Polynomials with Laguerre Weights in L^p

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Abstract. For each p $(0 < p \leq \infty)$, $s \geq 0$, and integer $m \geq 1$ we consider the extremal problem

$$E_{s,m,p} := \inf\{\| t^s e^{-t}(t^m - q_{m-1}(t)) \|_{L^p} : q_{m-1} \in \mathcal{P}_{m-1}\} ,$$

where the L^p-norm is taken over $[0,\infty)$ and $\mathcal{P}_{m-1}$ is the collection of polynomials of degree at most $m-1$. The asymptotic behavior of $E_{s,m,p}$ as $n := s+m \to \infty$ and $s/n \to \theta$ is determined along with the zero distribution for the associated Chebyshev polynomials. The paper includes the proofs of results announced in [7].

1. Introduction

Motivated by the theory of incomplete polynomials introduced by Lorentz [5], we proved in [6] that for every $\alpha > 0$ and integer $n \geq 0$, there is a constant $a(n,\alpha)$ such that for every polynomial P of degree at most n ,

$$\max_{x \in \mathbb{R}} |\exp(-|x|^\alpha)P(x)| = \max_{|y| \leq a(n,\alpha)} |\exp(-|y|^\alpha)P(y)| . \tag{1.1}$$

We found an explicit expression for $a(n,\alpha)$ in (1.1) which is asymptotically best possible (as $n \to \infty$) and also applied our results to the theory of certain orthogonal polynomials, now known as Freud polynomials. In order to develop a general theory unifying the previous results [3], [4], [5], [10], [11], [12] of Saff, Varga, Lachance, Lorentz, Kemperman, Ullman and others and our results in [6], we considered the case of the Laguerre weight function $x^s e^{-x}$ on $[0,\infty)$. The results for this weight were announced in [7]. In this paper, we shall provide the proofs.

During the last year, we did, in fact, make significant progress towards the development of a general theory ([8], [9]). Thus, instead

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of presenting the proofs which we had at the time of the publication of [7], we plan to use, as much as possible, the more general and more recent results in [8]. The novelty of the present paper is threefold. First, for the concrete case of the Laguerre weights, we can find explicitly the various quantities whose existence is asserted in [8]. In turn, we shall use these explicit expressions to prove the sharpness of our results. Second, we shall discuss some applications to the theory of Laguerre polynomials and other extremal polynomials. Third, the case of the Laguerre weights is, in a sense, "midway" between the case of the Jacobi weights studied in [10], [4] and the exponential weights investigated in [6]. Thus, it serves as an illustration of the connection between the two.

For the above reasons, we believe that Laguerre weights deserve a separate discussion, even in the presence of the general theory. In Section 2, we shall state the main theorems; the proofs are given in Section 3.

2. Main Results

For a Lebesgue measurable function g on an interval $I \subset [0,\infty)$, set

$$(2.1) \qquad \|g\|_{p,I} := \begin{cases} \left(\int_I |g(x)|^p dx \right)^{1/p} & \text{if } 0 < p < \infty, \\ \operatorname*{ess\,sup}_{x \in I} |g(x)| & \text{if } p = \infty. \end{cases}$$

The suffix I will be omitted if $I = [0,\infty)$ and the suffix p will be omitted if $p = \infty$.

Our first theorem is an analogue of Theorem 2.7 of [6] and, in part, follows as a consequence of the more general Theorem 2.2 of [8].

Theorem 2.1: *Let* $m \geq 0$ *be an integer,* $s \geq 0$, $\mu > 0$, $n := s + m > 0$ *and* $0 \not\equiv P_m \in \mathcal{P}_m$ *(the class of all polynomials of degree at most* m *). If* $\xi \in [0,\infty)$ *satisfies*

$$(2.2) \qquad |\xi^s e^{-\mu\xi} P_m(\xi)| = \|x^s e^{-\mu x} P_m(x)\|,$$

then

(2.3) $\quad a \le \xi \le b \quad ,$

where

$$a = a(s,n,\mu) := \frac{n}{\mu}\left(1 - \sqrt{1-(s/n)^2}\right) \quad ,$$

(2.3')

$$b = b(s,n,\mu) := \frac{n}{\mu}\left(1 + \sqrt{1-(s/n)^2}\right) \quad .$$

In particular,

(2.4) $$\|t^s e^{-\mu t} p_m(t)\|_{[a,b]} = \|x^s e^{-\mu x} p_m(x)\| \quad , \quad \forall\, p_m \in \mathcal{P}_m \quad .$$

In Section 3, we shall prove (2.4) using the more general results in [8]. However, a complete proof of Theorem 2.1 requires some explicit computations similar to those in [6].

Our next theorem concerns the asymptotic behavior of certain extremal polynomials. Before stating this result, we introduce some needed notation.

For each $p\,(0 < p \le \infty)$, $s \ge 0$ and integer $m \ge 1$, set

(2.5) $$E_{s,m,p} := \min\{\|t^s e^{-t}(t^m - g_{m-1}(t))\|_p \;:\; g_{m-1} \in \mathcal{P}_{m-1}\}$$

and let $T_{s,m,p}(t) = t^m + \ldots \in \mathcal{P}_m$ satisfy

(2.6) $$E_{s,m,p} = \|t^s e^{-t} T_{s,m,p}(t)\|_p \quad .$$

Theorem 2.2. *Let* $\theta\ (0 \le \theta < 1)$ *be fixed and suppose* $\{s_i\}$ *is a sequence of nonnegative real numbers and* $\{m_i\}$ *is a sequence of nonnegative integers such that* $n_i := s_i + m_i > 0$ *for each* i , $n_i \to \infty$ *and*

(2.7) $$s_i/n_i \to \theta \quad \textit{as} \quad i \to \infty \; .$$

Then,

(a) *For each* $p\ (0 < p \le \infty)$, *the minimal error defined in* (2.5) *satisfies*

(2.8) $$\lim_{i\to\infty} n_i^{-1} E_{s_i,m_i,p}^{1/n_i} = \left|\frac{(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}}{4e^2}\right|^{1/2} \quad .$$

(b) *There are* $m_i + 1$ *nonnegative numbers* $\xi_{1i} < \cdots < \xi_{m_i+1,i}$ *such that*

$$(2.9) \qquad |\xi_{ji}^{s_i} e^{-\xi_{ji}} T_{s_i,m_i,\infty}(\xi_{ji})| = E_{s_i,m_i,\infty} , \quad j = 1,\ldots,m_i+1 .$$

Furthermore,

$$(2.10) \qquad \xi_{1i}/n_i \to 1 - \sqrt{1-\theta^2} \quad \text{and} \quad \xi_{m_i+1,i}/n_i \to 1 + \sqrt{1-\theta^2} \quad \text{as} \quad i \to \infty .$$

(*In this sense, Theorem* 2.1 *is sharp*.)

(c) *All the zeros of the extremal polynomials* $T_{s_i,m_i,p}$ *are real and, if* $p \geq 1$, *simple*. *For each* p $(0 < p \leq \infty)$ *and interval* $[c,d] \subset [0,\infty)$, *let* $N_{i,p}(c,d)$ *denote the number of zeros of the normalized polynomial* $T_{s_i,m_i,p}(n_i t)$ *which lie in* $[c,d]$. *Then*

$$(2.11) \qquad \lim_{i\to\infty} \frac{N_{i,p}(c,d)}{m_i} = \int_c^d h(\theta,x)\,dx ,$$

where

$$(2.12) \qquad h(\theta,x) := \frac{1}{(1-\theta)\pi} \frac{\sqrt{(b^* - x)(x - a^*)}}{x} \quad \text{if} \quad x \in [a^*, b^*] ,$$

$$a^* := 1 - \sqrt{1-\theta^2} , \quad b^* := 1 + \sqrt{1-\theta^2} ,$$

and $h(\theta,x) = 0$ *if* $x \notin [a^*,b^*]$.

While the case $\theta > 0$ and $p = \infty$ of Theorem 2.2 follows immediately from the results in [8], the assertion for arbitrary L^p-norms requires a new argument.

3. Proofs

It is convenient to prove (2.4) of Theorem 2.1 first.

Proof of (2.4). Since the case $m = 0$ of (2.4) is trivial to verify, we assume hereafter that $m > 0$. Our goal is to find a finite interval $[a,b]$ such that for every $P_m \in \mathcal{P}_m$

$$(3.1) \qquad \| [w(t)]^m P_m(t) \|_{[a,b]} = \| [w(x)]^m P_m(x) \| ,$$

where

(3.2) $$w(t) := t^{s/m}e^{-\mu t/m} .$$

Since $Q(t) := \log(1/w(t)) = (\mu t - s\log t)/m$ is convex on $(0,\infty)$, we can now apply Theorem 2.2 of [8]. According to that theorem, the interval $[a,b]$ can be found by maximizing an "F-functional" defined by

(3.3) $$F(c,d) := \log\left(\frac{d-c}{4}\right) - \frac{1}{\pi}\int_c^d \frac{Q(t)\,dt}{\sqrt{(t-c)(d-t)}} , \quad 0 \le c < d .$$

To evaluate $F(c,d)$ explicitly, write

(3.4) $$t = \frac{c+d}{2} + \frac{d-c}{2}\cos\theta = \left(\frac{\sqrt{c}+\sqrt{d}}{2}\right)^2\left(1+\frac{e^{i\theta}}{\phi}\right)\left(1+\frac{e^{-i\theta}}{\phi}\right) ,$$

where $\phi := (\sqrt{d}+\sqrt{c})/(\sqrt{d}-\sqrt{c})$. Then, with $\alpha := s/m$, $\beta := \mu/m$, we have

(3.5) $$Q(t) = \beta t - \alpha\log t = \beta\left(\frac{c+d}{2}\right) - 2\alpha\log\left(\frac{\sqrt{c}+\sqrt{d}}{2}\right) + \beta\left(\frac{d-c}{2}\right)\cos\theta + 2\alpha\sum_{k=1}^{\infty}\frac{(-1)^k}{k\,\phi^k}\cos k\theta .$$

Substituting (3.5) and (3.4) into (3.3) we get

(3.6) $$F(c,d) = 2\alpha\log\left(\frac{\sqrt{c}+\sqrt{d}}{2}\right) - \beta\left(\frac{c+d}{2}\right) + \log\left(\frac{d-c}{4}\right) .$$

It is now elementary to check that the choice of c,d $(d > c)$ which maximizes $F(c,d)$ is given by

(3.7) $$c = a = \frac{1}{\mu}\left(n - \sqrt{n^2 - s^2}\right) , \quad d = b = \frac{1}{\mu}\left(n + \sqrt{n^2 - s^2}\right) ,$$

where $n := s + m$. □

A simple computation shows that the maximum value of F is

(3.8) $$\bar{F} := F(a,b) = \frac{1}{2(1-\gamma)}\log\left[\frac{n^2(1+\gamma)^{1+\gamma}(1-\gamma)^{1-\gamma}}{4e^2\mu^2}\right] , \quad \gamma := \frac{s}{n} .$$

Theorem 2.3 of [8] and the remark following Lemma 4.3 of [8] assert that there exists a (necessarily unique) unit measure ν with support $[a,b]$ such that

$$(3.9) \qquad \int_a^b \log|t-x|\,d\nu(x) = \frac{\mu}{m}t - \frac{s}{m}\log t + \bar{F}, \quad \forall\ t \in [a,b].$$

Further, if $P_m \in \mathcal{P}_m$ and

$$(3.10) \qquad |t^s e^{-\mu t} P_m(t)| \le M, \quad \forall\ t \in [a,b],$$

then

$$(3.11) \qquad |P_m(z)| \le M \exp\left(m\left[\int_a^b \log|z-t|\,d\nu(t) - \bar{F}\right]\right), \quad \forall\ z \in \mathbb{C}.$$

In order to complete the proof of Theorem 2.1, we shall explictly compute $d\nu$ and then estimate the right-hand side of (3.11). These technical results are summarized in the following lemma.

Lemma 3.1. *Set*

$$(3.12) \qquad g(x) := g(s,m,\mu;x) := \frac{\mu(b-x)(x-a)}{mx}, \quad x \in [a,b],$$

where a,b *are given in* (3.7) *with* $n := s+m$. *Then*

$$(3.13) \qquad \frac{1}{\pi}\int_a^b \frac{g(x)\,dx}{\sqrt{(b-x)(x-a)}} = 1,$$

$$(3.14) \qquad \frac{1}{\pi}\int_a^b \log|x-t|\,\frac{g(x)\,dx}{\sqrt{(b-x)(x-a)}} = \frac{\mu}{m}t - \frac{s}{m}\log t + \bar{F}, \quad \forall\ t \in [a,b],$$

$$(3.15) \qquad \frac{1}{\pi}\int_a^b \log|x-t|\,\frac{g(x)\,dx}{\sqrt{(b-x)(x-a)}} < \frac{\mu}{m}t - \frac{s}{m}\log t + \bar{F}, \ \forall\ t \in [0,\infty)\setminus[a,b].$$

Proof. Let

$$(3.16) \qquad x =: \frac{a+b}{2} + \frac{b-a}{2}\cos\phi$$

and observe from (3.7) that with $\alpha := s/m$, $\beta := \mu/m$ we have

$$(3.17) \qquad a = \frac{1}{\beta}\left[1+\alpha-\sqrt{1+2\alpha}\right], \quad b = \frac{1}{\beta}\left[1+\alpha+\sqrt{1+2\alpha}\right],$$

$$(3.18)\qquad \left(\frac{\sqrt{b}+\sqrt{a}}{2}\right)^2 = \frac{1+2\alpha}{2\beta}\ , \qquad \sqrt{ab} = \frac{\alpha}{\beta}\ ,$$

$$(3.19)\qquad \Phi := \frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}} = \sqrt{1+2\alpha} = \beta\left(\frac{b-a}{2}\right)\ ,$$

$$(3.20)\qquad x = \frac{1}{\beta}(1+\alpha+\Phi\cos\phi) = \frac{1+2\alpha}{2\beta}\left(1+\frac{e^{i\phi}}{\Phi}\right)\left(1+\frac{e^{-i\phi}}{\Phi}\right)\ .$$

Substituting (3.17) into (3.12) we get

$$g(x) = 2+2\alpha-\beta x-\frac{\alpha^2}{\beta x} = 2+\alpha-\beta x-\alpha\left[\frac{(\alpha/\beta)-x}{x}\right]$$

which, with the aid of (3.19) and (3.20), can be written as

$$(3.21)\qquad g(x) = 1-\Phi\cos\phi+2\alpha\left(\frac{1+\Phi\cos\phi}{\Phi^2\left|1+\frac{e^{i\phi}}{\Phi}\right|^2}\right)$$

$$= 1-\Phi\cos\phi+2\alpha\ \mathrm{Re}\left\{\frac{e^{i\phi}/\Phi}{1+e^{i\phi}/\Phi}\right\}$$

$$= 1-\Phi\cos\phi-2\alpha\sum_{k=1}^{\infty}\frac{(-1)^k}{\Phi^k}\cos k\phi\ .$$

Equation (3.13) is now clear. Also, (3.14) follows from the identity

$$(3.22)\qquad \frac{1}{\pi}\int_a^b \log|x-t|\,\frac{g(x)\,dx}{\sqrt{(b-x)(x-a)}} = \log\frac{b-a}{4} + \frac{1}{\pi}\int_{-\pi}^{\pi}\log|1-e^{i(\theta-\phi)}|\,g(x)\,d\phi\ ,$$

where θ is defined in (3.4), after comparing the Fourier series of the right-hand side of (3.22) with (3.5) for $c=a$, $d=b$.

To prove (3.15), let

$$(3.23)\qquad \psi(t) := \frac{d}{dt}\left\{\frac{1}{\pi}\int_a^b \log|x-t|\,\frac{g(x)\,dx}{\sqrt{(b-x)(x-a)}} - \beta t + \alpha\log t\right\}$$

$$= \frac{\beta}{\pi}\int_a^b \frac{\sqrt{(b-x)(x-a)}}{x(t-x)}\,dx - \beta + \frac{\alpha}{t}\ .$$

We shall show that $\psi(t) > 0$ if $t < a$. Write

$$(3.24) \qquad x = \frac{a+b}{2} + \frac{b-a}{2}\cos\phi \quad , \quad t = \frac{a+b}{2} - \frac{b-a}{2}\lambda \quad , \quad \lambda > 1 \quad ,$$

and set $R := (b+a)/(b-a)$. It requires only elementary computations to see that

$$\psi(t) = \frac{\beta}{R-\lambda}\left[\sqrt{\lambda^2-1} - \sqrt{R^2-1}\right] + \frac{2\alpha}{(b-a)(R-\lambda)}$$

$$= \frac{\beta\sqrt{\lambda^2-1}}{R-\lambda} + \frac{1}{R-\lambda}\left[\frac{2\alpha}{b-a} - \beta\sqrt{R^2-1}\right].$$

But, in view of (3.18), the last expression can be simplified to get

$$\psi(t) = \frac{\beta\sqrt{\lambda^2-1}}{R-\lambda} > 0 \quad .$$

A similar computation shows that $\psi(t) < 0$ if $t > b$. Hence from (3.14) it follows that (3.15) holds. □

<u>Proof of Theorem</u> 2.1. In view of Lemma 3.1, the measure ν of (3.9) and (3.11) is given by $g(x)dx/\pi\sqrt{(b-x)(x-a)}$. Hence, the proof is complete, using (2.4), (3.11) and (3.15). □

To facilitate the proof of Theorem 2.2 for the case $p = \infty$, $0 < \theta < 1$, we introduce some abbreviations. Let

$$w_i(t) := t^{s_i/m_i}e^{-t} \quad , \quad w_\theta(t) := t^{\theta/(1-\theta)}e^{-t} \quad ,$$

$$(3.25) \qquad \hat{T}_i(t) = t^{m_i} + \ldots \quad , \quad \hat{T}_{\theta,i}(t) = t^{m_i} + \ldots \in P_{m_i} \quad ,$$

$$E_{i,m_i} := \|[w_i(t)]^{m_i}\hat{T}_i(t)\|_\infty = \min_{P\in P_{m_i-1}} \|[w_i(t)]^{m_i}[t^{m_i} - P(t)]\|_\infty \, ,$$

$$E_{m_i} := \|[w_\theta(t)]^{m_i}\hat{T}_{\theta,i}(t)\|_\infty = \min_{P\in P_{m_i-1}} \|[w_\theta(t)]^{m_i}[t^{m_i} - P(t)]\|_\infty \quad .$$

Note that

$$(3.26) \qquad \hat{T}_i(t) = m_i^{-m_i}T_{s_i,m_i,\infty}(m_i t) \quad , \quad E_{i,m_i} = m_i^{-n_i}E_{s_i,m_i,\infty} \quad .$$

Also observe that, by Theorem 2.1, there exists a finite interval

$[\bar{a},\bar{b}] \subset (0,\infty)$ such that (with $\mu = m_i$, $s_i/n_i \to \theta$) all the sup norms in (3.25) are attained on $[\bar{a},\bar{b}]$ for i sufficiently large.

We divide the proof of Theorem 2.2 into several special cases.

Proof of Theorem 2.2 ($p=\infty$, $0 < \theta < 1$): (a) In view of Theorem 2.2 of [8] and the formula for $\bar{F}$ in (3.8) we have

$$\lim_{i\to\infty} E_{m_i}^{1/m_i} = (1-\theta)^{\frac{-1}{1-\theta}} \left\{ \frac{(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}}{4e^2} \right\}^{\frac{1}{2-2\theta}} =: \Delta \quad . \tag{3.27}$$

However, since $w_i(t)/w_\theta(t) \to 1$ uniformly on $[\bar{a},\bar{b}]$, where all the sup norms in (3.25) are actually attained, it is easy to see that (3.27) holds also for E_{i,m_i} replacing E_{m_i}. In view of (3.26), then, (2.8) follows by elementary computations.

(b) The existence of the extreme points is a consequence of the general theory for Haar systems. The limiting relations (2.10) will follow from part (c) together with Theorem 2.1.

(c) We first obtain the zero distribution for the polynomials $\hat{T}_i(t)$. For this purpose we show that

$$\limsup_{i\to\infty} \| [w_\theta(t)]^{m_i} \hat{T}_i(t) \|_\infty^{1/m_i} \le \Delta \ , \tag{3.28}$$

that is,

$$\limsup_{i\to\infty} \left\| [w_\theta(t)]^{m_i} \hat{T}_i(t) \right\|_{\infty,[\bar{a},\bar{b}]}^{1/m_i} \le \Delta \ . \tag{3.29}$$

Let $\varepsilon > 0$ and choose an integer I so large that $i \ge I$ implies

$$w_\theta(t)/w_i(t) \le 1+\varepsilon \ , \quad \forall\ t \in [\bar{a},\bar{b}] \ ,$$

and

$$E_{i,m_i}^{1/m_i} \le \Delta(1+\varepsilon) \quad .$$

Then, for $i \ge I$,

$$\begin{aligned} \| [w_\theta(t)]^{m_i} \hat{T}_i(t) \|_{\infty,[\bar{a},\bar{b}]} &\le (1+\varepsilon)^{m_i} \| [w_i(t)]^{m_i} \hat{T}_i(t) \| \\ &\le (1+\varepsilon)^{2m_i} \Delta^{m_i} \end{aligned}$$

from which (3.29) follows. Now, applying Theorem 2.4 of [8], we see from Lemma 3.1 that the limiting zero distribution of the $\hat{T}_i(t)$'s is given by $g(x)dx/\pi\sqrt{(b-x)(x-a)}$, where in the definition (3.12) of $g(x)$ we take $\mu = m$, $a = (1-\sqrt{1-\theta^2})/(1-\theta)$, $b = (1+\sqrt{1-\theta^2})/(1-\theta)$. Hence, from (3.26), this must also be the limiting zero distribution for the polynomials $T_{s_i,m_i,\infty}(m_i t)$. With a change of variable, we then obtain (2.11). □

Proof of Theorem 2.2: $(p = \infty, \theta = 0)$. In this case $\bar{a} = 0$ and $w_i(t)/w_\theta(t)$ need not converge to 1 uniformly on $[0,\bar{b}]$. However,

$$w_i(t) = t^{s_i/m_i} e^{-t} \le \bar{b}^{s_i/m_i} e^{-t} = \bar{b}^{s_i/m_i} w_\theta(t) \;, \quad t \in [0,\bar{b}] \;,$$

and hence

$$E_{i,m_i} = \| [w_i(t)]^{m_i} \hat{T}_i(t) \|_\infty \le \| [w_i(t)]^{m_i} \hat{T}_{\theta,i}(t) \|_\infty$$
$$\le \bar{b}^{s_i} \| [w_\theta(t)]^{m_i} \hat{T}_{\theta,i}(t) \|_\infty = \bar{b}^{s_i} E_{m_i} \;.$$

Since $s_i/m_i \to 0$ as $i \to \infty$, we get

$$\text{(3.30)} \qquad \limsup_{i\to\infty} E_{i,m_i}^{1/m_i} \le \lim_{i\to\infty} E_{m_i}^{1/m_i} = \frac{1}{2e} \;,$$

where the equality follows from (3.27) with $\theta = 0$. Furthermore, Theorem 2.1 of [8] gives

$$\liminf_{i\to\infty} E_{i,m_i}^{1/m_i} \ge \frac{1}{2e} \;.$$

Hence $E_{i,m_i}^{1/m_i} \to 1/2e$ as $i \to \infty$ which is equivalent to assertion (2.8) when $\theta = 0$.

Once more, part (b) will follow from part (c). To prove part (c), observe that since $s_i/m_i \to 0$ as $i \to \infty$, the distribution of zeros for $\hat{q}_i(t) := t^{s_i}\hat{T}_i(t)$ is the same as that for $\hat{T}_i(t)$. From (3.30) we get (with $w_\theta(t) = e^{-t}$)

$$\limsup_{i\to\infty} \left\| [w_\theta(t)]^{m_i+s_i} t^{s_i} \hat{T}_i(t) \right\|_{\infty,[0,\bar{b}]}^{1/(m_i+s_i)}$$
$$\le \limsup_{i\to\infty} \left\{ \| [w_i(t)]^{m_i} \hat{T}_i(t) \|_{\infty,[0,\bar{b}]}^{1/m_i} \right\}^{m_i/(m_i+s_i)}$$
$$\le \frac{1}{2e} \;.$$

Hence Theorem 2.4 of [8] gives the zero distribution for the sequence $t^{s_i}\hat{T}_i(t)$ and therefore for the $\hat{T}_i(t)$. Just as in the case $\theta > 0$, this leads to the desired zero distribution for the polynomials $T_{s_i,m_i,\infty}(n_i t)$. □

We now turn to the case when $p < \infty$. As in [6], our major tool will be Nikolskii-type inequalities. In order to use these, we need an estimate on the Christoffel function for the Laguerre weights $x^{\alpha}e^{-x}$ which is independent of α if $0 \le \alpha \le 1$. To this end, we prove

Lemma 3.2. Suppose k is a positive integer and $0 \le \alpha \le 1$. Let $\{L_n\}$ be the sequence of orthonormal Laguerre polynomials with respect to the weight $x^{\alpha k}e^{-x}$. Then for $x \in [0,\infty)$,

$$(3.31) \qquad |x^{\alpha k}e^{-x}\sum_{j=0}^{n-1} L_j^2(x)| \le cn^{2k+1} ,$$

where c is a constant independent of α .

Proof. Clearly, it suffices to estimate L_j when $j \ge k$. So, let $j \ge k$. Then, by Theorem 2.1 (with $b = b(\alpha k, 2j+\alpha k, 1)$), we have for all $x \in [0,\infty)$,

$$(3.32) \qquad |x^{\alpha k}e^{-x}L_j^2(x)| \le b^{\alpha k}\|e^{-x}L_j^2(x)\|_{\infty} \le (4j+2\alpha k)^{\alpha k}\|e^{-x}L_j^2(x)\|_{\infty}$$
$$\le (6j)^k\|e^{-x}L_j^2(x)\|_{\infty} .$$

Also from [2, §10.12, §10.18] we have for $x \in [0,\infty)$,

$$e^{-x}L_j^2(x) \le \frac{\Gamma(j+\alpha k+1)}{j!}\cdot\frac{1}{[\Gamma(\alpha k+1)]^2} \le (2j)^k .$$

Together with (3.32) this leads to (3.31). □

Using Lemma 3.2, Theorem 2.1 and also Theorems 6.1 and 6.4 of [6], we get the following:

Proposition 3.3: Let $0 \le \alpha \le 1$, $n \ge 1$ be an integer, $0 < p,r \le \infty$ and $P_n \in \mathcal{P}_n$. Then there are constants c and d depending upon p and r but not on α , n or P_n such that

(3.33) $$\|x^{\alpha}e^{-x}P_n(x)\|_p \le cn^d \|x^{\alpha}e^{-x}P_n(x)\|_r \quad .$$

We are now in a position to complete the proof of Theorem 2.2.

Proof of Theorem 2.2: ($p < \infty$) . In the proof of part (a) for the case when $p < \infty$, we use Proposition 3.3 with $x^{[s_i]}T_{s_i,m_i,p}(x)$ in place of P_n , where $[s_i]$ is the greatest integer less than s_i , to see that

$$\lim_{i\to\infty} n_i^{-1} E_{s_i,m_i,p}^{1/n_i} = \lim_{i\to\infty} n_i^{-1} E_{s_i,m_i,r}^{1/n_i} \quad , \quad 0 < p,r \le \infty \quad .$$

Part (a) then follows from the previously proved case $p = \infty$.

The proofs of parts (b) and (c) are now exactly the same as the proofs in [6] of the analogues of the parts (b) and (c) of Theorem 2.2. Hence, we omit the details. □

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