

WEIGHTED POLYNOMIALS ON FINITE AND INFINITE INTERVALS: A UNIFIED APPROACH

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1. Introduction. As described in the survey article [6], the study of "incomplete polynomials", as introduced by G. G. Lorentz [4] in 1976, leads to results on the asymptotic properties of polynomials orthogonal on an infinite interval (cf. [5]) and to theorems on the convergence of "ray sequences" of Padé approximants for Stieltjes functions. Here we present a generalization of the theory for incomplete polynomials which unifies many of the previous results. The essential question which serves as the starting point for the investigation is the following:¹

Suppose $w(x)$ is a nonnegative weight function continuous on its support $\Sigma \subset \mathbf{R} = (-\infty, \infty)$. (By the *support* of w we mean the *closure* of the set where w is positive.) Assume that $w(x)$ vanishes at points of Σ ; that is, $Z := \{x \in \Sigma : w(x) = 0\} \neq \emptyset$ (or, in case Σ is unbounded, then $|x|w(x) \rightarrow 0$ as $|x| \rightarrow \infty$). If P_n is an arbitrary polynomial of degree at most n , then the sup norm over Σ of the weighted polynomial $[w(x)]^n P_n(x)$ actually "lives" on some compact set $S \subset \Sigma - Z$ which is independent of n and P_n . The question is to determine the smallest such set S .

For example, if $w(x) = x^{\theta/(1-\theta)}$ with $\Sigma = [0, 1]$, $0 < \theta < 1$, then, as shown in [2, 8], S is the subinterval $[\theta^2, 1]$.

In this paper we use potential theoretic methods to show how S can be obtained for a class of weight functions. The assumptions on w are given in

DEFINITION 1.1. Let $w: \mathbf{R} \rightarrow [0, +\infty)$. We say that w is an *admissible weight function* if each of the following properties holds:

- (i) $\Sigma := \text{supp}(w)$ has positive capacity.
- (ii) The restriction of w to Σ is continuous on Σ .
- (iii) The set $Z := \{x \in \Sigma : w(x) = 0\}$ has capacity zero.
- (iv) If Σ is unbounded, then $|x|w(x) \rightarrow 0$ as $|x| \rightarrow \infty$, $x \in \Sigma$.

Here, and throughout the paper, the term "capacity" means inner logarithmic capacity (cf. [10, p. 55]). For any set $E \subset \mathbf{R}^2$, its capacity will be denoted by $C(E)$. If K is a compact set with positive capacity, then ν_K denotes the unique unit equilibrium measure on K with the property that (cf. [10, p. 60])

$$(1.1) \quad \int_K \log|x-t| d\nu_K(t) = \log C(K)$$

quasi-everywhere (q.e.) on K . (A property is said to hold q.e. on a set A if the subset E of A where it does not hold satisfies $C(E) = 0$.)

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For an admissible weight w , we always set

$$(1.2) \quad Q(x) := \log(1/w(x)).$$

Finally, if $K \subset \Sigma - Z$ is compact and $C(K) > 0$, we define the F -functional of K by the formula

$$(1.3) \quad F(K) := \log C(K) - \int_K Q d\nu_K.$$

The theorems of §2 show that, for a class of weight functions, S is derived by maximizing the F -functional. Also, if π_m denotes the collection of all polynomials of degree at most m and $\|\cdot\|_A$ denotes the sup norm over a set A , we describe the asymptotic behavior of the errors in the weighted Chebyshev problem

$$(1.4) \quad E_n(w) := \inf\{\|[w(x)]^n \{x^n - p_{n-1}(x)\}\|_\Sigma : p_{n-1} \in \pi_{n-1}\},$$

$$n = 1, 2, \dots,$$

as well as asymptotic properties (as $n \rightarrow \infty$) of the extremal polynomials $T_n(x; w) = x^n + \dots \in \pi_n$ which satisfy

$$(1.5) \quad E_n(w) = \|[w(x)]^n T_n(x; w)\|_\Sigma, \quad n = 1, 2, \dots$$

2. Statements of main results.

THEOREM 2.1. *Let w be an admissible weight function with support Σ . Then there exists a compact set $S \subset \Sigma - Z$ with $C(S) > 0$ that has the following properties.*

(a) *For every compact set $K \subset \Sigma - Z$ with $C(K) > 0$,*

$$(2.1) \quad F(K) \leq F(S),$$

where F is defined in (1.3).

(b) *If equality holds in (2.1), then $S \subset K$.*

(c) *For any positive integer n , if $P_n \in \pi_n$ and the inequality*

$$(2.2) \quad |[w(x)]^n P_n(x)| \leq M \quad (M = \text{constant})$$

holds q.e. on S , then it holds q.e. on Σ .

(d) *The errors $E_n(w)$ defined in (1.4) satisfy*

$$(2.3) \quad [E_n(w)]^{1/n} \geq \exp(F(S)), \quad \forall n = 1, 2, \dots$$

Clearly properties (a) and (b) uniquely determine the set $S = S(w)$ of Theorem 2.1. In the special case when $w(x) \equiv 1$ on Σ and Σ is compact, then S is just the support of the equilibrium measure ν_Σ for Σ .

Of practical importance is the characterization of S given in

THEOREM 2.2. *Assume that, in Theorem 2.1, the set $\Sigma - Z$ is the finite union of disjoint nondegenerate intervals and that $Q(x)$ of (1.2) is convex in each of the components of $\Sigma - Z$. Then the following additional properties hold.*

(a) *The compact set S of Theorem 2.1 is the finite union of nondegenerate disjoint closed intervals, at most one in each component of $\Sigma - Z$.*

(b) Equality holds in (2.1) if and only if $S \subset K$ and $C(K - S) = 0$.

(c) For any positive integer n , if $P_n \in \pi_n$, then

$$(2.4) \quad \|[w(x)]^n P_n(x)\|_\Sigma = \|[w(x)]^n P_n(x)\|_S.$$

(d) The errors $E_n(w)$ of (1.4) satisfy

$$(2.5) \quad \lim_{n \rightarrow \infty} [E_n(w)]^{1/n} = \exp(F(S)).$$

The proof of Theorem 2.1 follows by showing that S is actually the support of a measure which solves an extremal problem for generalized energy integrals, as we now describe. Let $M(\Sigma)$ denote the collection of all positive unit Borel measures μ with $\text{supp}(\mu) \subset \Sigma$, and define

$$(2.6) \quad I_w[\mu] := \int \int [\log|x-t| - Q(x) - Q(t)] d\mu(x) d\mu(t)$$

for $\mu \in M(\Sigma)$. Following methods of Frostman (cf. [10]) we obtain

THEOREM 2.3. *Let w be an admissible weight function with support Σ and let*

$$(2.7) \quad V_w := \sup\{I_w[\mu] : \mu \in M(\Sigma)\}.$$

Then there exists a unique measure $\mu_w \in M(\Sigma)$ such that $I_w[\mu_w] = V_w$. Moreover, $S_w := \text{supp}(\mu_w)$ satisfies all the properties stated in Theorem 2.1; that is, $S_w = S$.

Concerning the limiting distribution of the zeros of the extremal polynomials $T_n(x; w)$ we have

THEOREM 2.4. *With the assumptions of Theorem 2.2, let $\{x_{k,n}\}_{k=1}^n$ denote the zeros of the extremal polynomial $T_n(x; w)$ of (1.5), and let ν_n be the associated unit Borel measure defined by*

$$\nu_n(B) := (1/n) |\{k : x_{k,n} \in B\}|.$$

Then, in the weak star topology,

$$(2.8) \quad \lim_{n \rightarrow \infty} \nu_n = \mu_w,$$

where μ_w is the extremal measure of Theorem 2.3. Furthermore,

$$(2.9) \quad \lim_{n \rightarrow \infty} |T_n(z; w)|^{1/n} = \exp\left(\int \log|z-t| d\mu_w(t)\right)$$

uniformly on every compact set of the plane disjoint from the convex hull $[\lambda, \tau]$ of S .

3. Applications. For Jacobi weights of the form $w(x) = x^{\theta/(1-\theta)}$, $0 < \theta < 1$, $\Sigma = [0, 1]$, or $w(x) = (1-x)^{\lambda_1}(1+x)^{\lambda_2}$, $\lambda_1, \lambda_2 > 0$, $\Sigma = [-1, 1]$, maximizing the associated F -functional leads to the results of [2, 9, 3 and 7] concerning incomplete polynomials.

For a weight W on \mathbf{R} of the form $W(x) = \exp(-q(x))$, where $q(x)$ is even and convex on \mathbf{R} and $q(x)/\ln x \rightarrow \infty$ as $x \rightarrow \infty$, we can also analyze the extremal problems

$$(3.1) \quad e_n(W) := \inf\{\|W(x)\{x^n - p_{n-1}(x)\}\|_{\mathbf{R}} : p_{n-1} \in \pi_{n-1}\}, \quad n = 1, 2, \dots,$$

and the corresponding extremal polynomials $t_n(x; W) = x^n + \dots \in \pi_n$ satisfying $e_n(W) = \|W(x)t_n(x; W)\|_{\mathbf{R}}$. After maximizing the appropriate F -functional, Theorem 2.2(c) yields

$$(3.2) \quad \|W(x)P_n(x)\|_{\mathbf{R}} = \|W(x)P_n(x)\|_{[-a_n, a_n]}, \quad \forall P_n \in \pi_n,$$

where $a = a_n$ is a root of the equation

$$(3.3) \quad n = \frac{2}{\pi} \int_0^1 \frac{axq'(ax)}{\sqrt{1-x^2}} dx.$$

Letting $w_n(x) := \exp(-q(a_n x)/n)$ with $\Sigma_n := \text{supp}(w_n) = [-1, 1]$, it follows from (3.2) that

$$e_n(W) = a_n^n E_n(w_n), \quad t_n(a_n x; W) = a_n^n T_n(x; w_n).$$

If the weights w_n converge uniformly to an admissible weight w on $[-1, 1]$, it can be shown that the asymptotic behaviors (as $n \rightarrow \infty$) of $E_n(w_n)$ and $T_n(x; w_n)$ are the same as that for $E_n(w)$ and $T_n(x; w)$. These facts lead to the results of [5] for $W(x) = \exp(-|x|^\alpha)$, $\alpha \geq 1$, as well as to L^∞ -analogue of the L^2 -results in [1] for $W(x) = \exp(-\exp|x|)$.

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