ON EQUICONVERGENCE OF CERTAIN SEQUENCES OF RATIONAL INTERPOLANTS

E. B. Saff[1]
Center for Mathematical Services
University of South Florida
Tampa, Florida 33620

and

A. Sharma[2]
Mathematics Department
University of Alberta
Edmonton, Canada T6G 2G1

Abstract For a function $f(z)$ analytic on $|z| < \rho$, $\rho > 1$, we consider two schemes of rational interpolants which have poles equally spaced on the circle $|z| = \sigma$, $\sigma > 1$. The first scheme interpolates $f(z)$ in the roots of unity, while the second consists of best L^2-approximants to $f(z)$ on the unit circle. We obtain precise regions of equiconvergence for the two schemes of rational functions, thus extending a well-known result of J. L. Walsh.

1. Introduction

Let A_ρ denote the class of functions $f(z)$ which are analytic in the open disk $|z| < \rho$, but not on $|z| \leq \rho$. A fundamental result concerning the equiconvergence of certain sequences of polynomials is the following theorem of J. L. Walsh [5, p. 153]:

Theorem 1.1. *Suppose $f \in A_\rho$ with $\rho > 1$. For each positive integer n, let $L_{n-1}(z)$ denote the Lagrange polynomial interpolant to f in the n th roots of unity, and denote by $s_{n-1}(z)$ the $(n-1)$th order Taylor polynomial of f about the origin. Then*

$$(1.1) \qquad \lim_{n\to\infty} \{L_{n-1}(z) - s_{n-1}(z)\} = 0, \quad \forall\ |z| < \rho^2,$$

the convergence being uniform and geometric on any compact set in $|z| < \rho^2$. Moreover, the result is sharp in the sense that for any point z_0 on $|z| = \rho^2$, there is a function in A_ρ for which (1.1) does not hold at z_0.

[1]Research supported in part by the National Science Foundation.
[2]Research supported in part by the National Research Council of Canada.

Rational Approximation and Interpolation, Lecture Notes in Mathematics, Vol. 1105, Springer-Verlag, 1983.

For $f \in A_\rho$, $\rho > 1$, it is a simple consequence of the convergence properties of the two sequences $\{L_{n-1}(z)\}_1^\infty$ and $\{s_{n-1}(z)\}_1^\infty$ that (1.1) holds for $|z| < \rho$. The essential feature of Walsh's theorem is that equiconvergence holds in the larger disk $|z| < \rho^2$. A discussion of various extensions of Theorem 1.1 and related results can be found in [3] and [4].

The purpose of the present paper is to describe generalizations of Theorem 1.1 to the case of interpolating rational functions whose poles are equally spaced on a given circle $|z| = \sigma$, $\sigma > 1$. In place of the Lagrange polynomial $L_{n-1}(z)$, we will take the unique function $R_{n+m,n}(z)$ of the form

$$(1.2) \qquad R_{n+m,n}(z) = \frac{B_{n+m,n}(z)}{z^n - \sigma^n} \quad , \quad B_{n+m,n}(z) \in \pi_{n+m} \quad ,$$

which interpolates $f(z)$ in the $(n+m+1)$ th roots of unity; that is,

$$(1.3) \qquad B_{n+m,n}(z) = f(z)(z^n - \sigma^n) \quad , \quad \text{if} \quad z^{n+m+1} - 1 = 0 \quad .$$

(Here and below, π_k denotes the collection of all polynomials of degree at most k .) Since the $(n-1)$ th Taylor polynomial $s_{n-1}(z)$ is also the least squares approximation to $f(z)$ from π_{n-1} on the unit circle $|z| = 1$, we shall replace this polynomial by the unique rational function

$$(1.4) \qquad r_{n+m,n}(z) = \frac{P_{n+m,n}(z)}{z^n - \sigma^n} \quad , \quad P_{n+m,n}(z) \in \pi_{n+m} \quad ,$$

which minimizes the integral

$$(1.5) \qquad \int_{|z|=1} |f(z) - r(z)|^2 |dz|$$

over all rationals of the form $p(z)/(z^n - \sigma^n)$, $p(z) \in \pi_{n+m}$. From another elegant theorem of Walsh [5, §9.1], for each integer $m \geq -1$, the rational $r_{n+m,n}(z)$ must interpolate $f(z)$ in the $(n+m+1)$ roots of the equation $z^{m+1}(z^n - \sigma^{-n}) = 0$; that is, for $m \geq -1$,

$$(1.6) \qquad P_{n+m,n}(z) = f(z)(z^n - \sigma^n) \quad , \quad \text{if} \quad z^{m+1}(z^n - \sigma^{-n}) = 0 \quad .$$

In the spirit of Theorem 1.1, we shall examine the difference

$$R_{n+m,n}(z) - r_{n+m,n}(z) = \frac{B_{n+m,n}(z) - P_{n+m,n}(z)}{z^n - \sigma^n} ,$$

for each fixed integer m and show that the phenomenon of equiconvergence persists. In fact, if $\rho^2 > \sigma$, a new phenomenon arises which is described in Theorem 2.3 of §2.

Of special interest is the situation when $m < -1$ since, in this case, the interpolation property of (1.6) no longer holds. As we shall show in §4, the L^2-extremal rational function $r_{n+m,n}(z) = P_{n+m,n}(z)/(z^n - \sigma^n)$ for $m < -1$ has the following simple characterization. If we write

$$(1.7) \qquad P_{n-1,n}(z) = \sum_{k=0}^{n-1} b_{k,n} z^k ,$$

where (as in (1.6)) $P_{n-1,n}(z)$ interpolates $f(z)(z^n - \sigma^n)$ in the roots of $z^n - \sigma^{-n} = 0$, then for each $m = -2,-3,\ldots$ and $n \geq -m$, we have

$$(1.8) \qquad P_{n+m,n}(z) = \sum_{k=0}^{n+m} b_{k,n} z^k .$$

2. Equiconvergence of $\{R_{n+m,n}(z)\}$ and $\{r_{n+m,n}(z)\}$ for $m \geq -1$.

The first two theorems concern the separate convergence properties of the sequences $\{R_{n+m,n}(z)\}_{n=1}^{\infty}$ and $\{r_{n+m,n}(z)\}_{n=1}^{\infty}$ for fixed $m \geq -1$. We shall use the symbol $\|\cdot\|_A$ to denote the sup norm taken over the set A.

Theorem 2.1. Let $\rho > 1$, $\sigma > 1$ and an integer $m \geq -1$ be fixed. If $f \in A_\rho$ and if $R_{n+m,n}(z)$ is the rational function of the form (1.2) which interpolates $f(z)$ in the $(n+m+1)$ th roots of unity, then

$$(2.1) \qquad \lim_{n\to\infty} R_{n+m,n}(z) = f(z) , \quad \forall\ |z| < \min\{\sigma,\rho\} .$$

More precisely, if $\tau := \min\{\sigma,\rho\}$ and $K \subset \{z : |z| < \tau\}$ is compact, then

$$(2.2) \qquad \limsup_{n\to\infty} \|f(z) - R_{n+m,n}(z)\|_K^{1/n} \le \frac{1}{\tau}\max\{1, \|z\|_K\} < 1 .$$

Furthermore, if $\rho > \sigma$, *then for all* $|z| > \sigma$

$$(2.3) \qquad \lim_{n\to\infty} R_{n+m,n}(z) = \begin{cases} 0 , & \text{for } m = -1 \\ \sum_{k=0}^{m} a_k z^k , & \text{for } m = 0,1,\ldots , \end{cases}$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k$.

Theorem 2.2. Let $\rho > 1$, $\sigma > 1$ *and an integer* $m \ge -1$ *be fixed. If* $f \in A_\rho$ *and* $r_{n+m,n}(z)$ *is the rational function of (1.4) of least squares approximation to* f *on the unit circle, then the conclusions (2.1), (2.2) and (2.3) of Theorem 2.1 remain valid if* $R_{n+m,n}(z)$ *is replaced by* $r_{n+m,n}(z)$.

Remark 1. The proofs of Theorems 2.1 and 2.2 are immediate consequences of the following Hermite formula representations for $m \ge -1$:

$$(2.4) \qquad f(z) - R_{n+m,n}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(z^{n+m+1}-1)(t^n-\sigma^n)f(t)}{(z^n-\sigma^n)(t^{n+m+1}-1)(t-z)}\,dt ,$$

$$(2.5) \qquad f(z) - r_{n+m,n}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{z^{m+1}(z^n-\sigma^{-n})(t^n-\sigma^n)f(t)}{(z^n-\sigma^n)(t^n-\sigma^{-n})t^{m+1}(t-z)}\,dt ,$$

where Γ is the circle $|t| = \hat{\rho}$, $1 < \hat{\rho} < \rho$, and $|z| < \hat{\rho}$. In writing (2.5) we have used the interpolation property of (1.6). From (2.4) and (2.5), one can obtain integral formulae for $R_{n+m,n}(z)$ and $r_{n+m,n}(z)$, valid for all $z \in \mathbb{C}$, which imply (2.3).

It follows from Theorems 2.1 and 2.2 that if $\rho \le \sigma$, then

$$(2.6) \qquad \lim_{n\to\infty} \{R_{n+m,n}(z) - r_{n+m,n}(z)\} = 0 , \quad \forall\ |z| < \rho ,$$

and, if $\rho > \sigma$, then (2.6) holds $\forall\ |z| \ne \sigma$. A better result is given by

<u>Theorem 2.3</u>. <u>Let</u> $\rho > 1$, $\sigma > 1$ <u>and an integer</u> $m \geq -1$ <u>be fixed</u>. <u>If</u> $f \in A_\rho$, <u>then the rational functions of</u> (1.2) <u>and</u> (1.4) <u>satisfy</u>

$$(2.7) \qquad \lim_{n\to\infty} \{R_{n+m,n}(z) - r_{n+m,n}(z)\} = 0 \quad \begin{cases} \forall\ |z| < \rho^2, & \text{if } \sigma \geq \rho^2 \\ \forall\ |z| \neq \sigma, & \text{if } \rho^2 > \sigma . \end{cases}$$

<u>Moreover, the result is sharp</u>.

<u>Remark 2</u>. The proof of Theorem 2.3 follows from the representations (2.4) and (2.5) which yield

$$(2.8) \qquad R_{n+m,n}(z) - r_{n+m,n}(z) =$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\left(\frac{t^n-\sigma^n}{z^n-\sigma^n}\right)\left[\frac{t^{m+1}(t^n-\sigma^{-n})-z^{m+1}(z^n-\sigma^{-n})-t^{m+1}z^{m+1}(t^n-z^n)\sigma^{-n}}{t^{m+1}(t^n-\sigma^{-n})(t^{m+n+1}-1)}\right]dt.$$

One can also use (2.8) to obtain degree of convergence results. That (2.7) is sharp can be easily seen by taking $f(z) = (z-\rho e^{i\theta})^{-1}$.

<u>Remark 3</u>. Letting σ tend to infinity in Theorem 2.3 gives the classical result of Theorem 1.1.

<u>Remark 4</u>. For the case $\rho^2 > \sigma$, Theorem 2.3 asserts that equiconvergence holds at all points of the plane not on the circle $|z| = \sigma$. This is a new phenomenon which does not arise in Walsh's Theorem 1.1 where $\sigma = \infty$. (See also [2].)

3. Extension of Theorem 2.3.

Our next goal is to extend Theorem 2.3 in the spirit of Theorem 1 of [1]. The essence of the latter theorem is a representation of the Lagrange polynomial $L_{n-1}(z)$ interpolating $f(z)$ in the roots of $z^n - 1 = 0$. Namely, it is shown that, for each fixed n,

$$(3.1) \qquad L_{n-1}(z) = \sum_{\nu=0}^{\infty} s_{n-1}(z;\nu) ,$$

where $s_{n-1}(z;\nu) := \sum_{j=0}^{n-1} a_{j+\nu n} z^j$ is the shifted (n-1) th section of the Taylor expansion $\sum_{k=0}^{\infty} a_k z^k$ for $f(z)$. The representation (3.1) has two important properties. First, since $s_{n-1}(z;0) = s_{n-1}(z)$, equation (3.1) relates to an asymptotic formula for the difference $L_{n-1}(z) - s_{n-1}(z)$ occuring in Walsh's Theorem 1.1. Second, it yields a systematic way to construct the values of f in the nth roots of unity from the knowledge of the values of f and its derivatives at the origin; that is, if $\omega^n - 1 = 0$, then from (3.1) we have

$$f(\omega) = \sum_{\nu=0}^{\infty} \sum_{j=0}^{n-1} \frac{f^{(j+\nu n)}(0)}{(j+\nu n)!} \omega^j .$$

In a like manner, for the rational $R_{n+m,n}(z) = B_{n+m,n}(z)/(z^n - \sigma^n)$ which interpolates $f(z)$ in the roots of $z^{m+n+1}-1 = 0$, we seek a representation

$$(3.2) \qquad R_{n+m,n}(z) = \sum_{\nu=0}^{\infty} r_{n+m,n}(z;\nu) ,$$

where $r_{n+m,n}(z;0) = r_{n+m,n}(z)$ and, for each $\nu = 0,1,\ldots,$ $r_{n+m,n}(z;\nu)$ is a rational function of the form

$$(3.3) \qquad r_{n+m,n}(z;\nu) = \frac{P_{n+m,n}(z;\nu)}{z^n - \sigma^n} , \quad P_{n+m,n}(z;\nu) \in \pi_{n+m}$$

which is determined solely by the values of f and its derivatives in the roots of $z^{m+1}(z^n - \sigma^{-n}) = 0$.

For this purpose, it is convenient to have the following.

<u>Lemma 3.1</u>. <u>For fixed integers</u> $m \geq -1$, $n \geq 1$, <u>set</u> $N(\nu) := (\nu+1)(n+m+1)-1$, $\nu = 0,1,\ldots$, <u>and put</u>

$$(3.4) \qquad \alpha_{n,m}(z) := 1 - z^{m+1}\sigma^{-n} , \quad \beta_{n,m}(z) := z^{m+1}(z^n - \sigma^{-n}) , \quad \sigma > 1 .$$

<u>Let</u> $S_{N(\nu)}(z)$ <u>denote the unique polynomial in</u> $\pi_{N(\nu)}$ <u>which interpolates the function</u> $\{\alpha_{n,m}(z)\}^{\nu}(z^n - \sigma^n)f(z)$ <u>in the Hermite sense in the</u> $N(\nu)+1$ <u>roots of</u> $\{\beta_{n,m}(z)\}^{\nu+1} = 0$. <u>If</u> $f(z)$ <u>is analytic in</u>

$|z| \le 1$, then for each n sufficiently large,

$$\lim_{\nu \to \infty} \frac{S_{N(\nu)}(z)}{\{\alpha_{n,m}(z)\}^{\nu}} = (z^n - \sigma^n) f(z) \quad , \tag{3.5}$$

uniformly on $|z| \le 1$. Furthermore,

$$S_{N(\nu)}(z) - \alpha_{n,m}(z) S_{N(\nu-1)}(z) = \{\beta_{n,m}(z)\}^{\nu} P_{n+m,n}(z;\nu) \quad , \tag{3.6}$$

where $P_{n+m,n}(z;\nu) \in \pi_{n+m}$, $\nu = 1,2,\ldots$.

Consequently, for $|z| \le 1$,

$$(z^n - \sigma^n) f(z) = \sum_{\nu=0}^{\infty} \left\{\frac{\beta_{n,m}(z)}{\alpha_{n,m}(z)}\right\}^{\nu} P_{n+m,n}(z;\nu) \quad , \tag{3.7}$$

where $P_{n+m,n}(z;0) := S_{N(0)}(z)$.

Remark 5. Notice that since $S_{N(0)}(z)$ interpolates $(z^n - \sigma^n) f(z)$ in the zeros of $\beta_{n,m}(z)$, then $P_{n+m,n}(z;0) = P_{n+m,n}(z)$ which is the numerator polynomial in (1.4), i.e.,

$$\frac{P_{n+m,n}(z;0)}{z^n - \sigma^n} \equiv r_{n+m,n}(z) \quad . \tag{3.8}$$

Furthermore, since from (3.7), the polynomial $P_{n+m,n}(z;\nu) \in \pi_{n+m}$ interpolates the function

$$\left\{\frac{\alpha_{n,m}(z)}{\beta_{n,m}(z)}\right\}^{\nu} \left[(z^n - \sigma^n) f(z) - \sum_{k=0}^{\nu-1} \left\{\frac{\beta_{n,m}(z)}{\alpha_{n,m}(z)}\right\}^{k} P_{n+m,n}(z;k)\right]$$

in the zeros of $\beta_{n,m}(z)$, we see that $P_{n+m,n}(z;\nu)$ is determined only from the values of f and finitely many of its derivatives at these zeros.

Proof of Lemma 3.1. We first prove (3.6). Clearly, from the interpolation properties of the polynomials $S_{N(\nu)}(z)$, we see that $\{\beta_{n,m}(z)\}^{\nu}$ divides the polynomial $S_{N(\nu)}(z) - \alpha_{n,m}(z) S_{N(\nu-1)}(z)$. Hence

$$S_{N(\nu)}(z) - \alpha_{n,m}(z)S_{N(\nu-1)}(z) = \{\beta_{n,m}(z)\}^{\nu}P_{n+m,n}(z;\nu) \quad ,$$

where the degree of $P_{n+m,n}(z;\nu)$ is at most $N(\nu)-(n+m+1)\nu = n+m$.

In order to prove (3.5), we observe that for $|z| \le 1$,

$$E_{\nu}(z) := (z^n - \sigma^n)f(z) - \frac{S_{N(\nu)}(z)}{\{\alpha_{n,m}(z)\}^{\nu}} =$$

$$= \frac{1}{2\pi i}\int_{|t|=\hat{\rho}} \left\{\frac{\beta_{n,m}(z)}{\beta_{n,m}(t)}\right\}^{\nu+1} \left\{\frac{\alpha_{n,m}(t)}{\alpha_{n,m}(z)}\right\}^{\nu} \frac{(t^n-\sigma^n)f(t)}{t-z}\, dt \quad ,$$

where $\hat{\rho} > 1$ is selected so that $f(t)$ is analytic on $|t| \le \hat{\rho}$. Straightforward estimates then yield

$$\limsup_{\nu\to\infty} \|E_{\nu}(z)\|_{|z|\le 1}^{1/\nu} \le \frac{(1+\sigma^{-n})(1+\hat{\rho}^{m+1}\sigma^{-n})}{\hat{\rho}^{m+1}(\hat{\rho}^{n}-\sigma^{-n})(1-\sigma^{-n})} < 1$$

for $n > n_0(m,\hat{\rho},\sigma)$. This proves (3.5). Combining (3.5) and (3.6), we get (3.7). □

Corollary 3.2. Let $f \in A_{\rho}$, $\rho > 1$, and $B_{n+m,n} \in \pi_{n+m}$, $m \ge -1$, interpolate $(z^n - \sigma^n)f(z)$ in the $(n+m+1)$ th roots of unity. Then, for each n large $(n > n_0(m,\rho,\sigma))$, we have

$$B_{n+m,n}(z) = \sum_{\nu=0}^{\infty} P_{n+m,n}(z;\nu) \quad , \quad \forall \ z \in \mathbb{C} \quad , \tag{3.9}$$

where the polynomials $P_{n+m,n}(z;\nu) \in \pi_{n+m}$ are defined in (3.6).

Proof. If ω is an $(n+m+1)$th root of unity, then since $\beta_{n,m}(\omega) = \alpha_{n,m}(\omega)$, we deduce from (3.7) that

$$B_{n+m,n}(\omega) = (z^n - \sigma^n)f(z)\Big|_{z=\omega} = \sum_{\nu=0}^{\infty} P_{n+m,n}(\omega;\nu) \quad .$$

Thus, by the uniqueness of the interpolant, (3.9) follows. □

The next theorem gives a generalization of Theorem 2.3.

Theorem 3.3. *Let $\rho > 1$, $\sigma > 1$ and an integer $m \geq -1$ be fixed. If $f \in A_\rho$ and if ℓ is any given positive integer, then*

$$(3.10) \qquad \lim_{n\to\infty}\left\{R_{n+m,n}(z) - \sum_{\nu=0}^{\ell-1} r_{n+m,n}(z;\nu)\right\} = 0 \quad \begin{cases} \forall\ |z| < \rho^{\ell+1}, \text{ if } \sigma \geq \rho^{\ell+1} \\ \forall\ |z| \neq \sigma\ , \text{ if } \rho^{\ell+1} > \sigma\ , \end{cases}$$

where $R_{n+m,n}(z) = B_{n+m,n}(z)/(z^n - \sigma^n)$ is defined in (1.2) and

$$(3.11) \qquad r_{n+m,n}(z;\nu) := P_{n+m,n}(z;\nu)/(z^n - \sigma^n) \quad , \quad \nu = 0,1,\ldots \ .$$

The convergence in (3.10) is uniform and geometric on compact subsets of the regions described. Moreover, the result is sharp.

Notice from (3.8) that, in the case $\ell = 1$, Theorem 3.3 reduces to Theorem 2.3.

Proof of Theorem 3.3. For n sufficiently large, we have by Corollary 3.2,

$$(3.12) \qquad B_{n+m,n}(z) - \sum_{\nu=0}^{\ell-1} P_{n+m,n}(z;\nu) = \sum_{\nu=\ell}^{\infty} P_{n+m,n}(z;\nu) \quad , \quad \forall\ z \in \mathbb{C} \ .$$

Also, from the interpolating property of the polynomial $S_{N(\nu)}(z)$ defined in Lemma 3.1, we have

$$S_{N(\nu)}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)(t^n - \sigma^n)\{\alpha_{n,m}(t)\}^\nu[\{\beta_{n,m}(t)\}^{\nu+1} - \{\beta_{n,m}(z)\}^{\nu+1}]}{(t-z)\{\beta_{n,m}(t)\}^{\nu+1}}\,dt,$$

where $\Gamma : |t| = \tau$, $1 < \tau < \rho$, and $\alpha_{n,m}$, $\beta_{n,m}$ are given in (3.4). Using this representation and equation (3.6), we obtain after some algebra the following integral representation for $P_{n+m,n}(z;\nu)$, $\nu \geq 1$:

(3.13) $\quad P_{n+m,n}(z;\nu) =$

$$= \frac{1}{2\pi i}\int_{\Gamma} \frac{f(t)(t^n-\sigma^n)\{\alpha_{n,m}(z)\beta_{n,m}(t)-\alpha_{n,m}(t)\beta_{n,m}(z)\}}{(t-z)\alpha_{n,m}(t)\beta_{n,m}(t)} \left\{\frac{\alpha_{n,m}(t)}{\beta_{n,m}(t)}\right\}^{\nu} dt.$$

Thus, from (3.11), (3.12) and (3.13) we get, for n large and all $z \in \mathbb{C}$,

(3.14) $\quad R_{n+m,n}(z) - \sum_{\nu=0}^{\ell-1} r_{n+m,n}(z;\nu) =$

$$= \frac{1}{2\pi i}\int_{\Gamma} \frac{f(t)(t^n-\sigma^n)\{\alpha_{n,m}(z)\beta_{n,m}(t)-\alpha_{n,m}(t)\beta_{n,m}(z)\}}{(t-z)(t^{m+n+1}-1)\alpha_{n,m}(t)(z^n-\sigma^n)} \left\{\frac{\alpha_{n,m}(t)}{\beta_{n,m}(t)}\right\}^{\ell} dt.$$

A straightforward analysis of (3.14) then yields (3.10).

To prove the sharpness assertion of Theorem 3.3, we take $\hat{f}(z) := 1/(z-\rho)$. From (3.14), we obtain in this case

(3.15) $\quad R_{n+m,n}(z) - \sum_{\nu=0}^{\ell-1} r_{n+m,n}(z;\nu) =$

$$= \frac{(\rho^n-\sigma^n)\{\alpha_{n,m}(z)\beta_{n,m}(\rho)-\alpha_{n,m}(\rho)\beta_{n,m}(z)\}}{(z-\rho)(\rho^{m+n+1}-1)\alpha_{n,m}(\rho)(z^n-\sigma^n)} \left\{\frac{\alpha_{n,m}(\rho)}{\beta_{n,m}(\rho)}\right\}^{\ell} ,$$

from which it is easy to show that

$$\lim_{n\to\infty}\left\{R_{n+m,n}(\rho^{\ell+1}) - \sum_{\nu=0}^{\ell-1} r_{n+m,n}(\rho^{\ell+1};\nu)\right\} = \frac{1}{\rho-\rho^{\ell+1}} , \text{ if } \sigma > \rho^{\ell+1} ,$$

$$\lim_{n\to\infty}\left\{R_{n+m,n}(z) - \sum_{\nu=0}^{\ell-1} r_{n+m,n}(z;\nu)\right\} = \frac{z^{m+1}}{\sigma^{m+1}(z-\rho)} , \text{ if } \sigma = \rho^{\ell+1}, |z| > \sigma.$$

This completes the proof of Theorem 3.3. □

4. Equiconvergence of $\{R_{n-\mu,n}(z)\}$ and $\{r_{n-\mu,n}(z)\}$ for $\mu \geq 2$.

This case differs slightly from the case in §2 and §3. However, as we shall see, there is an essential continuity in the results which come out. We shall begin by proving a lemma.

Lemma 4.1. *Let* $\rho > 1$, $\sigma > 1$ *and an integer* μ , $2 \leq \mu \leq n$, *be fixed. Let* $P_{n-\mu,n}(z)$ *denote the polynomial in* $\pi_{n-\mu}$ *for which*

$$(4.1) \qquad \min_{Q \in \pi_{n-\mu}} \int_{|z|=1} \left| f(z) - \frac{Q(z)}{z^n - \sigma^n} \right|^2 |dz|$$

is attained, where $f(z) \in A_\rho$. *Then* $P_{n-\mu,n}(z)$ *is given by the formula*

$$(4.2) \qquad P_{n-\mu,n}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)t^{\mu-1}(t^{n-\mu+1} - z^{n-\mu+1})(t^n - \sigma^n)}{(t-z)(t^n - \sigma^{-n})} \, dt$$

where $\Gamma : |t| = \tau$, $1 < \tau < \rho$.

Proof. The minimization problem (4.1) is equivalent to finding

$$(4.3) \qquad \min_{\{a_j\}} \int_{|z|=1} |f(z) - \sum_{j=0}^{n-\mu} a_j f_j(z)|^2 |dz| ,$$

where $f_j(z) := z^j/(z^n - \sigma^n)$, $j = 0,1,\ldots,n-\mu$. It is easy to see that the minimum in (4.3) is attained if and only if

$$(4.4) \qquad \frac{1}{2\pi i} \int_{|z|=1} \{f(z) - \sum_{j=0}^{n-\mu} a_j f_j(z)\} \overline{f_\ell(z)} |dz| = 0 , \quad (\ell = 0,1,\ldots,n-\mu).$$

Since

$$f_\ell(z) = \frac{z^\ell}{z^n - \sigma^n} = \frac{1}{n\sigma^{n-\ell-1}} \sum_{k=0}^{n-1} \frac{\omega^{k+k\ell}}{z - \sigma\omega^k} , \quad \omega := e^{2\pi i/n} ,$$

$$(\ell = 0,1,\ldots,n-\mu)$$

it follows that

$$\overline{f_\ell(z)} = -\frac{1}{n\sigma^{n-\ell}} \sum_{k=0}^{n-1} \frac{\omega^{-k\ell} z}{z - \frac{\omega^k}{\sigma}} \quad , \quad |z| = 1 \quad .$$

From this observation, we see that

$$(4.5) \qquad \frac{1}{2\pi i} \int_{|z|=1} f(z)\overline{f_\ell(z)}\,|dz| = \frac{1}{2\pi i} \int_{|z|=1} f(z)\overline{f_\ell(z)}\,\frac{dz}{iz}$$

$$= \frac{i}{n\sigma^{n-\ell}} \sum_{k=0}^{n-1} \frac{\omega^{-k\ell}}{2\pi i} \int_{|z|=1} \frac{f(z)}{z - \omega^k/\sigma}\,dz$$

$$= \frac{i}{n\sigma^{n-\ell}} \sum_{k=0}^{n-1} \omega^{-k\ell} f\left(\frac{\omega^k}{\sigma}\right) .$$

Moreover, if we set $P_{n-\mu,n}(z) = \sum_{\nu=0}^{n-\mu} b_\nu z^\nu$, we get

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{P_{n-\mu,n}(z)}{z^n - \sigma^n}\,\overline{f_\ell(z)}\,|dz| = \frac{i}{n\sigma^{n-\ell}(\sigma^{-n}-\sigma^n)} \sum_{k=0}^{n-1} \omega^{-k\ell} P_{n-\mu,n}\left(\frac{\omega^k}{\sigma}\right)$$

$$= \frac{i}{n\sigma^{n-\ell}(\sigma^{-n}-\sigma^n)} \sum_{\nu=0}^{n-\mu} b_\nu \sigma^{-\nu} \sum_{k=0}^{n-1} \omega^{k(\nu-\ell)} \quad .$$

On using the properties of roots of unity, this yields

$$(4.6) \qquad \frac{1}{2\pi i} \int_{|z|=1} \frac{P_{n-\mu,n}(z)}{z^n - \sigma^n}\,\overline{f_\ell(z)}\,|dz| = \frac{ib_\ell}{\sigma^n(\sigma^{-n} - \sigma^n)}$$

$$(\ell = 0,1,\ldots,n-\mu) \quad .$$

From (4.4), (4.5) and (4.6), we see that

$$(4.7) \qquad b_j = \frac{1}{2\pi i}(\sigma^{-n} - \sigma^n) \int_\Gamma \frac{f(t)t^{n-1-j}}{t^n - \sigma^{-n}} dt \ , \ j = 0,1,\ldots,n-\mu \ ,$$

since

$$\frac{z^{n-1-j}}{z^n - \sigma^{-n}} = \frac{\sigma^j}{n} \sum_{k=0}^{n-1} \frac{\omega^{-kj}}{z - \sigma^{-1}\omega^k} \ .$$

We now easily see from (4.7) that

$$(4.8) \qquad P_{n-\mu,n}(z) = \frac{\sigma^{-n} - \sigma^n}{2\pi i} \int_\Gamma \frac{f(t)t^{\mu-1}(t^{n-\mu+1} - z^{n-\mu+1})}{(t-z)(t^n - \sigma^{-n})} dt \ .$$

Since

$$\sigma^{-n} - \sigma^n = \sigma^{-n} - t^n + t^n - \sigma^n \ ,$$

the above integral splits up into two integrals, one of which is

$$-\frac{1}{2\pi i} \int_\Gamma \frac{f(t)t^{\mu-1}(t^{n-\mu+1} - z^{n-\mu+1})}{t-z} dt = 0 \ .$$

This yields (4.2) and completes the proof. □

<u>Remark</u> 6. As stated in (1.6), the polynomial $P_{n-1,n}(z)$ interpolates $f(z)(z^n - \sigma^n)$ in the n roots of $z^n - \sigma^{-n} = 0$, from which it follows that

$$(4.9) \qquad P_{n-1,n}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)(t^n - z^n)(t^n - \sigma^n)}{(t-z)(t^n - \sigma^{-n})} dt \ .$$

Thus, equation (4.2) also holds for $\mu = 1$. Indeed, the derivation of (4.2) given above is valid in this case. Moreover, note that the formula (4.7) for the coefficients of $P_{n-\mu,n}(z)$ is <u>independent</u> of μ. Thus, if we write

$$(4.10) \qquad P_{n-1,n}(z) = \sum_{j=0}^{n-1} b_{j,n} z^j \ ,$$

it follows that $P_{n-\mu,n}(z)$ is just a partial sum of $P_{n-1,n}(z)$, i.e.,

$$(4.11) \qquad P_{n-\mu,n}(z) = \sum_{j=0}^{n-\mu} b_{j,n} z^j \quad , \quad \mu = 2,3,\ldots,n \quad ,$$

as claimed in (1.8). A similar situation arises in the case of discrete least squares approximation in the nth roots of unity. Namely, it is known (cf. [6,p.8]) that the polynomial $p_{n-\mu} \in \pi_{n-\mu}$, $\mu \geq 2$, for which the minimum

$$\min_{p \in \pi_{n-\mu}} \sum_{k=1}^{n} |F(\omega^k) - p(\omega^k)|^2 \quad , \quad \omega := e^{2\pi i/n} \quad ,$$

is attained is just a partial sum of the polynomial $p_{n-1} \in \pi_{n-1}$ which interpolates $F(z)$ in the nth roots of unity. This known characterization can be viewed as a limiting case of (4.11) where $\sigma \to 1$ and $f(z) = F(z)/(z^n - \sigma^n)$.

We can now prove that Theorem 2.3 holds for all negative integers m.

Theorem 4.2. *Let* $\rho > 1$, $\sigma > 1$ *and an integer* $\mu \geq 2$ *be fixed. If* $f \in A_\rho$, *then the rational functions* $R_{n-\mu,n}(z) = B_{n-\mu,n}(z)/(z^n - \sigma^n)$, $r_{n-\mu,n}(z) = P_{n-\mu,n}(z)/(z^n - \sigma^n)$ *defined in* (1.2) *and* (1.4) (*with* $m = -\mu$) *satisfy*

$$(4.12) \qquad \lim_{n\to\infty} \{R_{n-\mu,n}(z) - r_{n-\mu,n}(z)\} = 0 \quad \begin{cases} \forall\ |z| < \rho^2 \ , \ \text{if} \ \ \sigma \geq \rho^2 \\ \forall\ |z| \neq \sigma \ , \ \text{if} \ \ \rho^2 > \sigma \end{cases} .$$

Moreover, the result is sharp.

Proof. From formula (4.2) and the representation

$$(4.13) \qquad B_{n-\mu,n}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)(t^n - \sigma^n)(t^{n-\mu+1} - z^{n-\mu+1})}{(t-z)(t^{n-\mu+1} - 1)} \, dt \quad ,$$

where $\Gamma : |t| = \hat{\rho}$, $1 < \hat{\rho} < \rho$, we find

(4.14) $B_{n-\mu,n}(z) - P_{n-\mu,n}(z) =$

$$= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)(t^n - \sigma^n)(t^{n-\mu+1} - z^{n-\mu+1})(t^{\mu-1} - \sigma^{-n})}{(t-z)(t^n - \sigma^{-n})(t^{n-\mu+1} - 1)} dt .$$

Equation (4.12) then follows by estimating the integral in (4.14).

To prove the sharpness assertion, take $\hat{f}(z) = 1/(z-\rho)$. Then it is easy to verify from the interpolating properties that, for $n \geq 2(\mu - 1)$, we have

$$(4.15) \qquad B_{n-\mu,n}(z) = B_{n-\mu,n}(z;\hat{f}) = \frac{\sigma^n - z^{\mu-1}}{\rho - z} + \frac{z^{n-\mu+1}-1}{\rho - z}\left(\frac{\rho^{\mu-1} - \sigma^n}{\rho^{n-\mu+1} - 1}\right) .$$

Moreover, from (4.8), we find for $\mu \geq 1$,

$$(4.16) \qquad P_{n-\mu,n}(z) = P_{n-\mu,n}(z;\hat{f}) = \rho^{\mu-1}\left(\frac{\sigma^n - \sigma^{-n}}{\rho^n - \sigma^{-n}}\right)\left(\frac{\rho^{n-\mu+1} - z^{n-\mu+1}}{\rho - z}\right) .$$

On subtracting (4.16) from (4.15), it can be shown that

$$(4.17) \qquad \lim_{n\to\infty} \{R_{n-\mu,n}(\rho^2;\hat{f}) - r_{n-\mu,n}(\rho^2;\hat{f})\} = \frac{1}{\rho - \rho^2} , \text{ if } \sigma > \rho^2 ,$$

$$(4.18) \qquad \lim_{n\to\infty} \{R_{n-\mu,n}(z;\hat{f}) - r_{n-\mu,n}(z;\hat{f})\} = \frac{z^{1-\mu}}{z-\rho} , \text{ if } \sigma = \rho^2 , \ |z| > \sigma ,$$

which proves that (4.12) is sharp. □

Theorem 4.2 can be extended in a manner similar to the generalization of Theorem 2.3, given in Theorem 3.3 by introducing the corresponding polynomials $P_{n-\mu,n}(z;\nu)$ defined by

$$(4.19) \quad P_{n-\mu,n}(z;\nu) := \frac{1}{2\pi i} \int_{|t|=1} \frac{f(t)t^{\mu-1}(t^n - \sigma^n)(t^{n-\mu+1} - z^{n-\mu+1})(t^{\mu-1} - \sigma^{-n})^{\nu}}{(t-z)(t^n - \sigma^{-n})^{\nu+1}} dt, \qquad \nu = 0,1,2,\ldots .$$

The details are left for the reader.

References

1. A. S. Cavaretta Jr., A. Sharma and R. S. Varga. Interpolation in the roots of unity: An extension of a theorem of J. L. Walsh. Resultate der Math. 1981, vol. 3, 155-191.

2. G. López Lagomasino and René Piedra de la Torre, Sobre un teorema de sobreconvergencia de J. L. Walsh. Revista Ciencias Matemáticas. 1983, vol. IV, No. 3, 67-78.

3. E. B. Saff, A. Sharma and R. S. Varga, An extension to rational functions of a theorem of J. L. Walsh on differences of interpolating polynomials, R.A.I.R.O. Analyse numérique, 1981, vol. 15, 371-390.

4. R. S. Varga. Topics in Polynomial and Rational Interpolation and Approximation. Séminaire de Math. Supérieures, Les Presses de L'Univ. de Montréal, Montréal, 1982, 69-93.

5. J. L. Walsh. Interpolation and Approximation by Rational Functions in the Complex Domain. A.M.S. Colloq. Publ. Vol XX, Providence, R.I., 5th ed. 1969.

6. A. Zygmund, Trigonometric Series, Vol. II, University Press, Cambridge, 2nd ed. 1959.