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CONSTRUCTIVE APPROXIMATION

Where Does the Sup Norm of a Weighted Polynomial Live?

(A Generalization of Incomplete Polynomials)

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Abstract. A characterization is given of the sets supporting the uniform norms of weighted polynomials $[w(x)]^n P_n(x)$, where P_n is any polynomial of degree at most n. The (closed) support Σ of $w(x)$ may be bounded or unbounded; of special interest is the case when $w(x)$ has a nonempty zero set Z. The treatment of weighted polynomials consists of associating each admissible weight with a certain functional defined on subsets of $\Sigma - Z$. One main result of this paper states that there is a unique compact set (independent of n and P_n) maximizing this functional that contains the points where the norms of weighted polynomials are attained. The distribution of the zeros of Chebyshev polynomials corresponding to the weights $[w(x)]^n$ is also studied. The main theorems give a unified method of investigating many particular examples. Applications to weighted approximation on the real line with respect to a fixed weight are included.

Dedicated to Professor Geza Freud

1. Introduction

As described in the survey article [11], the study of "incomplete polynomials," as introduced by G.G. Lorentz [8] in 1976, leads to results on the asymptotic properties of polynomials orthogonal on an infinite interval (cf. [10]) and to theorems on the convergence of "ray sequences" of Padé approximations for Stieltjes functions. We present here a generalization of the theory for incomplete polynomials that unifies many of the previous results. The essential problem that serves as the starting point for the investigation is the following:

Suppose that $w(x)$ is a nonnegative weight function that is continuous on its support $\Sigma \subset \mathbf{R} = (-\infty, \infty)$. (By the *support* of w we mean the *closure* of the set where w is positive.) Assume that $w(x)$ vanishes at points of Σ; that is, $Z := \{x \in \Sigma: w(x) = 0\} \neq \emptyset$ (or, in case Σ is unbounded, then $|x| w(x) \to 0$ as $|x| \to \infty$). If P_n is an arbitrary polynomial of degree at most n, then the sup norm over Σ of the weighted polynomial

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$[w(x)]^n P_n(x)$ actually "lives" on some compact set $\mathcal{S} \subset \Sigma - Z$ which is independent of n and P_n. The problem is to determine the smallest such set $\mathcal{S}$.

For example, if $w(x) = x^{\theta/(1-\theta)}$ with $\Sigma = [0, 1]$, $0 < \theta < 1$, then, as shown in [5] and [14], the set $\mathcal{S}$ is the subinterval $[\theta^2, 1]$. This special fact plays a key role in the study of polynomials that are incomplete in the original sense of Lorentz.

In this paper we use potential theoretic methods to show how the set $\mathcal{S}$ can be obtained for a class of weight functions. The assumptions on w are given in

Definition 1.1. *Let $w: \mathbf{R} \to [0, +\infty)$. We say that w is an admissible weight function if each of the following properties holds:*
(i) $\Sigma := \operatorname{supp}(w)$ *has positive capacity.*
(ii) *The restriction of w to Σ is continuous on Σ.*
(iii) *The set $Z := \{x \in \Sigma: w(x) = 0\}$ has capacity zero.*
(iv) *If Σ is unbounded, then $|x|w(x) \to 0$ as $|x| \to \infty$, $x \in \Sigma$.*

Here and throughout this paper, the term "capacity" means inner logarithmic capacity (cf. [17, p. 55]). For any set $E \subset \mathbf{R}^2$, its capacity will be denoted by $C(E)$. If K is a compact set with positive capacity, then ν_K shall denote the unique unit equilibrium measure on K with the property that

$$\int_K \log |x-t| \, d\nu_K(t) = \log C(K) \tag{1.1}$$

quasi-everywhere (q.e.) on K (cf. [17, p. 60]). A property is said to hold q.e. on a set A if the subset E of A where it does not hold satisfies $C(E) = 0$.

For an admissible weight w, we always set

$$Q(x) := \log [1/w(x)]. \tag{1.2}$$

Finally, if $K \subset \Sigma - Z$ is compact and $C(K) > 0$, we define the *F-functional* of K by the formula

$$F(K) := \log C(K) - \int_K Q \, d\nu_K. \tag{1.3}$$

The theorems of Section 2 show that, for a class of weight functions, the set $\mathcal{S}$ mentioned above is derived by maximizing the F-functional. Also, if π_m denotes the collection of all polynomials of degree at most m and $\|\cdot\|_A$ denotes the sup norm over a set A, we describe the asymptotic behavior of the errors in the weighted Chebyshev problem

$$E_n(w) := \inf \{\|[w(x)]^n\{x^n - p_{n-1}(x)\}\|_\Sigma: p_{n-1} \in \pi_{n-1}\}, \qquad n = 1, 2, \ldots, \tag{1.4}$$

as well as asymptotic properties (as $n \to \infty$) of the extremal polynomials $T_n(x; w) = x^n + \ldots \in \pi_n$ that satisfy

$$E_n(w) = \|[w(x)]^n T_n(x; w)\|_\Sigma, \qquad n = 1, 2, \ldots. \tag{1.5}$$

The outline of the paper is as follows. In Section 2, we state our main theorems, and in Section 3 we give some applications of our results. These applications include several known theorems that heretofore were treated by separate analysis, as well as some new results for weighted Chebyshev problems on infinite intervals. Section 4 contains the proofs of the theorems stated in Section 2.

2. Statements of Main Results

With the notation of Section 1, we now describe our main theorems.

Theorem 2.1. *Let w be an admissible weight function with support Σ. Then there exists a compact set $\mathcal{S} \subset \Sigma - Z$ with $C(\mathcal{S}) > 0$ that has the following properties:*

(a) *For every compact set $K \subset \Sigma - Z$ with $C(K) > 0$,*

$$F(K) \leq F(\mathcal{S}), \tag{2.1}$$

where F is defined in (1.3).

(b) *If equality holds in* (2.1), *then $\mathcal{S} \subset K$.*

(c) *For any positive integer n, if $P_n \in \pi_n$ and the inequality*

$$|[w(x)]^n P_n(x)| \leq M \qquad (M = \text{constant}) \tag{2.2}$$

holds q.e. on $\mathcal{S}$, then it holds q.e. on Σ.

(d) *The errors $E_n(w)$ defined in* (1.4) *satisfy*

$$[E_n(w)]^{1/n} \geq \exp(F(\mathcal{S})), \qquad \forall\, n = 1, 2, \ldots. \tag{2.3}$$

Clearly properties (a) and (b) uniquely determine the set $\mathcal{S} = \mathcal{S}(w)$ of Theorem 2.1. In the special case when $w(x) \equiv 1$ on Σ and Σ is compact, then $\mathcal{S}$ is just the support of the equilibrium measure ν_Σ for Σ.

If Σ is regular, i.e. for all n large, $\Sigma \cap [-n, n]$ is regular with respect to the Dirichlet problem for its complement on the Riemann sphere, then part (c) can be strengthened.

Theorem 2.1 (c′). *Assume Σ is regular. For any positive integer n, if $P_n \in \pi_n$, then*

$$\|[w(x)]^n P_n(x)\|_\Sigma = \|[w(x)]^n P_n(x)\|_{\mathcal{S}}. \tag{2.4}$$

Of practical importance is the characterization of $\mathcal{S}$ given in

Theorem 2.2. *Assume that, in Theorem 2.1, the set $\Sigma - Z$ is the finite union of disjoint nondegenerate intervals and that $Q(x)$ of* (1.2) *is convex in each of the components of $\Sigma - Z$. Then the following additional properties hold:*

(a) *The compact set $\mathcal{S}$ of Theorem 2.1 is the finite union of nondegenerate disjoint closed intervals, at most one in each component of $\Sigma - Z$.*

(b) *Equality holds in* (2.1) *if and only if $\mathcal{S} \subset K$ and $C(K - \mathcal{S}) = 0$.*

(c) *For any positive integer n, if $P_n \in \pi_n$, then* (2.4) *holds.*

(d) *The errors $E_n(w)$ of* (1.4) *satisfy*

$$\lim_{n\to\infty} [E_n(w)]^{1/n} = \exp(F(\mathcal{S})). \tag{2.5}$$

With the assumptions on w stated in Theorem 2.2, the set $\mathcal{S}$ can be determined as follows. Maximize the F-functional over only those sets J having the same form as $\mathcal{S}$ [cf. Theorem 2.2.(a)]. If this maximum occurs for a set J_0, then $F(\mathcal{S}) = F(J_0)$ and, from part (b) of Theorem 2.2, it follows that $\mathcal{S} = J_0$. In Section 3 we carry out this procedure for various weights w.

Remark. The limit (2.5) holds under milder assumptions than those given in Theorem 2.2. To be precise, if w is an admissible weight for which the set $\mathcal{S}$ of Theorem 2.1 is regular with respect to the Dirichlet problem and $\Sigma - Z$ has the property that every set $A \subset \Sigma - Z$ satisfying $C(\Sigma - Z - A) = 0$ is dense in $\Sigma - Z$, then (2.5) holds.

The proof of Theorem 2.1 follows from showing that the set $\mathcal{S}$ is actually the support of a measure that solves an extremal problem for generalized energy integrals as we now describe. Let $\mathcal{M}(\Sigma)$ denote the collection of all positive unit Borel measures μ with $\operatorname{supp}(\mu) \subset \Sigma$, and define

$$(2.6) \qquad I_w[\mu] := \iint [\log |x-t| - Q(x) - Q(t)]\, d\mu(x) d\mu(t),$$

for $\mu \in \mathcal{M}(\Sigma)$. Following methods of Frostman (cf. [4], [17]) we shall prove

Theorem 2.3. *Let w be an admissible weight function with support Σ, and let*

$$(2.7) \qquad V_w := \sup \{I_w[\mu] \colon \mu \in \mathcal{M}(\Sigma)\}.$$

Then the following properties hold.

(a) *V_w is finite.*

(b) *There exists a unique element $\mu_w \in \mathcal{M}(\Sigma)$ such that*

$$(2.8) \qquad I_w[\mu_w] = V_w.$$

(c) *$\mathcal{S}_w := \operatorname{supp}(\mu_w)$ is compact, $\mathcal{S}_w \subset \Sigma - Z$, and $C(\mathcal{S}_w) > 0$.*

(d) *The inequality*

$$(2.9) \qquad \int \log |x-t|\, d\mu_w(t) \le Q(x) + V_w + \int Q\, d\mu_w$$

holds q.e. on Σ.

(e) *The inequality*

$$(2.10) \qquad \int \log |x-t|\, d\mu_w(t) \ge Q(x) + V_w + \int Q\, d\mu_w$$

holds for all $x \in \mathcal{S}_w$.

(f) *The F-functional of the set $\mathcal{S}_w$ is given by*

$$(2.11) \qquad F(\mathcal{S}_w) = V_w + \int Q\, d\mu_w.$$

(g) *For any positive integer n, if $P_n \in \pi_n$ and*

$$(2.12) \qquad |[w(x)]^n P_n(x)| \le M \qquad \text{q.e. on } \mathcal{S}_w,$$

then

$$(2.13) \qquad |P_n(z)| \le M \exp\left(n\left[\int \log |z-t|\, d\mu_w(t) - F(\mathcal{S}_w)\right]\right), \qquad \forall\, z \in \mathbf{C},$$

where **C** *denotes the complex plane.*

(h) *The set* $\mathcal{S}_w$ *satisfies all the properties stated in Theorem 2.1; that is,* $\mathcal{S}_w = \mathcal{S}$.

Under certain conditions we can show that the measure μ_w of Theorem 2.3 gives the limiting distribution of the zeros of the extremal polynomials $T_n(x;w)$ of (1.5).

Theorem 2.4. *Let* w *be an admissible weight function and suppose that* $\mathcal{I} \subset \mathbf{R}$ *is a closed bounded interval containing the set* $\mathcal{S} = \mathcal{S}_w$ *of Theorem 2.1. Let* $\{t_{k,n}\}_{k=1}^{n}$, $n = 1, 2, \ldots$, *be a triangular scheme of points lying in* $\mathcal{I}$. *With this scheme associate the sequence of polynomials*

$$q_n(x) := \prod_{k=1}^{n} (x - t_{k,n}), \qquad n = 1, 2, \ldots,$$

and the sequence of elementary unit measures $\{\nu^{(n)}\}_{n=1}^{\infty}$, *where for any Borel set* $\mathcal{B}$,

(2.14) $$\nu^{(n)}(\mathcal{B}) := \frac{1}{n} \,|\{k\colon t_{k,n} \in \mathcal{B}\}|, \qquad n = 1, 2, \ldots.$$

Assume that

(2.15) $$\limsup_{n\to\infty} \|[w(x)]^n q_n(x)\|_{\mathcal{S}}^{1/n} \le \exp(F(\mathcal{S})).$$

Then, in the weak-star topology,

(2.16) $$\lim_{n\to\infty} \nu^{(n)} = \mu_w,$$

where μ_w *is the extremal measure of Theorem 2.3. Furthermore,*

(2.17) $$\lim_{n\to\infty} |q_n(z)|^{1/n} = \exp\left[\int \log|z-t| \, d\mu_w(t)\right]$$

uniformly on every compact set of the complex plane disjoint from $\mathcal{I}$.

Corollary 2.5. *With the assumptions of Theorem 2.2, if* $\mathcal{I}$ *is the convex hull of* $\mathcal{S}$ *and* $\{t_{k,n}\}_{k=1}^{n}$ *are the zeros of the extremal polynomial* $T_n(x;w)$ *of* (1.5), *then* (2.16) *and* (2.17) *hold with* $q_n(z) = T_n(z;w)$.

Remark. For a weight w satisfying the hypotheses of Theorem 2.2, property (2.16) implies that the zeros of the extremal polynomials $\{T_n(x;w)\}_{n=1}^{\infty}$ are dense in the set $\mathcal{S}$. Hence, from the equioscillation theorem for Chebyshev approximation, the same is true of the points where the norm of $[w(x)]^n T_n(x;w)$ over Σ is attained. Thus, $\mathcal{S}$ is the *smallest* compact set such that equation (2.4) holds for all n and all $P_n \in \pi_n$.

3. Applications

Example 1. Incomplete Polynomials of Lorentz. Let $w(x) = x^{\theta/(1-\theta)}$, where $0 < \theta < 1$ and $\Sigma = [0, 1]$. Since $Q(x) = -[\theta/(1-\theta)] \log x$ is convex on $(0, 1] = \Sigma - Z$, then according to Theorem 2.2 we can determine the set $\mathcal{S}$ by maximizing the F-

functional over intervals $K = [a, b] \subset (0, 1]$. It is easily verified that

$$F([a, b]) = \log\left(\frac{b-a}{4}\right) + \frac{2\theta}{1-\theta}\log(\sqrt{b}+\sqrt{a}) - \frac{\theta}{1-\theta}\log 4, \tag{3.1}$$

which is maximized when $[a, b] = [\theta^2, 1]$. Hence, as stated in the introduction, $\mathcal{S} = [\theta^2, 1]$. From Theorem 2.2 we also obtain

$$\|x^{n\theta/(1-\theta)}P_n(x)\|_{[0,1]} = \|x^{n\theta/(1-\theta)}P_n(x)\|_{[\theta^2,1]}, \qquad \forall\, P_n \in \pi_n, \tag{3.2}$$

and

$$\lim_{n\to\infty} [E_n(w)]^{1/n} = \frac{(1-\theta)(1+\theta)^{(1+\theta)/(1-\theta)}}{4^{1/(1-\theta)}} = \exp(F(\mathcal{S})), \tag{3.3}$$

which gives the known results of Kemperman and Lorentz [5] and Saff and Varga [13]. The generalized energy problem in this special case was studied in the dissertation of H. Stahl [16]. The corresponding extremal measure is

$$\mu_w(x)\,dx = \frac{1}{(1-\theta)\pi x}\sqrt{\frac{x-\theta^2}{1-x}}\,dx, \qquad \operatorname{supp}(\mu_w) = [\theta^2, 1] = \mathcal{S}, \tag{3.4}$$

which, by Corollary 2.5, gives the limiting distribution of the zeros of the extremal polynomials $T_n(x;w)$.

Example 2. Jacobi weights. Let $w(x) = (1-x)^{\lambda_1}(1+x)^{\lambda_2}$ with $\Sigma = [-1, 1]$ and $\lambda_1, \lambda_2 > 0$. Then $Q(x) = \log[1/w(x)]$ is convex on $(-1, 1) = \Sigma - Z$ and straightforward calculations show that the associated F-functional for intervals $[a, b] \subset (-1, 1)$ is maximized for $[a, b] = [\sin(\phi_1 - \phi_2), \sin(\phi_1 + \phi_2)]$, where

$$\sin\phi_1 := \frac{\lambda_1+\lambda_2}{1+\lambda_1+\lambda_2}\quad\left(0 < \phi_1 < \frac{\pi}{2}\right), \qquad \cos\phi_2 := \frac{\lambda_2-\lambda_1}{1+\lambda_1+\lambda_2}\quad(0 < \phi_2 < \pi).$$

Hence $\mathcal{S} = [\sin(\phi_1 - \phi_2), \sin(\phi_1 + \phi_2)]$, which was first proved in [6]. Further results of [6] and [12] likewise follow from the theorems of Section 2.

Example 3. Exponential weights on $\mathbf{R}$. Let $W(x) = \exp[-q(x)]$, where $q(x)$ is even and convex on $\mathbf{R}$ and $q(x)/\ln x \to \infty$ as $x \to \infty$. To analyze the extremal problems

$$\mathcal{E}_n(W) := \inf\{\|W(x)\{x^n - p_{n-1}(x)\}\|_{\mathbf{R}}: p_{n-1} \in \pi_{n-1}\}, \qquad n = 1, 2, \ldots, \tag{3.5}$$

and the corresponding extremal polynomials $\mathcal{T}_n(x;W) = x^n + \ldots \in \pi_n$ satisfying

$$\mathcal{E}_n(W) = \|W(x)\mathcal{T}_n(x;W)\|_{\mathbf{R}}, \qquad n = 1, 2, \ldots, \tag{3.6}$$

we proceed as follows. Let $W_n(x) := \exp[-q(x)/n]$. Since q is even, the F-functional, $F_n(K)$, associated with the weight W_n need be considered only for intervals of the form $[-a, a]$, $a > 0$. It is easily verified that

$$F_n([-a, a]) = \log\frac{a}{2} - \frac{2}{n\pi}\int_0^1 \frac{q(ax)}{\sqrt{1-x^2}}\,dx,$$

which is maximized at a root $a = a_n$ of the equation

$$n = \frac{2}{\pi}\int_0^1 \frac{axq'(ax)}{\sqrt{1-x^2}}\,dx. \tag{3.7}$$

By Theorem 2.2(c), for any $P_n \in \pi_n$,

$$\|W(x)P_n(x)\|_{\mathbf{R}} = \|[W_n(x)]^n P_n(x)\|_{\mathbf{R}} = \|[W_n(x)]^n P_n(x)\|_{[-a_n,a_n]} = \|W(x)P_n(x)\|_{[-a_n,a_n]}. \tag{3.8}$$

Now let $w_n(x) := W_n(a_n x)$ with $\Sigma = \mathrm{supp}(w_n) = [-1, 1]$. Then from (3.8) it follows that

$$\mathcal{E}_n(W) = a_n^n E_n(w_n), \qquad \mathcal{T}_n(a_n x; W) = a_n^n T_n(x; w_n). \tag{3.9}$$

If the weights w_n converge uniformly to an admissible weight w on $[-1, 1]$, it can be shown that the asymptotic behaviors (as $n \to \infty$) of $[E_n(w_n)]^{1/n}$ and the zeros of $T_n(x;w_n)$ are the same as for $[E_n(w)]^{1/n}$ and the zeros of $T_n(x;w)$. (A detailed proof of this fact will appear, along with further applications, in a future paper.)

In particular, if $q(x) = e^{|x|}$, then an analysis of (3.7) via Laplace's method gives $a_n \sim \log n$ and $w_n(x) \to 1$ uniformly on $[-1, 1]$. From (3.9) and the asymptotic properties for the weight $w(x) \equiv 1$ on $\Sigma = [-1, 1]$, we find

$$\lim_{n\to\infty} \frac{1}{\log n}[\mathcal{E}_n(W)]^{1/n} = \frac{1}{2}, \qquad W(x) = \exp(-\exp|x|). \tag{3.10}$$

Furthermore, the limiting distribution of the *contracted zeros of* $\mathcal{T}_n$ [i.e., the zeros of $\mathcal{T}_n(a_n x; W)$] is the arcsine distribution. (For orthonormal polynomials associated with this weight, these properties were obtained by Erdös [1].) In fact, the arcsine distribution arises whenever q satisfies

$$\lim_{n\to\infty} \frac{q(a_n)}{n} = 0.$$

If $q(x) = |x|^\alpha$, $\alpha \geq 1$, then it is easily verified from (3.7) that

$$a_n = (n/\lambda_\alpha)^{1/\alpha}, \qquad \lambda_\alpha := \frac{\Gamma(\alpha)}{2^{\alpha-2}\{\Gamma(\alpha/2)\}^2}.$$

In this case, $w_n(x) = \exp(-|x|^\alpha/\lambda_\alpha)$, $n = 1, 2, \ldots$, and we deduce the results of [10], namely,

$$\lim_{n\to\infty} n^{-1/\alpha}[\mathcal{E}_n(W)]^{1/n} = \frac{1}{2}\left(\frac{1}{e\lambda_\alpha}\right)^{1/\alpha}, \qquad W(x) = \exp(-|x|^\alpha), \tag{3.11}$$

and the limiting distribution of the contracted zeros of the extremal polynomials is given by the Ullman distribution

$$v(\alpha;t) := \frac{\alpha}{\pi}\int_{|t|}^1 \frac{y^{\alpha-1}}{\sqrt{y^2-t^2}}\,dy, \qquad -1 \leq t \leq 1. \tag{3.12}$$

More generally, the limiting distribution (3.12) arises whenever q satisfies the additional property

$$\lim_{R\to\infty} \frac{q'(Rx)}{q'(R)} = g(x), \qquad \forall\, x\in[0, \infty),$$

where $g(x)$ is continuous on $[0, \infty)$.

The technique of Example 3 can also be applied to Laguerre weights of the form $W(x) = x^{\beta} \exp[-q(x)]$, $x\in[0, \infty)$, $\beta > 0$, with $q(x)$ convex on $[0, \infty)$ and $q(x)/\ln x \to \infty$ as $x\to\infty$. This leads to the results of [9].

4. Proofs

We begin with the proof of Theorem 2.3, after which we establish Theorems 2.1, 2.2, and 2.4 and Corollary 2.5. Several portions of the proofs given below are standard arguments from classical potential theory. For the reader's convenience, we include most of the details and especially emphasize the new features.

Proof of Theorem 2.3(a). First note that, from properties (ii) and (iv) of Definition 1.1, the function

(4.1) $$G(x,t) := \log|x-t| - Q(x) - Q(t) = \log[w(x)w(t)|x-t|]$$

is bounded from above on $\Sigma\times\Sigma$. Hence the integral $I_w[\mu]$ in (2.6) is well-defined for every $\mu\in\mathcal{M}(\Sigma)$ and $V_w < \infty$. Next, from properties (i) and (iii) of Definition 1.1, we have $C(\Sigma - Z) > 0$. Thus, there exists a compact set $K\subset\Sigma - Z$ such that $C(K) > 0$. For the unit equilibrium measure ν_K associated with K we therefore have

$$\iint \log|x-t|\, d\nu_K(x)\, d\nu_K(t) > -\infty.$$

Furthermore, by continuity, w has a positive minimum on K and so

$$\iint \log[w(x)w(t)]\, d\nu_K(x)\, d\nu_K(t) > -\infty.$$

Consequently $I_w[\nu_K] > -\infty$ and, since $\nu_K\in\mathcal{M}(\Sigma)$, it follows that $V_w > -\infty$. ■

Proof of Theorem 2.3(b). We first show that, for the extremal problem (2.7), it is sufficient to consider only unit measures that are supported on $\Sigma_N := \Sigma\cap[-N, N]$, where N is a suitably chosen positive integer. This is trivial if Σ is compact, so assume Σ is unbounded.

Clearly, in (2.7), we need consider only measures $\mu\in\mathcal{M}(\Sigma)$ for which

(4.2) $$\lambda := V_w - 1 < I_w[\mu].$$

Since, by property (iv) of Definition 1.1, we have

(4.3) $$G(x, t)\to -\infty \quad \text{as} \quad \max(|x|, |t|)\to\infty,$$

then there exists a positive integer N such that

(4.4) $$G(x, t) - \lambda < 0, \quad \forall\ (x, t) \in \mathbf{R}^2 - [-N, N] \times [-N, N].$$

Now, if $\mu \in \mathcal{M}(\Sigma)$ satisfies (4.2) and

$$\operatorname{supp}(\mu) \cap (\mathbf{R} - \Sigma_N) \neq \emptyset,$$

then, from (4.4),

(4.5) $$0 < I_w[\mu] - \lambda = \iint [G - \lambda]\, d(\mu \times \mu) =$$

$$= \iint_{\Sigma_N \times \Sigma_N} [G - \lambda]\, d(\mu \times \mu) + \iint_{\mathbf{R}^2 - \Sigma_N \times \Sigma_N} [G - \lambda]\, d(\mu \times \mu)$$

$$< \mu(\Sigma_N)^2 \{I_w[\hat{\mu}] - \lambda\} \leq I_w[\hat{\mu}] - \lambda,$$

where $\hat{\mu} \in \mathcal{M}(\Sigma_N)$ is defined by $\hat{\mu} := \mu/\mu(\Sigma_N)$. (Necessarily, $\mu(\Sigma_N) > 0$.) But (4.5) implies that $I_w[\mu] < I_w[\hat{\mu}]$. Hence the equation $I_w[\mu] = V_w$ can hold only if $\operatorname{supp}(\mu) \subset \Sigma_N$ and, moreover,

(4.6) $$V_w = \sup \{I_w[\mu] : \mu \in \mathcal{M}(\Sigma_N)\}, \qquad \Sigma_N = \Sigma \cap [-N, N].$$

The proof of part (b) of Theorem 2.3 is now essentially the same as the proof of Theorem 16.4.3 in [4]. The existence of μ_w follows from the fact that $\mathcal{M}(\Sigma_N)$ is compact in the weak-star topology. We also note that, since μ_w has compact support and V_w is finite, then μ_w *has finite logarithmic energy*. To prove uniqueness, suppose that $\bar{\mu} \in \mathcal{M}(\Sigma_N)$ also satisfies $I_w[\bar{\mu}] = V_w$. Then we have

$$I_w\left[\frac{\mu_w - \bar{\mu}}{2}\right] = \iint \log |x - t|\, d\left(\frac{\mu_w - \bar{\mu}}{2}\right)(x)\, d\left(\frac{\mu_w - \bar{\mu}}{2}\right)(t),$$

which is the negative of the ordinary logarithmic energy integral of the compactly supported signed measure $(\mu_w - \bar{\mu})/2$ with total mass zero. Thus, from Theorem 16.4.2 of [4],

(4.7) $$I_w\left[\frac{\mu_w - \bar{\mu}}{2}\right] \leq 0,$$

with equality if and only if $\bar{\mu} = \mu_w$. But

(4.8) $$I_w\left[\frac{\mu_w + \bar{\mu}}{2}\right] + I_w\left[\frac{\mu_w - \bar{\mu}}{2}\right] = \frac{1}{2}(I_w[\mu_w] + I_w[\bar{\mu}]) = V_w,$$

and since $I_w[(\mu_w + \bar{\mu})/2] \leq V_w$, we have from (4.7) that $I_w[(\mu_w - \bar{\mu})/2] = 0$. Thus $\bar{\mu} = \mu_w$. ∎

Proof of Theorem 2.3(c). From the proof of part (b), we know that $\mathcal{S}_w = \operatorname{supp}(\mu_w)$ is contained in the compact set Σ_N. Furthermore, as μ_w has finite energy, then $C(\mathcal{S}_w) > 0$. Thus it only remains to show that $\mathcal{S}_w \cap Z_N = \emptyset$, where $Z_N := Z \cap [-N, N]$. For this purpose, define

(4.9) $$\mathcal{H} := \{x \in \Sigma_N : G(x, t) < V_w,\ \forall\ t \in \Sigma_N\}.$$

Then $\mathcal{H}$ is open relative to Σ_N and $Z_N \subset \mathcal{H}$. Now suppose that $\mu_w(\mathcal{H}) > 0$. Then, from (2.8),

$$0 = \iint_{\Sigma_N \times \Sigma_N} [G - V_w] \, d(\mu_w \times \mu_w) = \tag{4.10}$$

$$= \iint_{(\Sigma_N - \mathcal{H}) \times (\Sigma_N - \mathcal{H})} [G - V_w] \, d(\mu_w \times \mu_w) + \iint_{\Lambda_N} [G - V_w] \, d(\mu_w \times \mu_w),$$

where $\Lambda_N := \Sigma_N \times \Sigma_N - (\Sigma_N - \mathcal{H}) \times (\Sigma_N - \mathcal{H})$. By symmetry, $G(x, t) < V_w$, $\forall (x, t) \in \Lambda_N$. Furthermore, $(\mu_w \times \mu_w)(\Lambda_N) > 0$. Hence (4.10) implies

$$0 < [\mu_w(\Sigma_N - \mathcal{H})]^2 \{I_w[\hat{\mu}] - V_w\}, \tag{4.11}$$

where $\hat{\mu} \in \mathcal{M}(\Sigma_N - \mathcal{H})$ is defined by $\hat{\mu} := \mu_w / \mu_w(\Sigma_N - \mathcal{H})$. (Necessarily, $\mu_w(\Sigma_N - \mathcal{H}) > 0$.) But (4.11) implies that $V_w < I_w[\hat{\mu}]$, which is absurd. Thus $\mu_w(\mathcal{H}) = 0$, and so $\mathcal{S}_w \cap \mathcal{H} = \emptyset$. As $Z_N \subset \mathcal{H}$, then

$$\mathcal{S}_w \subset \Sigma_N - Z_N \subset \Sigma - Z.$$

This completes the proof of part (c). ■

Proof of Theorem 2.3(d). Since this is similar to the classical proof of Frostman for the potential of an equilibrium distribution (cf. Theorem III.12 in [17]), we merely sketch the argument. Define

$$\mathcal{U}_w(x) := \int \log |x - t| \, d\mu_w(t) - Q(x), \qquad x \in \Sigma. \tag{4.12}$$

Then $\mathcal{U}_w$ is an upper semi-continuous extended real-valued function on Σ. Consequently, the set $\{x \in \Sigma\colon \mathcal{U}_w(x) \geq \alpha\}$ is closed for each $\alpha \in \mathbf{R}$. Now suppose, to the contrary, that the set

$$A := \left\{x \in \Sigma\colon \mathcal{U}_w(x) > V_w + \int Q \, d\mu_w\right\} \tag{4.13}$$

has positive capacity. Then there exists a large positive integer n_0 such that the compact set

$$E_1 := \left\{x \in \Sigma \cap [-n_0, n_0]\colon \mathcal{U}_w(x) \geq V_w + \int Q \, d\mu_w + \frac{1}{n_0}\right\} \tag{4.14}$$

also has positive capacity. On the other hand, since

$$\int \mathcal{U}_w \, d\mu_w = \iint \log |x - t| \, d\mu_w(x) \, d\mu_w(t) - \int Q \, d\mu_w \tag{4.15}$$

$$= V_w + \int Q \, d\mu_w,$$

there exists a compact set $E_2 \subset \mathcal{S}_w$, disjoint from E_1, such that

$$\mathcal{U}_w(x) < V_w + \int Q\, d\mu_w + \frac{1}{2n_0}, \qquad \forall\, x \in E_2, \tag{4.16}$$

and such that

$$m := \mu_w(E_2) > 0.$$

Now let σ be a positive measure supported on E_1 such that $I_w[\sigma]$ is finite and $\sigma(E_1) = m$. The existence of σ follows from the facts that $C(E_1) > 0$ and $E_1 \subset \Sigma - Z$ [see the proof of Theorem 2.3(a)]. But then, for the signed measure σ_1 on Σ defined by

$$\sigma_1 := \sigma \text{ on } E_1, \quad \sigma_1 := -\mu_w \text{ on } E_2, \quad \sigma_1 := 0 \text{ elsewhere}, \tag{4.17}$$

it can be verified that for $\eta > 0$ sufficiently small, the measure $\mu_w + \eta\sigma_1 \in \mathcal{M}(\Sigma)$ satisfies

$$I_w[\mu_w + \eta\sigma_1] > V_w.$$

As this contradicts the definition of V_w, the set A of (4.13) has capacity zero. Thus (2.9) follows. ■

Proof of Theorem 2.3(e). Again, this follows Frostman's argument. Let $x_0 \in \mathcal{S}_w$ and suppose that

$$\mathcal{U}_w(x_0) < V_w + \int Q\, d\mu_w, \tag{4.18}$$

where $\mathcal{U}_w$ is given by (4.12). From the upper semi-continuity of $\mathcal{U}_w$, there exists an open interval $\mathcal{N}(x_0)$ about x_0 such

$$\mathcal{U}_w(x) < V_w + \int Q\, d\mu_w - \epsilon, \qquad \forall\, x \in E := \mathcal{N}(x_0) \cap \mathcal{S}_w, \tag{4.19}$$

where ϵ is some positive number. Now since μ_w has finite logarithmic energy, the inequality (2.9) holds μ_w-almost everywhere (cf. Theorem III.7 in [17]). Hence, from (4.15), (4.19), and (2.9), we have

$$\begin{aligned} V_w + \int Q\, d\mu_w &= \int \mathcal{U}_w\, d\mu_w = \int_E \mathcal{U}_w\, d\mu_w + \int_{\mathcal{S}_w - E} \mathcal{U}_w\, d\mu_w \\ &\le \mu_w(E)\left\{V_w + \int Q\, d\mu_w - \epsilon\right\} + [1 - \mu_w(E)]\left\{V_w + \int Q\, d\mu_w\right\}, \end{aligned} \tag{4.20}$$

which implies that $\mu_w(E) = 0$. But this is absurd, because E is a nonempty open subset of the support $\mathcal{S}_w$. Hence the assertion of (2.10) follows. ■

Remark. From (2.9) and (2.10) of Theorem 2.3, we have

$$\int \log|x - t|\, d\mu_w(t) = Q(x) + V_w + \int Q\, d\mu_w, \qquad \text{q.e. on } \mathcal{S}_w. \tag{4.21}$$

Proof of Theorem 2.3(f). Let ν_w be the unit equilibrium measure for $\mathcal{S}_w$. Then, since (4.21) holds ν_w-almost everywhere, integration with respect to ν_w yields

$$\text{(4.22)} \qquad \int Q\,d\nu_w + V_w + \int Q\,d\mu_w = \iint \log|x-t|\,d\mu_w(t)\,d\nu_w(x) = \iint \log|x-t|\,d\nu_w(x)\,d\mu_w(t),$$

where the change in the order of integration can be justified by Tonelli's theorem. Using (1.1), we therefore find

$$\int Q\,d\nu_w + V_w + \int Q\,d\mu_w = \log C(\mathcal{S}_w),$$

and so [cf. (1.3)]

$$F(\mathcal{S}_w) = \log C(\mathcal{S}_w) - \int Q\,d\nu_w = V_w + \int Q\,d\mu_w.$$

■

To establish part (g) of Theorem 2.3 it is convenient to have for reference the following generalized maximum principle for potentials.

Lemma 4.1. *Assume that μ is a positive unit measure with finite logarithmic energy and compact support $S \subset \mathbf{R}$. If $P_n \in \pi_n$ $(n \geq 1)$ and, for some constant L,*

$$\text{(4.23)} \qquad \frac{1}{n}\log|P_n(z)| \leq L + \int \log|z-t|\,d\mu(t), \qquad \textit{q.e. on } S,$$

then inequality (4.23) *holds for all $z \in \mathbf{C}$.*

The proof of Lemma 4.1 shall be omitted because it relies on standard arguments similar to those of [7, Theorem 1.10] and [3, Theorem 11.7].

Proof of Theorem 2.3(g). Assumption (2.12) is equivalent to

$$\frac{1}{n}\log|P_n(x)| \leq \frac{1}{n}\log M + Q(x), \qquad \text{q.e. on } \mathcal{S}_w.$$

Hence, by (2.10) and (2.11) of Theorem 2.3,

$$\text{(4.24)} \qquad \frac{1}{n}\log|P_n(x)| \leq \frac{1}{n}\log M + \int \log|x-t|\,d\mu_w(t) - F(\mathcal{S}_w), \quad \text{q.e. on } \mathcal{S}_w.$$

Thus, from Lemma 4.1, inequality (4.24) holds throughout $\mathbf{C}$, which is the assertion of (2.13). ■

The proof of Theorem 2.3(h) is included in the following proof of Theorem 2.1. In view of part (c) of Theorem 2.3, we need to show that properties (a), (b), (c), and (d) of Theorem 2.1 hold for $\mathcal{S} = \mathcal{S}_w$.

Proof of Theorem 2.1(a). Suppose $K \subset \Sigma - Z$ is a compact set with $C(K) > 0$. Let ν_K be the unit equilibrium measure associated with K and let $S_K := \operatorname{supp}(\eta_K)$. Since inequality (2.9) holds ν_K-almost everywhere and (cf. [17, Theorem III.12])

$$(4.25) \qquad U(t;\nu_K) := \int \log |x-t| \, d\nu_K(x) \geq \log C(K), \qquad \forall\, t \in \mathbf{C},$$

we obtain, on integrating (2.9) with respect to ν_K,

$$(4.26) \qquad \int Q \, d\nu_K + V_w + \int Q \, d\mu_w \geq \int_{S_K} \int_{\mathcal{S}_w} \log |x-t| \, d\mu_w(t) \, d\nu_K(x)$$
$$= \int_{\mathcal{S}_w} \int_{S_K} \log |x-t| \, d\nu_K(x) \, d\mu_w(t)$$
$$\geq \log C(K).$$

Thus, from part (f) of Theorem 2.3,

$$F(\mathcal{S}_w) = V_w + \int Q \, d\mu_w \geq \log C(K) - \int Q \, d\nu_K = F(K),$$

which establishes (2.1). ∎

Proof of Theorem 2.1(b). Assume that $F(K) = F(\mathcal{S}_w)$, where $K \subset \Sigma - Z$ is compact and $C(K) > 0$. Then equality must hold throughout (4.26). Thus

$$(4.27) \qquad \int_{\mathcal{S}_w} \int_{S_K} \log |x-t| \, d\nu_K(x) \, d\mu_w(t) = \log C(K)$$

and

$$(4.28) \qquad Q(x) + V_w + \int Q \, d\mu_w = \int \log |x-t| \, d\mu_w(t), \qquad \nu_K\text{-a.e.}$$

To prove that $\mathcal{S}_w \subset K$ we shall utilize only equation (4.27); equation (4.28) is stated here for later use in the proof of Theorem 2.2.

Suppose there exists a point $x_0 \in \mathcal{S}_w$ such that $x_0 \notin S_K$. Then there is an open disk $\mathcal{D}$ centered at x_0 for which $\mathcal{D} \cap S_K = \emptyset$. Since $\mathcal{D}$ is contained in $\mathbf{C} - S_K$ and $\mathbf{C} - S_K$ is an unbounded open connected set, then (4.25) and the maximum principle for harmonic functions imply

$$(4.29) \qquad U(t;\nu_K) > \log C(K), \qquad \forall\, t \in \mathcal{D}.$$

Furthermore, as x_0 belongs to the support of μ_w, we have $\mu_w(\mathcal{D}) > 0$. Thus, from (4.29) and (4.25),

$$\int_{\mathcal{S}_w} U(t;\nu_K)\, d\mu_w(t) = \int_{\mathcal{D}} U(t;\nu_K)\, d\mu_w(t) + \int_{\mathcal{S}_w - \mathcal{D}} U(t;\nu_K)\, d\mu_w(t)$$

$$> \mu_w(\mathcal{D}) \log C(K) + [1 - \mu_w(\mathcal{D})] \log C(K)$$

$$= \log C(K),$$

which contradicts (4.27). Hence $\mathcal{S}_w \subset S_K$ and since $S_K \subset K$, the assertion of Theorem 2.1(b) is proved for $\mathcal{S} = \mathcal{S}_w$. ■

Proof of Theorem 2.1(c). Assuming that (2.2) holds q.e. on $\mathcal{S} = \mathcal{S}_w$, we get, from (2.13) and Theorem 2.3(d),

$$|P_n(x)| \leq M \exp [nQ(x)], \qquad \text{q.e. on } \Sigma,$$

which is equivalent to saying that (2.2) holds q.e. on Σ. ■

Proof of Theorem 2.1(d). Define

(4.30) $\hat{E}_n(w) := \inf \{\|[w(x)]^n\{x^n - p_{n-1}(x)\}\|_{\mathcal{S}} : p_{n-1} \in \pi_{n-1}\}, \qquad n = 1, 2, \ldots,$

and let $\hat{T}_n(x) = \hat{T}_n(x;w) = x^n + \ldots \in \pi_n$ satisfy

(4.31) $$\hat{E}_n(w) = \|[w(x)]^n \hat{T}_n(x)\|_{\mathcal{S}}.$$

From part (g) of Theorem 2.3 [with $P_n = \hat{T}_n$ and $M = \hat{E}_n(w)$], we obtain, on letting $z \to \infty$ in (2.13),

$$1 = \lim_{z\to\infty} \frac{|\hat{T}_n(z)|}{\exp [n \int \log |z-t|\, d\mu_w(t)]} \leq \hat{E}_n(w) \exp [-nF(\mathcal{S})].$$

Thus,

(4.32) $$E_n(w) \geq \hat{E}_n(w) \geq \exp [nF(\mathcal{S})], \qquad \forall\, n = 1, 2, \ldots.$$

■

Proof of Theorem 2.1 (c′). It follows from (2.13) that

$$\|[w(x)]^n P_n(x)\|_\Sigma = \|[w(x)]^n P_n(x)\|_{\mathcal{S} \cup \mathcal{A}},$$

where

$$\mathcal{A} := \{x \in \Sigma : \int \log |x - t|\, d\mu_w(t) > Q(x) + F(\mathcal{S})\}.$$

But, it is easily seen that $\mathcal{A} - \mathcal{S}$ is contained in the set of irregular points of Σ. Hence, from the assumption on Σ, we have $\mathcal{A} - \mathcal{S} = \emptyset$ and so (2.4) follows. ■

Before giving the proof of Theorem 2.2, we establish

Theorem 4.2. *Let w be an admissible weight function with support* Σ *and suppose that the set* $\mathcal{S}$ *of Theorem 2.1 is regular in the sense that the complement* $\mathbf{C} - \mathcal{S}$ *possesses a classical Green's function* $\mathcal{G}(z)$ *with pole at* ∞. *Then*

$$\lim_{n\to\infty} [\hat{E}_n(w)]^{1/n} = \exp[F(\mathcal{S})], \tag{4.33}$$

where $\hat{E}_n(w)$ *is defined in* (4.30).

By the classical Green's function $\mathcal{G}(z)$ with pole at ∞ we mean that $\mathcal{G}(z)$ is harmonic in $\mathbf{C} - \mathcal{S}$, $\mathcal{G}(z) \to 0$ as $z \to \mathcal{S}$, $z \in \mathbf{C} - \mathcal{S}$, and $\mathcal{G}(z) - \ln|z|$ is harmonic at ∞.

In the proof of Theorem 4.2 we shall appeal to

Lemma 4.3. *With the assumptions of Theorem 4.2, the potential*

$$u(z) := \int \log|z-t|\, d\mu_w(t), \tag{4.34}$$

where μ_w *is the extremal measure of Theorem 2.3, is continuous on the whole plane* $\mathbf{C}$.

Proof of Lemma 4.3. First note that since $\operatorname{supp}(\mu_w) = \mathcal{S} \subset \Sigma - Z$, then $Q(x)$ is continuous on $\mathcal{S}$. Hence, since $\mathcal{S}$ is regular, there exists a function $h(z)$ continuous on $\mathbf{C}^* = \mathbf{C} \cup \{\infty\}$ and harmonic in $\mathbf{C}^* - \mathcal{S}$ such that

$$h(x) = Q(x) + F(\mathcal{S}), \qquad \forall\, x \in \mathcal{S}. \tag{4.35}$$

We claim that

$$u(z) = h(z) + \mathcal{G}(z), \qquad \forall\, z \in \mathbf{C}, \tag{4.36}$$

from which Lemma 4.3 follows.

To establish (4.36), note that the function

$$k(z) := u(z) - h(z) - \mathcal{G}(z) \tag{4.37}$$

is harmonic in $\mathbf{C}^* - \mathcal{S}$ (including $z = \infty$). That $k(z)$ is also bounded in $\mathbf{C}^* - \mathcal{S}$ can be shown as follows. If $D_R := \{z\colon |z| \le R\}$ is a disk that contains $\mathcal{S}$ in its interior, then clearly $k(z)$ is bounded in the complement of D_R. Moreover, on $D_R - \mathcal{S}$, the functions $h(z)$ and $\mathcal{G}(z)$ are bounded and the potential $u(z)$ is bounded from above. The potential $u(z)$ is also bounded below on $D_R - \mathcal{S}$, since inequality (2.10) and equation (2.11) imply that for all $z \in \mathcal{S}$,

$$u(z) \ge \min_{x \in \mathcal{S}} \{Q(x) + F(\mathcal{S})\} > -\infty, \tag{4.38}$$

and hence, by the maximum principle for potentials (cf. [17, Theorem III.1]), (4.38) holds for all $z \in \mathbf{C}$. Thus $k(z)$ is a bounded harmonic function in $\mathbf{C}^* - \mathcal{S}$.

Now consider a point $\xi \in \mathcal{S}$ such that

$$u(\xi) = Q(\xi) + F(\mathcal{S}). \tag{4.39}$$

As $u(z)$ is upper semi-continuous on $\mathbf{C}$, then

$$\limsup_{z\to\xi} u(z) \le u(\xi).$$

Also, from (4.39) and part (e) of Theorem 2.3,

$$\liminf_{\substack{z\to\xi \\ z\in\mathcal{S}}} u(z) \geq u(\xi).$$

Hence $u(z)$ restricted to the set $\mathcal{S}$ is continuous at $z = \xi$, and therefore (cf. [17, Theorem III.2]) $u(z)$ considered as a function on $\mathbf{C}$ is continuous at $z = \xi$. Thus, from (4.35) and (4.39),

$$\lim_{z\to\xi} k(z) = 0. \tag{4.40}$$

But, since (4.39) holds q.e. for ξ on $\mathcal{S}$[cf. (4.21) and (2.11)], so does (4.40), and the maximum principle for harmonic functions (cf. [17, Theorem III.28]) implies

$$k(z) = 0, \qquad \forall\, z\in\mathbf{C}^* - \mathcal{S}. \tag{4.41}$$

Thus (4.36) holds for all $z\notin\mathcal{S}$.

To show that (4.36) also holds on $\mathcal{S}$, let $x_0\in\mathcal{S}$ and $\epsilon > 0$. Since $h + \mathcal{G}$ is continuous at x_0, there exists a disk $D_\delta(x_0) := \{z\colon |z - x_0| \leq \delta\}$ such that

$$h(z) + \mathcal{G}(z) < h(x_0) + \mathcal{G}(x_0) + \epsilon = h(x_0) + \epsilon, \qquad \forall\, z\in D_\delta(x_0).$$

Then, by the Mean Value Inequality for subharmonic functions and the fact that (4.36) holds for all $z\in\mathbf{C} - \mathcal{S}$, we have

$$\begin{aligned} u(x_0) &\leq \frac{1}{\pi\delta^2}\iint_{D_\delta(x_0)} u(z)\,dx\,dy = \frac{1}{\pi\delta^2}\iint_{D_\delta(x_0)} [h(z) + \mathcal{G}(z)]\,dx\,dy \\ &\leq h(x_0) + \epsilon. \end{aligned}$$

Letting $\epsilon \to 0$, we find $u(x_0) \leq h(x_0)$. Finally, from (2.10), we also have $u(x_0) \geq h(x_0)$ and so $u(x_0) = h(x_0)$ for all $x_0\in\mathcal{S}$. Thus (4.36) holds for all $z\in\mathbf{C}$. ■

Remark. From the above lemma it follows that if $\mathcal{S}$ is regular then

$$u(x) = Q(x) + F(\mathcal{S}), \qquad \forall\, x\in\mathcal{S}. \tag{4.42}$$

Proof of Theorem 4.2. Lemma 4.3 implies, in particular, that the potential $u(x)$ is a continuous function on $\mathbf{R}$. Hence, by Lemma 4.4 of [10], for each $\epsilon > 0$ there exist an integer n_ϵ and a sequence of monic polynomials $p_n(x) = x^n + \ldots \in\pi_n$, $n = n_\epsilon$, $n_\epsilon + 1, \ldots$, such that

$$\frac{1}{n}\log |p_n(x)| \leq \epsilon + u(x) = \epsilon + Q(x) + F(\mathcal{S}), \qquad \forall\, x\in\mathcal{S}, \qquad n \geq n_\epsilon.$$

Thus

$$w(x)|p_n(x)|^{1/n} \leq \exp\,[\epsilon + F(\mathcal{S})], \qquad \forall\, x\in\mathcal{S}, \qquad n \geq n_\epsilon,$$

which implies

$$\limsup_{n\to\infty}\, [\hat{E}_n(w)]^{1/n} \leq \limsup_{n\to\infty}\, \|[w(x)]^n p_n(x)\|_{\mathcal{S}}^{1/n} \leq \exp\,[\epsilon + F(\mathcal{S})].$$

As $\epsilon > 0$ is arbitrary, we obtain

$$\limsup_{n\to\infty} [\hat{E}_n(w)]^{1/n} \le \exp[F(\mathcal{S})]$$

which, together with inequalities (4.32), yields (4.33). ■

If, in addition to the hypotheses of Theorem 4.2, the set $\Sigma - Z$ has the property that every set $A \subset \Sigma - Z$ satisfying $C(\Sigma - Z - A) = 0$ is dense in $\Sigma - Z$, then $E_n(w) = \hat{E}_n(w)$ for every n, and so

(4.43) $$\lim_{n\to\infty} [E_n(w)]^{1/n} = \exp [F(\mathcal{S})].$$

This is the situation that occurs in Theorem 2.2. In giving the proof of Theorem 2.2, it is more convenient to prove part (b) first.

Proof of Theorem 2.2(b). First assume that $K \subset \Sigma - Z$ is compact, $C(K) > 0$, and $F(K) = F(\mathcal{S})$. [Recall that $\mathcal{S} = \mathcal{S}_w = \text{supp}(\mu_w)$.] Our goal is to show that $C(K - \mathcal{S}) = 0$. [We have already proved in Theorem 2.1(b) that $\mathcal{S} \subset K$.] As in the proof of Theorem 2.1(a), let $S_K := \text{supp}(\nu_K)$, where ν_K is the unit equilibrium measure for K. It follows immediately from the definition of the F-functional in (1.3) that $F(S_K) = F(K)$, and so $F(S_K) = F(\mathcal{S})$.

Now suppose $S_K - \mathcal{S} \ne \emptyset$. Then since $\mathcal{S} \subset S_K$ and $\mathcal{S}$ is compact, we have $\nu_K(S_K - \mathcal{S}) > 0$. Consequently, there exists a point $x_0 \in S_K - \mathcal{S}$ such that equation (4.28) holds at x_0 and x_0 lies in the interior of some (nondegenerate) component of $\Sigma - Z$. Thus, there exists an open interval J containing x_0 such that $J \subset \Sigma - Z$ and $J \cap \mathcal{S} = \emptyset$. Since $x_0 \in S_K$, then $\nu_K(J) > 0$ and so there exists a point $x_1 \in J$, $x_1 \ne x_0$, such that (4.28) holds at x_1. Let $\hat{J}$ be the open interval with endpoints x_0 and x_1. Since the function

(4.44) $$\mathcal{U}_w(x) := \int \log |x - t| \, d\mu_w(t) - Q(x)$$

is strictly concave on $\hat{J}$ and equals $F(\mathcal{S})$ at the endpoints, then

(4.45) $$\mathcal{U}_w(x) > F(\mathcal{S}), \qquad \forall\, x \in \hat{J}.$$

But $C(\hat{J}) > 0$, and so (4.45) contradicts Theorem 2.3(d). Hence $S_K - \mathcal{S} = \emptyset$, which implies $S_K = \mathcal{S}$. Now, since $K \subset \mathbf{R}$, then $C(K - S_K) = 0$ and, consequently, $C(K - \mathcal{S}) = 0$. We have thus established the "only if" assertion of part (b). The "if" assertion is trivial. ■

Proof of Theorem 2.2(a). First we prove that there exists a compact set $\mathcal{S}^*$ satisfying $F(\mathcal{S}^*) = F(\mathcal{S})$ and that $\mathcal{S}^*$ is the union of disjoint closed intervals, with at most one interval in each component of $\Sigma - Z$. Let J_i be a component of $\Sigma - Z$ such that $J_i \cap \mathcal{S} \ne \emptyset$, and set

(4.46) $$x_1^{(i)} := \inf \{x \in J_i \cap \mathcal{S}\}, \qquad x_2^{(i)} := \sup \{x \in J_i \cap \mathcal{S}\}.$$

Then, since $\mathcal{S} \subset \Sigma - Z$ is compact, $x_1^{(i)}, x_2^{(i)} \in J_i \cap \mathcal{S}$. Moreover, as $\mathcal{S}$ cannot have isolated points, then $x_1^{(i)} < x_2^{(i)}$. Put

$$\mathcal{S}_i^* := [x_1^{(i)}, x_2^{(i)}].$$

We claim that

$$(4.47) \qquad \mathcal{U}_w(x) = \int \log|x-t|\, d\mu_w(t) - Q(x) \geq F(\mathcal{S}), \qquad \forall\, x \in \mathcal{S}_i^*.$$

Indeed, if $x \in \mathcal{S} \cap \mathcal{S}_i^*$, then (4.47) holds by (2.10) and (2.11). So assume $x \in \mathcal{S}_i^* - \mathcal{S}$ and define

$$(4.48) \qquad y_1 := \inf\{y\colon (y, x] \cap \mathcal{S} = \emptyset\}, \qquad y_2 := \sup\{y\colon [x, y) \cap \mathcal{S} = \emptyset\}.$$

Then $y_1, y_2 \in \mathcal{S}$ and $(y_1, y_2) \cap \mathcal{S} = \emptyset$. Since (4.47) holds at y_1 and y_2 and the function $\mathcal{U}_w(x)$ is concave on (y_1, y_2), then (4.47) also holds at the point $x \in (y_1, y_2)$. This establishes the claim of (4.47).

Now let $\mathcal{S}^*$ be the union of all such closed intervals $\mathcal{S}_i^*$. By construction, the $\mathcal{S}_i^*$ are pairwise disjoint and it is easy to see that $\mathcal{S}^*$ is compact, $\mathcal{S}^* \subset \Sigma - Z$, and, because $\mathcal{S} \subset \mathcal{S}^*$, $C(\mathcal{S}^*) > 0$. Since (4.47) holds on $\mathcal{S}^*$, we integrate this inequality with respect to $\nu_{\mathcal{S}^*}$, the unit equilibrium measure of $\mathcal{S}^*$, to obtain

$$F(\mathcal{S}^*) = \log C(\mathcal{S}^*) - \int Q\, d\nu_{\mathcal{S}^*} = \iint \log|x-t|\, d\nu_{\mathcal{S}^*}(x)\, d\mu_w(t) - \int Q\, d\nu_{\mathcal{S}^*}$$
$$\geq F(\mathcal{S}),$$

where we have used the fact that the equation

$$\int \log|x-t|\, d\nu_{\mathcal{S}^*}(x) = \log C(\mathcal{S}^*)$$

holds q.e. on $\mathcal{S}^*$ and hence q.e. on $\mathcal{S}$. Thus $F(\mathcal{S}^*) = F(\mathcal{S})$ and so, by part (b) of Theorem 2.2, proved above, $C(\mathcal{S}^* - \mathcal{S}) = 0$. Consequently, from the construction of $\mathcal{S}^*$, it follows that $\mathcal{S}^* = \mathcal{S}$. ■

Proof of Theorem 2.2(c). This follows immediately from Theorem 2.1 (c′). ■

Proof of Theorem 2.2(d). From part (a) of Theorem 2.2, the set $\mathcal{S}$ is regular. Hence (2.5) is a consequence of Theorem 4.2 and the remark following its proof. ■

Proof of Theorem 2.4. We first prove that the polynomials

$$(4.49) \qquad q_n(z) = \prod_{k=1}^{n} (z - t_{k,n})$$

satisfy

$$(4.50) \qquad \lim_{n\to\infty} \frac{1}{n} \log|q_n(z)| = \int \log|z-t|\, d\mu_w(t)$$

uniformly on every compact subset of $\mathbf{C} - \mathcal{J}$. Let

$$(4.51) \qquad u_n(z) := \frac{1}{n} \log|q_n(z)|, \qquad u(z) := \int \log|z-t|\, d\mu_w(t),$$

and notice that, for each $n \geq 1$, the function

(4.52) $$h_n(z) := u_n(z) - u(z)$$

is harmonic in $\mathbf{C}^* - \mathcal{J}$ and satisfies $h_n(\infty) = 0$. Let $\epsilon > 0$ be given. The assumption of (2.15) implies that for each n sufficiently large,

$$\|[w(x)]^n q_n(x)\|_{\mathcal{S}} \leq \exp\,(n[F(\mathcal{S}) + \epsilon]),$$

and hence, from part (g) of Theorem 2.3, it easily follows that

(4.53) $$h_n(z) \leq \epsilon, \qquad \forall\ z \in \mathbf{C}^*.$$

Therefore, the sequence $\{h_n\}$ forms a normal family of harmonic functions in $\mathbf{C}^* - \mathcal{J}$. Let $H(z)$ be an arbitrary limit function of this family. Then, by (4.53) and the arbitrariness of ϵ, we have

(4.54) $$H(z) \leq 0, \qquad \forall\ z \in \mathbf{C}^* - \mathcal{J}.$$

But H is harmonic in $\mathbf{C}^* - \mathcal{J}$ and $H(\infty) = 0$, and so, by the maximum principle, $H(z) \equiv 0$. Since this is true for every limit $H(z)$, we have $h_n(z) \to 0$ uniformly on every compact subset of $\mathbf{C} - \mathcal{J}$. This proves (4.50) which is equivalent to (2.17).

Next notice that, for the measures $\nu^{(n)}$ of (2.14), we have

(4.55) $$\frac{1}{n}\frac{q_n'(z)}{q_n(z)} = \int \frac{d\nu^{(n)}(t)}{z-t}, \qquad \forall\ z \in \mathbf{C} - \mathcal{J}.$$

Consequently, from (4.50),

(4.56) $$\lim_{n\to\infty} \int \frac{d\nu^{(n)}(t)}{z-t} = \int \frac{d\mu_w(t)}{z-t}, \qquad \forall\ z \in \mathbf{C} - \mathcal{J}.$$

Since $\nu^{(n)}(\mathcal{J}) = 1$ for all n, it follows from Helly's theorem (cf. [15]), that there exists a subsequence $\nu^{(n_i)}$ and a measure ν^* supported on $\mathcal{J}$ such that $\nu^{(n_i)} \to \nu^*$ weakly. For this subsequence,

(4.57) $$\lim_{i\to\infty} \int \frac{d\nu^{(n_i)}(t)}{z-t} = \int \frac{d\nu^*(t)}{z-t}, \qquad \forall\ z \in \mathbf{C} - \mathcal{J},$$

and hence, from (4.56),

(4.58) $$\int \frac{d\nu^*(t)}{z-t} = \int \frac{d\mu_w(t)}{z-t}, \qquad \forall\ z \in \mathbf{C} - \mathcal{J}.$$

By uniqueness of the Cauchy transform (cf. [2]), we then have $\nu^* = \mu_w$. As this is true for every weak limit ν^*, then $\nu^{(n)} \to \mu_w$ as claimed in (2.16). ■

Proof of Corollary 2.5. Note that, from part (c) of Theorem 2.2, we have

(4.59) $$\hat{E}_n(w) = \|[w(x)]^n T_n(x;\ w)\|_{\mathcal{S}},$$

where $\hat{E}_n(w)$ is defined in (4.30). It is easily seen that this extremal property implies that all the zeros of $T_n(x;\ w)$ are real and lie in the convex hull of the compact set $\mathcal{S}$. Furthermore, by part (d) of Theorem 2.2, the polynomials $T_n(x;\ w)$ satisfy (2.15). Thus, with $\{t_{k,n}\}_{k=1}^{n}$ taken as the zeros of $T_n(x;\ w)$, the conclusions (2.16) and (2.17) hold. ■

Remark. With the assumptions of Theorem 2.2, if $\mathcal{N}_n([c, d])$ denotes the number of zeros of $T_n(x; w)$ that lie in an arbitrary interval $[c, d]$, then by Corollary 2.5 and the fact that $\mu_w(\{c\}) = \mu_w(\{d\}) = 0$,

$$\lim_{n\to\infty} n^{-1}\mathcal{N}_n([c, d]) = \int_c^d d\mu_w(t). \tag{4.60}$$

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Note Added in Proof. The authors have received a communication from Dr. H. Stahl asserting that the asymptotic error estimate of (2.5) holds, more generally, for any admissible weight function w.

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