COMPUTING WITH THE FABER TRANSFORM

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Abstract Some theoretical and computational aspects of Faber-Padé approximants are discussed. In particular, a Montessus type theorem is proved and a new method for computing the approximants is presented. Results of numerical tests for the latter are included.

1. Introduction

The purpose of this paper is to further discuss the Faber-Padé (FP) approximants introduced in [4]. In this section we review some of their basic properties and, in Section 2, we prove a Montessus type theorem. A new method for computing the FP approximants is presented in Section 3.

Let E be a closed bounded point set (not a single point) in the z-plane whose complement K is simply connected on the sphere. By the Riemann mapping theorem, there exists a conformal map $w = \phi(z)$ of K onto $|w| > 1$ with the property that, in a neighborhood of infinity,

$$\phi(z) = \frac{z}{c} + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots , \quad c > 0 . \tag{1.1}$$

If $F(w)$ is analytic on $|w| \leq 1$, then the Faber transform of F is defined by

$$f(z) = T(F)(z) := \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{F(\phi(\xi))}{\xi - z}\, d\xi , \quad z \text{ inside } \Gamma_\rho , \tag{1.2}$$

where $\Gamma_\rho := \{\xi : |\phi(\xi)| = \rho\}$ and $\rho (>1)$ is chosen so that $F(w)$ is analytic on $|w| \leq \rho$. When E is a Jordan region bounded by

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a rectifiable Jordan curve C, then $T(F)$ is further defined for any F analytic in $|w| < 1$ and continuous on $|w| \leq 1$ by replacing Γ_ρ by C in (1.2). (Further extensions are discussed in the paper [1] by Anderson in this volume.)

Denoting by ψ the inverse of the mapping ϕ, it is straightforward to prove that the Faber transform has the following "singularity preserving property."

<u>Lemma</u> 1.1. <u>Let</u> F <u>be analytic on the closed disk</u> $|w| \leq 1$, <u>and let</u> $f = T(F)$. <u>Then</u> $F(w) - f(\psi(w))$ <u>can be extended analytically to</u> $|w| > 1$, <u>including the point at infinity</u>.

As a consequence of the above property we have

<u>Proposition</u> 1.2. <u>Let</u> $R(w)$ <u>be a type</u> (m,n) <u>rational function with all its poles in</u> $|w| > 1$. <u>Then</u> $r(z) := T(R)(z)$ <u>is a type</u> $(\tilde{m},n)$ <u>rational function, where</u> $\tilde{m} := \max(m,n-1)$, <u>and the poles of</u> $r(z)$ <u>are the images under</u> ψ <u>of those of</u> $R(w)$, <u>with corresponding multiplicities</u>.

<u>Proof</u>. It follows from (1.2) and Lemma 1.1 that r is meromorphic in the extended plane and hence is rational. The second part of the proposition is also an easy consequence of Lemma 1.1. (Different proofs of this result are given in [1] and [5].) □

We observe also that $\phi_n(z) := T(w^n)(z)$, $n = 0,1,\ldots,$ is a polynomial of degree n, the so-called <u>Faber polynomial</u>. If

$$F(w) = \sum_{k=0}^{\infty} a_k w^k ,$$

then $T(F)$ has the expansion

$$f(z) = T(F)(z) = \sum_{k=0}^{\infty} a_k \phi_k(z) .$$

Indeed, this property is often taken as the definition of the Faber transform. In practice, of course, we will be given the function f rather than F, but provided the mapping function ψ is known the coefficients a_n can easily be computed from the former (see [3]).

Observe that if $R(w)$ is a normal type (m,n) Padé approximant to $F(w)$, and has all its poles in $|w| > 1$, then $r = T(R)$ satisfies

$$(1.3) \qquad f(z) - r(z) = \sum_{k=m+n+1}^{\infty} b_k \phi_k(z)$$

for suitable coefficients b_k. This (with an obvious modification if R is not normal) is the Faber-Padé approximant of $f(z)$ as introduced in [4] (see also [5]). In the special case when E is the real interval $[-1,1]$, the FP approximant reduces to the Chebyshev-Padé approximant. For arbitrary point sets E, the FP approximant has two apparent drawbacks. First, it need not be of the "correct" type if $m < n-1$; second, the associated rational $R(w)$ is required to have no poles in the unit disk. Although, as is well-known, the first difficulty can be overcome in the special case of Chebyshev-Padé approximation, there appears to be no simple technique to extend this to the general setting. The second problem is, for the case of meromorphic functions F, addressed in the next section.

2. A Montessus Theorem

The following theorem guarantees the existence of the Faber-Padé under certain conditions and also shows that it behaves in the expected manner. The proof is a straightforward application of the singularity preserving property (Lemma 1.1) and it is possible to generalize other properties of the classical Padé approximants in a similar manner. With E and ϕ as described in the introduction, we have

Theorem 2.1. *Let f be analytic on E and meromorphic with precisely n poles (counting multiplicities) in the Jordan region E_ρ bounded by the level curve $|\phi(z)| = \rho$, $\rho > 1$. Then for each m sufficiently large, the type (m,n) Faber-Padé approximant $r_{m,n}$ exists satisfying (1.3) on E. The $r_{m,n}$ have precisely n finite poles, and as $m \to \infty$, these poles approach, respectively, the n poles of f in E_ρ. Moreover, the sequence $r_{m,n}$ converges uniformly to f on every compact subset of E_ρ which excludes the poles of f.*

Proof. $F = T^{-1}(f)$ exists since f is analytic on the closed set E and hence has a Faber series expansion that converges in an open set containing E . In view of Lemma 1.1, the function F is analytic on $|w| \leq 1$ and meromorphic with n poles in $|w| < \rho$ (these are the images of the poles of f under the map $w = \phi(z)$). From the classical Montessus theorem (see e.g. [2, p. 246]), it follows that there exists a sequence $R_{m,n}$, $m \geq m_0$, of type (m,n) Padé approximants to F with the following properties:

(A) For each $m \geq m_0$,

$$F(w) - R_{m,n}(w) = O(w^{m+n+1}) \quad \text{as} \quad w \to 0 \ ;$$

(B) For each $m \geq m_0$, $R_{m,n}$ has precisely n finite poles which approach the n poles of F in $|w| < \rho$ (with corresponding multiplicities);

(C) $\lim_{m\to\infty} R_{m,n}(w) = F(w)$ uniformly on every compact subset of $|w| < \rho$ which contains no poles of F .

From property (B), we see that for each m large, $R_{m,n}$ is analytic on $|w| \leq 1$ and hence its Faber transform exists. With

$$r_{m,n} := T(R_{m,n}) \ ,$$

we note from Proposition 1.2 that $r_{m,n}$ is a type (m,n) rational for each m large. In view of property (A) we have

$$(2.1) \qquad f(z) - r_{m,n}(z) = \sum_{k=m+n+1}^{\infty} b_k^{(m)} \phi_k(z) \ , \quad z \in E \ ,$$

and, since the poles of $r_{m,n}$ are the images under ψ of the poles of $R_{m,n}$, the assertion of the theorem regarding the poles of $r_{m,n}$ follows immediately.

To prove convergence, observe that from (1.2) we have

$$(2.2) \qquad f(z) - r_{m,n}(z) = \frac{1}{2\pi i} \int_{\Gamma_\sigma} \frac{F(\phi(\xi)) - R_{m,n}(\phi(\xi))}{\xi - z} \, d\xi \ ,$$

for z inside $\Gamma_\sigma : |\phi(\xi)| = \sigma$, where $\sigma(>1)$ is suitably chosen. Equation (2.2) is valid for z on any compact set $K \subset E_\rho \setminus \{n \text{ poles of } f\}$ provided m is sufficiently large and Γ_σ is replaced by the curve $\Gamma_\tau : |\phi(\xi)| = \tau$, with $\rho - \tau > 0$ sufficiently small, together with small circles around the poles of f. Since, from property (C), the sequence $R_{m,n}(\phi(\xi))$ converges uniformly to $F(\phi(\xi))$ on these curves, the convergence of $r_{m,n}(z)$ to $f(z)$ on K follows. □

3. Computing the Faber Transform of a Rational Function

A crucial stage in computing Faber-Padé or Faber-CF approximants is the computation of the transform of a rational function R analytic on the unit disk. In [4] this was carried out by computing the poles of R and applying Proposition 1.2 in an obvious fashion. We describe here an alternative and much faster method based on the integral representation (1.2) of the transform. In so doing, we suppose that E is bounded by a rectifiable Jordan curve C and that the origin lies interior to C. Then, for any $f = T(F)$, we have

$$\frac{f^{(k)}(0)}{k!} = \frac{-1}{4\pi^2} \int_{|z|=\delta} \frac{1}{z^{k+1}} \int_C \frac{F(\phi(\xi))}{\xi - z}\, d\xi dz \quad ,$$

where $\delta > 0$ is sufficiently small. By interchanging the order of integration and computing the integral with respect to z we obtain

$$(3.1) \qquad \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \int_C \frac{F(\phi(\xi))}{\xi^{k+1}}\, d\xi \quad .$$

If F is entire, we may replace the curve C by a circle and evaluate as many of the integrals as we require simultaneously by the trapezium rule and the fast Fourier transform. However, this is not in general possible for the case required here where $F = R$ is a type (m,n) rational function. Instead, we make the substitution $w = \phi(\xi)$ in (3.1) and obtain, for $r = T(R)$,

$$(3.2) \qquad \frac{r^{(k)}(0)}{k!} = \frac{1}{2\pi i} \int_{|w|=\rho} \frac{R(w)\psi'(w)}{[\psi(w)]^{k+1}}\, dw \quad , \qquad k = 0,1,\ldots \quad .$$

In (3.2), the constant $\rho > 1$ is chosen sufficiently small to ensure that the circle $|w| = \rho$ does not enclose any poles of R. Note that in practice this may be easily checked by evaluating (to the nearest integer) the integral

$$\frac{1}{2\pi i} \int_{|w|=\rho} \frac{Q'(w)}{Q(w)} \, dw \quad ,$$

where Q is the denominator of R, since this integral gives the number of zeros of Q inside the circle.

To calculate the transform of a type (m,n) rational function R (with $m \geq n-1$) we first evaluated the integrals (3.2) for $k = 0,1,\ldots,m+n+1$ using the 512 point trapezium rule on a VAX 11 using double precision (about 16 decimal digits). These $m+n+2$ values uniquely determine the type (m,n) rational $r = T(R) = p/q$, and p,q can be computed from the Padé equations. Some of our numerical results are given below. Although these refer only to real poles (the correct position of the pole is easier to calculate in this case), the method has also been used successfully with conjugate pairs of poles.

Example 1: $\psi(w) = w + 1/4w$ (an ellipse).
Since this curve is analytic we may choose $\rho = 1$ here.

i) R is type (2,2) with a pole at 1.1. The corresponding pole of the transformed rational r was calculated to be 1.327272727272730 which is correct to 16 figures.

ii) R is type (4,5), near degenerate, with a pole at 2.0 and a zero at 2.01. The corresponding pole of the transformed rational r was calculated to be 2.124999998829719 which is correct to 10 figures.

iii) R is type (4,4), degenerate with a pole and zero at 2.0. The spurious pole and zero of the transformed rational r agreed to 15 figures, but they were inside the ellipse. This suggests that it would be advisable to check for degeneracy before making use of these approximants.

iv) R is type (2,2), with a double pole at 2. The poles of the transformed rational r were only calculated correct to eight figures, but this turned out to be due to the ill-conditioning inherent in the determination of multiple zeros of a polynomial; examination of the coefficients of the rational function revealed them to be correct to 16 figures. Thus this is another reason for preferring this method of calculating the approximants over that given in [4].

Example 2. $\psi(w) = \int (1 + w^{-4})^{1/2}\, dw$ (a square).

In this example we chose $\rho = 1.1$ and evaluated $\psi(w)$ on the circle $|w| = \rho$ by expanding as a series which was then summed using the fast Fourier transform. For an example where R has poles at 2 and 5, the poles of the transformed rational r agreed with the true values as accurately as the latter could be computed, which was to about eight figures.

The above examples therefore indicate that the method described here is an effective way to evaluate the Faber transform of a rational function when computing Faber-Padé and Faber-CF approximants.

References

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