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A de MONTESSUS THEOREM FOR VECTOR VALUED RATIONAL INTERPOLANTS

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Abstract   A convergence theorem for vector valued Padé approximants (simultaneous Padé approximants) is established. The theorem is a natural extension of the theorem of de Montessus de Ballore for a row sequence of (scalar) Padé approximants. The result is also generalised to the case of vector valued (N-point) rational interpolants.

1. Introduction

The theorem of R. de Montessus de Ballore [7] is a remarkably elegant theorem on the convergence of row sequences of Padé approximants to a meromorphic function. Here, in section 2, we present its extension to the case of simultaneous Padé approximation (see Theorem 1) and to vector valued Padé approximation (see Theorem 2). The generalisations of de Montessus' theorem to multipoint rational interpolation, as distinct from Padé approximation, derived by Saff [8] and Warner [9] are extended to the case of vector valued rational interpolation in Theorem 3.

Simultaneous Padé approximation involves approximation of several functions $\{f_i(z),\ i=1,2,\dots,d\}$ by rationals of the form $\{P_{N,i}(z)/Q_N(z),\ i=1,2,\dots,d\}$, where the denominator polynomial $Q_N(z)$ is common to each of the d rational approximants. A full specification of the problem of constructing such polynomials was given by Mahler [5] in 1968. He also considered the extension to the case of interpolating rationals,

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and such problems were called "German Polynomial Approximation Problems", because a Gothic font was originally used for printing the polynomials. For the simultaneous Padé approximation problem, an explicit solution in terms of determinants was given by de Bruin [1], and the explicit solution for the corresponding vector valued rational interpolation problem was given by Graves-Morris [4].

The first extension of de Montessus' theorem to simultaneous Padé approximation was given by Mall [6] in 1934. His results are a special case of the theorems of this paper, as we point out in remark 3 of section 2. Gončar and Rahmanov [3] have recently presented a powerful convergence theorem for simultaneous Padé approximants of Stieltjes functions. It is an extension of the work of Chebyshev and Markov on the convergence of an $[N-1/N]$ sequence of Padé approximants to a Stieltjes function in the cut plane $\mathbb{C}^- := \mathbb{C}-(-\infty,0]$. In the theorem of Gončar and Rahmanov, the Stieltjes functions are generated by measures supported on mutually disjoint intervals of the real axis. (It should be noted that there is a small but significant difference in the use of the parameters ρ_i in the equivalent definition of a Simultaneous Padé Approximant used by Gončar and Rahmanov and our own usage (see (2.6).)

The more elegant proof of de Montessus' theorem uses complex variable methods [8]. Nevertheless, the original proof using Hadamard determinants is also instructive. This is also true for our extension of de Montessus' theorem to vector valued rational interpolants. For conciseness, we present the proofs using complex variable methods only, knowing that results such as (2.8) below may be proved, and the detail of Definition 1 motivated by determinantal representations.

2. Extensions of de Montessus' Theorem

As stated in the introduction, the vector valued Padé approximation problem is concerned with simultaneous rational approximation of d functions, $f_1(z)$, $f_2(z),\ldots,f_d(z)$, which are analytic at the origin. The degrees of the polynomials involved in forming the approximants are specified by non-negative integers N and $\rho_1,\rho_2,\ldots,\rho_d$. We use the symbol $\partial\{\pi(x)\}$ to denote the degree of a polynomial $\pi(x)$. By inspection of the determinants which occur in the construction of these approximants from the power series coefficients of $f_1,f_2,\ldots,f_d$, (see Mall [6], de Bruin [1] or Graves-Morris [4]), we see that $f_1(z)$, $f_2(z)$, $\ldots,f_d(z)$ must, in some sense, be quite different from each other for the set of rational approximants to be unique. In the context of de Montessus type theorems, the concept is made precise by the following.

Definition 1 Let each of the functions $f_1(z), f_2(z), \ldots, f_d(z)$ be meromorphic in the disc $\mathcal{D}_R := \{z : |z| < R\}$ and let non-negative integers $\rho_1, \rho_2, \ldots, \rho_d$ be given for which

$$(2.1) \qquad \sum_{i=1}^{d} \rho_i > 0 .$$

Then the functions $f_i(z)$ are said to be polewise independent, with respect to the numbers ρ_i, in $\mathcal{D}_R$ if there *do not exist* polynomials $\pi_1(z), \pi_2(z), \ldots, \pi_d(z)$, at least one of which is non-null, satisfying

$$(2.2a) \qquad \partial\{\pi_i(z)\} \leq \rho_i - 1, \quad \text{if } \rho_i \geq 1$$

$$(2.2b) \qquad \pi_i(z) \equiv 0 \qquad , \quad \text{if } \rho_i = 0$$

and such that

$$(2.3) \qquad \Phi(z) := \sum_{i=1}^{d} \pi_i(z)\, f_i(z)$$

is *analytic* throughout $\mathcal{D}_R$.

Remark 1 Under the assumptions of Definition 1, each f_i must have poles of total multiplicity at least ρ_i in $\mathcal{D}_R$. On the other hand, a particular f_i may be analytic throughout $\mathcal{D}_{\bar{R}}$, in which case, necessarily, $\rho_i = 0$. The power series coefficients of such an f_i do not appear in the standard determinantal representation of the denominator polynomial.

The theorem of de Montessus de Ballore [7] applies to the case where the degree of the denominator precisely matches the number of poles (counting multiplicity) of the given function in some disc $\mathcal{D}_R$. This is generalised to the case of simultaneous Padé approximation in the following main result.

Theorem 1 Suppose that each of the d functions $f_1(z), f_2(z), \ldots, f_d(z)$ is analytic in the disc $\mathcal{D}_R := \{z : |z| < R\}$, except for possible poles at the M (not necessarily distinct) points $z_1, z_2, \ldots, z_M$ in $\mathcal{D}_R$ which are different from the origin. (If z_k is repeated exactly p times, then each $f_i(z)$ is permitted to have a pole of order at most p at z_k.) Let $\rho_1, \rho_2, \ldots, \rho_d$ be non-negative integers such that

(2.4) $$M = \sum_{i=1}^{d} \rho_i$$

and such that the functions $f_i(z)$ are polewise independent in $\mathcal{D}_R$ with respect to the ρ_i's in the sense of Definition 1. Then, for each integer N sufficiently large, there exist polynomials $Q_N(z)$, $\{P_{N,i}(z)\}_{i=1}^{d}$ with

(2.5) $$\partial\{Q_N(z)\} = M ,$$

(2.6) $$\partial\{P_{N,i}(z)\} \leq N-\rho_i , \quad i=1,2,\ldots,d ,$$

such that

(2.7) $$f_i(z) - P_{N,i}(z)/Q_N(z) = O(z^{N+1}), \qquad i=1,2,\ldots,d .$$

The denominator polynomials (suitably normalised) satisfy

(2.8) $$\lim_{N\to\infty} Q_N(z) = Q(z) := \prod_{j=1}^{M} (z-z_j), \qquad \forall z \in \mathbb{C} .$$

Let $\mathcal{D}_R^- := \mathcal{D}_R - \bigcup_{j=1}^{M}\{z_j\}$. Then, for $i=1,2,\ldots,d$,

(2.9) $$\lim_{N\to\infty} P_{N,i}(z)/Q_N(z) = f_i(z), \qquad \forall z \in \mathcal{D}_R^- ,$$

the convergence being uniform on compact subsets of $\mathcal{D}_R^-$. More precisely, if K is any compact subset of the plane,

(2.10) $$\limsup_{N\to\infty} \| Q_N - Q \|_K^{1/N} \leq \frac{1}{R} \max_{j=1}^{M} \{|z_j|\} < 1 ,$$

and if E is any compact subset of $\mathcal{D}_R^-$,

(2.11) $$\limsup_{N\to\infty} \| f_i - P_{N,i}/Q_N \|_E^{1/N} \leq \|z\|_E / R < 1$$

for $i=1,2,\ldots,d$.

In (2.10) and (2.11), the norm is taken to be the sup norm over the indicated set.

Remark 2 By the assumptions of Theorem 1, each $f_i(z)$ has poles in $\mathcal{D}_R$ of total multiplicity at most M. Furthermore, if z_k is repeated exactly p times, then at least one $f_i(z)$ has a pole of order p at z_k. The latter assertion is a consequence of the assumption of polewise independence, as is revealed in the following preliminary lemma.

Lemma 1 With the assumptions of Theorem 1, write the list $z_1, z_2, \ldots, z_M$ in the form $\{\zeta_k\}_{k=1}^{\nu}$, where the ζ_k's are distinct and each ζ_k is of multiplicity m_k, so that

$$(2.12)\qquad Q(z) = \prod_{j=1}^{M} (z-z_j) = \prod_{k=1}^{\nu} (z-\zeta_k)^{m_k}, \qquad \sum_{k=1}^{\nu} m_k = M.$$

Then for each $k=1,2,\ldots,\nu$ and each $s=1,2,\ldots,m_k$, there exists a function $F_{k,s}(z)$ of the form

$$(2.13)\qquad F_{k,s}(z) = \sum_{i=1}^{d} \pi_i(z)\, f_i(z),$$

where the π_i's satisfy (2.2), which is analytic in $\mathcal{D}_R$, except for a pole of order s at the point ζ_k.

Naturally, the polynomials $\pi_i(z)$ in (2.13) will, in general, depend on k and s.

Proof Consider the linear problem of finding d polynomials $\pi_i(z)$, satisfying (2.2), such that for each $j=1,2,\ldots,\nu$, $j \neq k$,

$$(2.14)\qquad \int_{|z-\zeta_j|=\varepsilon} (z-\zeta_j)^{\ell} \left\{ \sum_{i=1}^{d} \pi_i(z) f_i(z) \right\} dz = 0, \quad \ell = 0,1,\ldots,m_j-1,$$

and

$$(2.15)\qquad \int_{|z-\zeta_k|=\varepsilon} (z-\zeta_k)^{\ell} \left\{ \sum_{i=1}^{d} \pi_i(z) f_i(z) \right\} dz = 0, \quad \ell = 0,1,\ldots,m_k-1, \ \ell \neq s-1,$$

where ε (>0) is sufficiently small. The system (2.14) and (2.15) has M unknowns (the coefficients of the π_i's) and consists of $M-1$ homogeneous equations. Hence it has a non-trivial solution. For such a solution, the function defined by

$$(2.16)\qquad F_{k,s}(z) := \sum_{i=1}^{d} \pi_i(z) f_i(z)$$

is either analytic throughout $\mathcal{D}_R$ or is analytic in $\mathcal{D}_R$ except for a pole of precise order s at the point ζ_k. The former possibility is excluded by the hypothesis of polewise independence. Thus $F_{k,s}(z)$ is the desired function. □

Having established the preliminary lemma, we now give the

Proof of Theorem 1 It is well known [6,5,1,4] that, for each integer N ($\geq M$), polynomials $q_N(z)$ and $\{p_{N,i}(z)\}_{i=1}^{d}$ exist which satisfy $\partial\{q_N(z)\} \leq M$, $\partial\{p_{N,i}(z)\} \leq N-\rho_i$ for $i=1,2,\ldots,d$, and

$$(2.17)\qquad q_N(z)f_i(z) - p_{N,i}(z) = O(z^{N+1}), \qquad i=1,2,\ldots,d,$$

with $q_N(z)\neq 0$. We normalise $q_N(z)$ by setting

$$(2.18a)\qquad q_N(z) = \sum_{j=0}^{M} b_{N,j}z^j,$$

$$(2.18b)\qquad \sum_{j=0}^{M} |b_{N,j}| = 1, \qquad N=M,M+1,\ldots,$$

and then the $q_N(z)$ are uniformly bounded on each compact subset of the plane.

We first show that, for $k=1,2,\ldots,\nu$,

$$(2.19)\qquad \limsup_{N\to\infty} |q_N^{(j)}(\zeta_k)|^{1/N} \leqslant |\zeta_k|/R, \qquad j=0,1,\ldots,m_k-1,$$

where

$$(2.20)\qquad q_N^{(j)}(z) := \left(\frac{d}{dz}\right)^j q_N(z).$$

To establish (2.19), fix k and consider $F_{k,1}(z)$ of the lemma which is analytic in $\mathcal{D}_R$ except for a simple pole at ζ_k. Write

$$(2.21)\qquad F_{k,1}(z) = \frac{g_{k,1}(z)}{z-\zeta_k},$$

where $g_{k,1}(z)$ is analytic in $\mathcal{D}_R$ and $g_{k,1}(\zeta_k)\neq 0$. By using the polynomials $\pi_i(z)$ defined by (2.13) when s=1, we deduce from (2.17) that

$$(2.22)\qquad q_N(z)\,F_{k,1}(z) - \tilde{p}_{N,1}(z) = O(z^{N+1}),$$

$$(2.23)\qquad \tilde{p}_{N,1}(z) := \sum_{i=1}^{d} \pi_i(z)\,p_{N,i}(z).$$

From (2.2) and the fact that $\partial\{p_{N,i}(z)\}\leqslant N-\rho_i$, it follows that

$$(2.24)\qquad \partial\{\tilde{p}_{N,1}(z)\} \leqslant N-1,$$

and hence, from (2.22), $(z-\zeta_k)\tilde{p}_{N,1}(z)$ is the unique polynomial of degree at most N which interpolates $q_N(z)g_{k,1}(z)$ to order N (inclusively) at the origin. Thus, since $q_N(z)g_{k,1}(z)$ is analytic in $\mathcal{D}_R$, we use Hermite's formula to show that

$$(2.25)\qquad q_N(z)g_{k,1}(z) - (z-\zeta_k)\tilde{p}_{N,1}(z) = \frac{1}{2\pi i}\int_{C_{R'}} \frac{z^{N+1}}{t^{N+1}}\frac{q_N(t)g_{k,1}(t)}{t-z}\,dt ,$$

for all $|z|<R'$, where $|\zeta_k|<R'<R$ and $C_{R'}: |t| = R'$. On taking $z=\zeta_k$ in (2.25), we obtain by straightforward estimation of the integral and then by letting $R'\to R$,

$$\limsup_{N\to\infty} |q_N(\zeta_k)g_{k,1}(\zeta_k)|^{1/N} \leq |\zeta_k|/R .$$

As $g_{k,1}(\zeta_k)\neq 0$, this implies that

$$(2.26)\qquad \limsup_{N\to\infty} |q_N(\zeta_k)|^{1/N} \leq |\zeta_k|/R .$$

Proceeding by induction, we take $s \leq m_k$ and assume that

$$(2.27)\qquad \limsup_{N\to\infty} |q_N^{(j)}(\zeta_k)|^{1/N} \leq |\zeta_k|/R,\quad j=0,1,\ldots,s-2 ,$$

and we must show that (2.27) holds for $j=s-1$. Utilizing the function $F_{k,s}(z)$ of the lemma, we obtain as above,

$$(2.28)\qquad q_N(z)F_{k,s}(z) - \tilde{p}_{N,s}(z) = O(z^{N+1}) ,$$

where $\partial\{\tilde{p}_{N,s}(z)\} \leq N-1$. Express

$$(2.29)\qquad F_{k,s}(z) = \frac{g_{k,s}(z)}{(z-\zeta_k)^s} ,$$

where $g_{k,s}(z)$ is analytic in $\mathcal{D}_R$ and $g_{k,s}(\zeta_k)\neq 0$. By (2.28), the polynomial $(z-\zeta_k)\tilde{p}_{N,s}(z)$ interpolates $q_N(z)g_{k,s}(z)/(z-\zeta_k)^{s-1}$ to order N inclusively at the origin. For any given compact set K, where $K\subset\mathcal{D}_R^-$, we may choose $R'(<R)$ and $\varepsilon(>0)$, so that, for all $z\in K$,

$$(2.30)\qquad q_N(z)\frac{g_{k,s}(z)}{(z-\zeta_k)^{s-1}} - (z-\zeta_k)\tilde{p}_{N,s}(z) = I_N(z) - J_N(z) ,$$

where

$$(2.31)\qquad I_N(z) := \frac{1}{2\pi i}\int_{C_{R'}} \frac{z^{N+1}}{t^{N+1}}\frac{q_N(t)g_{k,s}(t)}{(t-\zeta_k)^{s-1}(t-z)}\,dt ,$$

$$(2.32)\qquad J_N(z) := \frac{1}{2\pi i}\int_{|t-\zeta_k|=\varepsilon} \frac{z^{N+1}}{t^{N+1}}\frac{q_N(t)g_{k,s}(t)}{(t-\zeta_k)^{s-1}(t-z)}\,dt.$$

The result (2.30) is established using the Hermite formula and the residue theorem. To estimate $J_N(z)$ for $z \in K$, express

$$q_N(t) = \sum_{j=0}^{N} \frac{q_N^{(j)}(\zeta_k)}{j!} (t-\zeta_k)^j \quad .$$

Then

$$(2.33) \qquad J_N(z) = \sum_{j=0}^{s-2} \frac{1}{2\pi i} \int_{|t-\zeta_k|=\varepsilon} \frac{z^{N+1}}{t^{N+1}} \frac{q_N^{(j)}(\zeta_k) g_{k,s}(t)}{j!(t-\zeta_k)^{s-1-j}(t-z)} dt.$$

By straightforward estimation of the integral in (2.33) and using the inductive hypothesis (2.27), we obtain

$$(2.34) \qquad \lim_{\varepsilon\to 0} \limsup_{N\to\infty} \|J_N(z)\|_K^{1/N} \leq \frac{\|z\|_K}{|\zeta_k|} \cdot \frac{|\zeta_k|}{R} = \frac{\|z\|_K}{R} .$$

Similarly,

$$(2.35) \qquad \lim_{R'\to R} \limsup_{N\to\infty} \|I_N(z)\|_K^{1/N} \leq \|z\|_K/R .$$

Hence, from (2.30), (2.34) and (2.35), we find that, for compact $K \subset \mathcal{D}_R$,

$$(2.36) \qquad \limsup_{N\to\infty} \|q_N(z) g_{k,s}(z) - (z-\zeta_k)^s \tilde{p}_{N,s}(z)\|_K^{1/N} \leq \|z\|_K/R .$$

But since the function

$$\phi(z) := q_N(z) g_{k,s}(z) - (z-\zeta_k)^s \tilde{p}_{N,s}(z) ,$$

which appears in the left-hand side of (2.36) is analytic throughout $\mathcal{D}_R$, then (2.36) also holds for any $K \subset \mathcal{D}_R$. By using Cauchy's contour integral formula for the function $(d/dz)^{s-1}\phi(z)$, and (2.36), we find that

$$(2.37) \qquad \limsup_{N\to\infty} \left[\left(\frac{d}{ds}\right)^{s-1} \left[q_N(z) g_{k,s}(z) \right] \Big|_{z=\zeta_k} \right]^{1/N} \leq \frac{|\zeta_k|}{R} .$$

Using Leibniz's formula for differentiating the product (2.37), and the inductive hypothesis (2.27), we get

$$(2.38) \qquad \limsup_{N\to\infty} |g_{k,s}(\zeta_k) - q_N^{(s-1)}(\zeta_k)|^{1/N} \leq |\zeta_k|/R .$$

As $g_{k,s}(\zeta_k)\neq 0$, it follows from (2.38) that (2.27) holds also for $j=s-1$, which completes the induction. This proves the claim (2.19).

Next, consider a basis of polynomials

$$B = \{B_{k,s}(z),\ k=1,2,\ldots,\nu,\ s=0,1,\ldots,m_k-1\}$$

such that both

(2.39) $\quad \partial\{B_{k,s}(z)\} \leq M-1 \quad$ for all k,s

and the polynomials interpolate at the points ζ_i according to

(2.40) $$\left[\left(\frac{d}{dz}\right)^j B_{k,s}(z)\right]_{z=\zeta_i} = \delta_{ik}\cdot\delta_{js}, \quad 1\leq i\leq\nu, \quad 0\leq j\leq m_i-1 .$$

Then we can write (see (2.12), (2.18a) and (2.39))

(2.41) $$q_N(z) = \sum_{k=1}^{\nu}\sum_{s=0}^{m_k-1} q_N^{(s)}(\zeta_k)\, B_{k,s}(z) + b_{N,M}Q(z) .$$

By (2.18b), we have $|b_{N,M}| \leq 1$. More importantly, however,

(2.42) $$\liminf_{N\to\infty} |b_{N,M}| > 0 ;$$

indeed, if this were not the case, (2.19) shows that some subsequence of indices $\{N_i\}$ exists for which

$$\lim_{i\to\infty} q_{N_i}(z) = 0, \quad \text{for all } z\in\mathbb{C} ,$$

contradicting (2.18). Thus, for N sufficiently large, we define

(2.43) $\quad Q_N(z) := q_N(z)/b_{N,M}$

(2.44) $\quad P_{N,i}(z) := p_{N,i}(z)/b_{N,M}, \qquad i=1,2,\ldots,d ,$

and the assertions (2.5)-(2.8) all follow. Assertion (2.10) follows from (2.19) and (2.41).

Finally, to establish (2.11), (and hence (2.9)), let E be a compact subset of $\mathcal{D}_R$. Then, for $z\in E$ and $i=1,2,\ldots,d$,

(2.45) $$Q_N(z)f_i(z) - P_{N,i}(z) = I_{N,i}(z) - \sum_{k=1}^{\nu} J_{N,i,k}(z) ,$$

where

$$(2.46) \qquad I_{N,i}(z) := \frac{1}{2\pi i} \int_{C_{R'}} \frac{z^{N+1}}{t^{N+1}} \frac{Q_N(t) f_i(t)}{t-z} dt ,$$

$$(2.47) \qquad J_{N,i,k}(z) := \frac{1}{2\pi i} \int_{|t-\zeta_k|=\varepsilon} \frac{z^{N+1}}{t^{N+1}} \frac{Q_N(t) f_i(t)}{t-z} dt, \quad k=1,2,\ldots,\nu.$$

Since the inequalities (2.19) also hold for the polynomials $Q_N(t)$, the integrals of (2.46) and (2.47) can be estimated in the same manner as used in the inductive portion of the proof. This gives

$$\limsup_{N\to\infty} \|Q_N(z)f_i(z) - P_{N,i}(z)\|_E^{1/N} \leq \|z\|_E/R, \quad i=1,2,\ldots,d .$$

The inequalities (2.11) now follow from (2.8). □

Corollary 1 (Uniqueness). Under the assumptions of Theorem 1, there exist, for each sufficiently large integer N, a *unique* set of rationals $\{R_{N,i}(z)\}_{i=1}^{d}$ of the form

$$R_{N,i}(z) = P_{N,i}(z)/Q_N(z) ,$$

such that

$$\partial\{Q_N(z)\} \leq M , \qquad \partial\{P_{N,i}(z)\} \leq N-\rho_i ,$$

and

$$(2.48) \qquad f_i(z) - R_{N,i}(z) = O(z^{N+1}), \quad i=1,2,\ldots,d .$$

Proof Assume, to the contrary, that, for some subsequence $\mathcal{N}$ of integers N, another set of rationals $\{\tilde{R}_{N,i}(z)\}_{i=1}^{d}$ exists of the form

$$\tilde{R}_{N,i}(z) = \tilde{P}_{N,i}(z)/\tilde{Q}_N(z) ,$$

such that

$$\partial\{\tilde{Q}_N(z)\} \leq M , \qquad \partial\{\tilde{P}_{N,i}(z)\} \leq N-\rho_i ,$$

$$(2.49) \qquad f_i(z) - \tilde{R}_{N,i}(z) = O(z^{N+1}) , \quad i=1,2,\ldots,d ,$$

but such that $\tilde{R}_{N,j}(z) \neq R_{N,j}(z)$ for some j. Then, necessarily, $Q_N(z)$ is not a constant multiple of $\tilde{Q}_N(z)$. The proof of Theorem 1 shows that, for $N \in \mathcal{N}$ and N sufficiently large, both $Q_N(z)$ and $\tilde{Q}_N(z)$ must be of precise degree M, and so, without loss of generality, we assume that both are monic and of degree M. From (2.48) and (2.49), we have

$$\{Q_N(z) - \tilde{Q}_N(z)\}f_i(z) - \{P_{N,i}(z) - \tilde{P}_{N,i}(z)\} = O(z^{N+1}) ,$$

for $i=1,2,\ldots,d$. The proof of Theorem 1 now implies that

$$\lim_{N\to\infty} c_N\{Q_N(z) - \tilde{Q}_N(z)\} = Q(z) \tag{2.50}$$

for $N \in \mathcal{N}$, $z \in \mathbb{C}$ and a suitable choice of normalizing constants c_N. But $Q(z)$ is of degree M, whereas the polynomials $\{Q_N(z)-\tilde{Q}_N(z)\}$ are of degree M-1 at most, making (2.50) absurd. □

Remark 3 Consider the special case of Theorem 1, in which each $f_i(z)$ has poles of total multiplicity precisely equal to ρ_i in $\mathcal{D}_R$. Further, assume that the pole sets of each f_i (within $\mathcal{D}_R$) are mutually disjoint sets. Then it is clear that the f_i's are polewise independent with respect to the ρ_i's, and so the conclusions of Theorem 1 and its corollary are valid. In this special case, if we *further* assume that all the poles involved are simple, then Theorem 1 yields the result (Theorem VIII) in the dissertation of Mall [6].

Theorem 2 (for Directed Vector Padé Approximants) Let each of the d functions $f_1(z), f_2(z), \ldots, f_d(z)$ be analytic in the disc $\mathcal{D}_R$ except for possible poles in the M (not necessarily distinct) points $z_1, z_2, \ldots, z_M$ in $\mathcal{D}_R$ which are different from the origin. Given ℓ constant d-dimensional vectors $\underline{w}^{(1)}, \underline{w}^{(2)}, \ldots, \underline{w}^{(\ell)}$, define a column vector $\underline{f}$ by

$$\underline{f} = (f_1(z), f_2(z), \ldots, f_d(z))^T$$

and set

$$F_j(z) := \underline{f}^T.\underline{w}^{(j)}, \quad j=1,2,\ldots,\ell, \tag{2.51}$$

where T denotes transpose. Let $k_1, k_2, \ldots, k_\ell$ be positive integers for which $\sum_{j=1}^{\ell} k_j = M$ and such that the functions $F_j(z)$ are polewise independent with respect to the k_j's in $\mathcal{D}_R$. Then, for each N large enough, there exist polynomials $Q_N(z)$, $\{P_{N,i}(z)\}_{i=1}^{d}$, for which

(2.52) $\quad \partial\{Q_N(z)\} = M, \quad \partial\{P_{N,i}(z)\} \leq N, \qquad i=1,2,\ldots,d,$

(2.53) $\quad f_i(z) - P_{N,i}(z)/Q_N(z) = O(z^{N+1}), \quad i=1,2,\ldots,d,$

(2.54) $\quad \partial\{\underline{P}_N^T\cdot\underline{w}^{(j)}\} \leq N-k_j, \qquad j=1,2,\ldots,\ell,$

where $\underline{P}_N^T := (P_{N,1}(z),P_{N,2}(z),\ldots,P_{N,d}(z))$. Furthermore, the conclusions (2.8)-(2.11) of Theorem 1 hold.

Remark 4 The assumption of polewise independence implies that the vectors $\underline{w}^{(1)},\underline{w}^{(2)},\ldots,\underline{w}^{(\ell)}$ are linearly independent, and consequently $\ell\leq d$.

Proof In the case that $\ell<d$, the definitions need to be extended by taking

(2.55) $\quad k_j := 0, \quad j=\ell+1,\ell+2,\ldots,d$.

By suitably defining vectors $\underline{w}^{(\ell+1)},\underline{w}^{(\ell+2)},\ldots,\underline{w}^{(d)}$ such that $\underline{w}^{(1)},\underline{w}^{(2)},\ldots,\underline{w}^{(d)}$ are linearly independent, we define a non-singular matrix W whose columns are the vectors $\underline{w}^{(j)}$, i.e.

$$W_{ij} = w_i^{(j)}, \qquad i,j=1,2,\ldots,d .$$

In this way, we may extend (2.51) so that it becomes

(2.56) $\quad F_j(\underline{z}) = \sum_{i=1}^{d} w_i^{(j)} f_i(\underline{z}) = (\underline{f}^T W)_j, \qquad j=1,2,\ldots,d .$

Furthermore, $\{F_j(z)\}_{j=1}^d$ remain polewise independent in $\mathcal{D}_R$ with respect to $\{k_j\}_{j=1}^d$, by (2.2b) and (2.55). We now infer from Theorem 1 that polynomials $Q_N(z)$ and $\{\tilde{P}_{N,i}(z)\}_{i=1}^d$ exist for which

(2.57) $\quad \partial\{Q_N(z)\} = M$

(2.58) $\quad \partial\{\tilde{P}_{N,i}(z)\} \leq N-k_i \qquad i=1,2,\ldots,d ,$

(2.59) $\quad F_i(z) - \tilde{P}_{N,i}(z)/Q_N(z) = O(z^{N+1}) \qquad i=1,2,\ldots,d ,$

and the corresponding equivalents of (2.8)-(2.11) hold too.

From (2.56), we have

$$f_i = (\underline{f}^T)_i = (\underline{F}^T W^{-1})_i = \sum_{j=1}^{d} F_j(z)(W^{-1})_{ji} ,$$

and so we are led to define

$$P_{N,i}(z) := \sum_{j=1}^{d} (W^{-1})_{ji} \tilde{P}_{N,j}(z), \quad i=1,2,\ldots,d .$$

With this definition, (2.52) and (2.53) follow immediately; (2.54) follows from (2.58), and the other equivalent properties of Theorem 1 follow similarly by linearity. □

Theorems 1 and 2 can immediately be extended to include more general cases of Lagrange and Hermite rational interpolation. These extensions are analogues of the theorem of Saff [8] for equilibrium distributions of the interpolating points and its generalisation by Warner [9] for the case of regular interpolation schemes. Padé approximation problems are associated with osculatory interpolation at the origin, and we next consider the generalisation of Theorems 1 and 2 to Lagrange and Hermite interpolation on points which belong to a compact set S .

For each positive integer N, we consider an interpolating point set

$$S_N := \{\beta_{N,i}, \quad i=0,1,\ldots,N : \beta_{N,i} \in S\}$$

where the $\beta_{N,i}$'s need not necessarily be distinct, so that partially confluent cases of Hermite interpolation are included. Following Warner [9], we assume that the points $\{\{\beta_{N,i}\}_{i=0}^{N}\}_{N=1}^{\infty}$ have an associated sequence of elementary measures μ_N which is regular in the sense that $\mu_N \to \mu$. This ensures that the logarithmic potentials

$$u(z,\mu_N) := -\int \log|z-\zeta|\, d\mu_N(\zeta) , \qquad N=1,2,\ldots,$$

have the property that

$$\lim_{N\to\infty} u(z,\mu_N) = u(z,\mu), \quad \forall \; z \in \mathbb{C} \backslash S ,$$

where

$$u(z,\mu) := -\int \log|z-\zeta|\, d\mu(\zeta) .$$

The convergence theorems are based on a nested set of regions $\mathbf{D}_\lambda$, defined for each $\lambda > 0$ by

$$\mathbf{D}_\lambda = \{z : e^{-u(z,\mu)} < \lambda\} .$$

With these preliminaries and the obvious extension of Definition 1 to the region $\mathbf{D}_\lambda$, we state

Theorem 3. Let each of the functions $f_1(z), f_2(z), \ldots, f_d(z)$ be analytic on S and also in the larger region $\mathbf{D}_R$, except for possible poles at the points $z_1, z_2, \ldots, z_M$ in $\mathbf{D}_R$. Define

$$\mathbf{D}_R^- := \mathbf{D}_R - \bigcup_{i=1}^{M} \{z_i\}$$

and let E by any compact subset of $\mathbf{D}_R^-$. Given d non-negative integers $\rho_1, \rho_2, \ldots, \rho_d$ satisfying $M = \sum_{i=1}^{d} \rho_i$, assume that the $f_i(z)$ are polewise independent with respect to the ρ_i's in $\mathbf{D}_R$. Then, for each N large enough, there exist polynomials $Q_N(z)$, $\{P_{N,i}(z)\}_{i=1}^{d}$, with

$$\partial\{Q_N(z)\} = M, \quad \partial\{P_{N,i}(z)\} \leq N-\rho_i, \qquad i=1,2,\ldots,d ,$$

such that $P_{N,i}(z)/Q_N(z)$ interpolates as

$$P_{N,i}(z)/Q_N(z) = f_i(z) \quad \forall \ z \in S_N , \qquad i=1,2,\ldots,d ,$$

in the Hermite sense. The denominator polynomials obey

$$\lim_{N\to\infty} Q_N(z) = Q(z) := \prod_{i=1}^{M} (z-z_i), \quad \forall \ z \in \mathbb{C}.$$

Furthermore,

$$\lim_{N\to\infty} P_{N,i}(z)/Q_N(z) = f_i(z), \qquad \forall \ z \in \mathbf{D}_R^- , \quad i=1,2,\ldots,d,$$

the convergence being uniform on E, which is an arbitrary compact subset of $\mathbf{D}_R^-$.

Define r to be the smallest number for which it is true that

$$\bigcup_{i=1}^{M} \{z_i\} \subset \mathbf{D}_{r'}, \qquad \forall \ r' > r ,$$

and define σ to be the smallest number for which it is true that

$\bar{E} \subset \mathbf{D}_{\sigma'} \qquad \forall\ \sigma' > \sigma$.

Then the rates of convergence of the interpolants are given by

$$\limsup_{N\to\infty} \|Q_N - Q\|_K^{1/N} \leqslant r/R ,$$

where K is any compact subset of $\mathbb{C}$, and

$$\limsup_{N\to\infty} \|f_i(z) - P_{N,i}(z)/Q_N(z)\|_E^{1/N} \leqslant \sigma/R, \quad i=1,2,\dots,d .$$

A uniqueness assertion and the generalisation to directed vector valued rational interpolants also hold for Theorem 3.

Remark 5 According to the hypotheses of the theorem, each $f_i(z)$ is analytic in $\bar{\mathbf{D}}_R$. A possible configuration is shown in Fig.1.

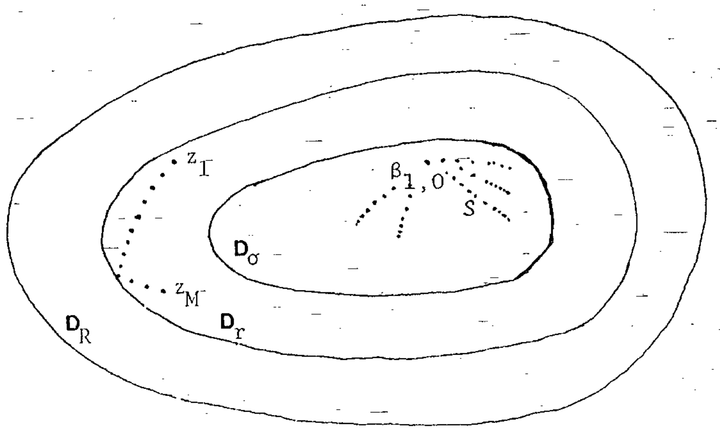

Fig. 1 The poles $\{z_i\}$, the point sequence $\{\beta_{N,i}\} \subset S$, and the boundaries of the domains $\mathbf{D}_\sigma$, $\mathbf{D}_r$ and $\mathbf{D}_R$ are shown schematically.

<u>Postscript</u> After this paper and the paper [W] by Hans Wallin had been presented at the Conference, we saw that we had adopted quite similar approaches to estimation of the rate of convergence of the denominator polynomials and related quantities.

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