$$

and a positive number $\varepsilon_2 \leqq \varepsilon_0$ such that

$$\operatorname{Re}(Q_{\delta,2}(z)) \leqq 2\delta, \qquad \text{for all} \quad z \in B(\varepsilon_2) \backslash J(\varepsilon_1), \tag{5.30}$$

where

$$B(\varepsilon_2) := \{z \in B: z = \tilde{\mu}(\xi), |\xi| \geqq 1 - \varepsilon_2\}.$$

Since $\operatorname{Re}(Q_{\delta,v}(z))$ is harmonic in B, it follows by (5.27) and the definition of $Q_{\delta,v}(z)$ that for all $z \in E$,

$$\operatorname{Re}(Q_{\delta,1}(z)) \geqq -\delta \qquad \text{and} \qquad \operatorname{Re}(Q_{\delta,2}(z)) \leqq 1 + \delta. \tag{5.31}$$

Moreover, by definition we have

$$\operatorname{Re}(Q_{\delta,2}(z)) \leqq \delta, \qquad \text{for all} \quad z \in E \backslash \bar{B}. \tag{5.32}$$

Assumption (3.7) yields

$$Z_n(B \backslash B(\varepsilon_2)) = o(n), \qquad \text{as} \quad n \to \infty. \tag{5.33}$$

Hence, by (5.29) and (5.31), we obtain

$$\frac{1}{n} \operatorname{Re}\left[\sum_{z_{n,k} \in \mathring{E}} Q_{\delta,1}(z_{n,k}) + \sum_{z_{n,k} \in \bar{K}} Q_{\delta,1}(\psi(e^{i\varphi_{n,k}}))\right]$$
$$\geqq \frac{Z_n(J(\varepsilon_1))}{n}(1-2\delta) - \delta \geqq \frac{Z_n(J(\varepsilon_1))}{n} - 3\delta$$

or, using (5.24) and (5.28):

$$\overline{\lim_{n \to \infty}} \frac{Z_n(J(\varepsilon_1))}{n} \leqq \frac{\beta - \alpha}{2\pi} + R_1(\delta) + 3\delta. \tag{5.34}$$

Concerning the function $Q_{\delta,2}(z)$, we obtain by (5.30)–(5.33) that

$$\frac{1}{n} \operatorname{Re}\left[\sum_{z_{n,k} \in \mathring{E}} Q_{\delta,2}(z_{n,k}) + \sum_{z_{n,k} \in \bar{K}} Q_{\delta,2}(\psi(e^{i\varphi_{n,k}}))\right]$$
$$\leqq \frac{Z_n(J(\varepsilon_1))}{n}(1+\delta) + 2\delta + o(n),$$

and, again by (5.24), (5.28):

$$\underline{\lim_{n \to \infty}} \frac{Z_n(J(\varepsilon_1))}{n} \geqq \frac{\beta - \alpha}{2\pi} + R_2(\delta) - 3\delta + 0(n). \tag{5.35}$$

Since δ can be arbitrarily small, (3.9) follows from (5.34), (5.35), and (3.7). ∎

Proof of Theorem 3.4. Let J be a subarc in the interior of the free two-sided boundary arc J_1,

$$J_1 = \{\psi(t) \colon t = e^{i\varphi}, \alpha^* \leqq \varphi \leqq \beta^*\},$$

where $\alpha^* < \alpha < \beta < \beta^*$. For

$$\delta < \min\left(\alpha - \alpha^*, \beta^* - \beta, \frac{\beta - \alpha}{2}\right),$$

we consider the real-valued functions $g_{\delta,1}(\varphi)$, whose graph is made up by straight-line segments connecting the points

$$(\beta+\delta-2\pi, 0), \quad (\alpha-\delta, 0), \quad (\alpha, 1), (\beta, 1), (\beta+\delta, 0)$$

and the function $g_{\delta,2}(\varphi)$ whose graph is made up by straight-line segments between the points

$$(\beta-2\pi, 0), \quad (\alpha, 0), (\alpha+\delta, 1), (\beta-\delta, 1), (\beta, 0).$$

We define for $z \in E$ and $v = 1, 2$ the functions

$$\begin{aligned} f_{\delta,v}(z) &:= g_{\delta,v}(\varphi), \qquad && \text{for } z = \psi(t) \text{ with } t = e^{i\varphi} \text{ and } \varphi \in [\alpha^*, \beta^*], \\ &:= 0, && \text{for } \; z \in E \backslash J_1. \end{aligned}$$

$f_{\delta,1}(z)$ and $f_{\delta,2}(z)$ are analytic on $\mathring{E}$, continuous on E, and we may apply the Eq. (5.24). Since $f_{\delta,1}(z) = f_{\delta,2}(z) = 0$ in $\mathring{E}$, we conclude from (5.24) that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{z_{n,k} \in \bar{K}} & f_{\delta,v}(\psi(e^{i\varphi_{n,k}})) \\ &= \frac{1}{2\pi} \int_0^{2\pi} f_{\delta,v}(\psi(e^{i\varphi}))\, d\varphi \\ &= \frac{\beta-\alpha+\tilde{\beta}-\tilde{\alpha}}{2\pi} + R_v(\delta), \end{aligned} \tag{5.36}$$

where $\lim_{\delta\to 0} R_v(\delta) = 0$. Inserting in (5.36) the inequalities

$$\frac{1}{n} \sum_{z_{n,k} \in \bar{K}} f_{\delta,1}(\psi(e^{i\varphi_{n,k}})) \geqq \frac{Z_n(\tilde{J}(\varepsilon))}{n}$$

and

$$\frac{1}{n} \sum_{z_{n,k} \in \bar{K}} f_{\delta,2}(\psi(e^{i\varphi_{n,k}})) \leqq \frac{Z_n(\tilde{J}(\varepsilon))}{n},$$

we obtain, since $\delta > 0$ can be chosen arbitrarily small:

$$\lim_{n\to\infty} \frac{Z_n(\tilde{J}(\varepsilon))}{n} = \frac{\beta-\alpha+\tilde{\beta}-\tilde{\alpha}}{2\pi}. \quad \blacksquare$$

Proof of Corollary 3.1. Let J be a subarc in the interior of the free one-sided boundary arc J_1, with J_1 represented as in (5.25). We fix again $\delta > 0$

such that (5.26) holds. Then there exists $\varepsilon_0 > 0$ such that $\psi(t)$ can be extended to a conformal mapping on the set

$$\{t = \rho e^{i\varphi}: \rho \geqq 1 - \varepsilon_0, \alpha - \delta \leqq \varphi \leqq \beta + \delta\}.$$

Let $0 < \varepsilon \leqq \varepsilon_0$ and define

$$J_\delta(\varepsilon) := \{\psi(t): t = \rho e^{i\varphi}, 1 - \varepsilon \leqq \rho \leqq 1 + \varepsilon, \alpha - \delta \leqq \varphi \leqq \beta + \delta\}.$$

Then there exists $\varepsilon_1 > 0$ such that $J_\delta(\varepsilon) \subset \bar{K} \cup B$ and

$$(\bar{D} \cap E_{1+\varepsilon}) \subset (J_\delta(\varepsilon) \cup B)$$

for all $0 < \varepsilon \leqq \varepsilon_1$. For such ε the set

$$C := \bar{D} \cap \overline{(B \backslash J_\delta(\varepsilon))}$$

is compact in B. Hence, by (3.7), we get for any $0 < \varepsilon \leqq \varepsilon_1$

$$\varlimsup_{n \to \infty} \frac{Z_n(\bar{D} \cap E_{1+\varepsilon})}{n} \leqq \lim_{n \to \infty} \frac{Z_n(J_\delta(\varepsilon))}{n}. \tag{5.37}$$

Analogously, let

$$J_\delta^*(\varepsilon) := \{\psi(t): t = \rho e^{i\varphi}, 1 - \varepsilon \leqq \rho \leqq 1 + \varepsilon, \alpha + \delta \leqq \varphi \leqq \beta - \delta\}.$$

Then there exists a real number ε_2, $0 < \varepsilon_2 \leqq \varepsilon_1$, such that

$$J_\delta^*(\varepsilon_2) \subset D.$$

Consequently,

$$\varlimsup_{n \to \infty} \frac{Z_n(D)}{n} \geqq \lim_{n \to \infty} \frac{Z_n(J_\delta^*(\varepsilon_2))}{n}. \tag{5.38}$$

Since δ can be chosen arbitrary small, we conclude from (3.1) in Theorem 3.1, Theorem 3.3, and the inequalities (5.37) and (5.38) that (3.12) is true. ∎

Corollary 3.2 is obtained by using just the same arguments.

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