EXTREMAL PROBLEMS FOR POLYNOMIALS WITH LAGUERRE WEIGHTS

H. N. Mhaskar and E. B. Saff

In a recent paper [2] we studied certain extremal problems on the real line $\mathbb{R}$ for expressions of the form $\exp(-|x|^\alpha)P_n(x)$, where $\alpha > 0$ and $P_n \in \mathcal{P}_n$, the class of all real polynomials of degree at most n. These expressions resemble the incomplete polynomials introduced by G. G. Lorentz [1] in the sense that the weight function $\exp(-|x|^\alpha)$ vanishes at the endpoints $(\pm\infty)$ of the interval. Analogous to the "θ^2-result" of Lorentz for incomplete polynomials, we proved in [2] the following result.

THEOREM 1. *Let* $n \geq 0$ *be an integer,* $\alpha > 0$ *and* $P_n \in \mathcal{P}_n$ *with* $P_n(x) \not\equiv 0$. *If* $\xi \in \mathbb{R}$ *satisfies*

$$(1) \qquad \exp(-|\xi|^\alpha)|P_n(\xi)| = \|\exp(-|x|^\alpha)P_n(x)\|_{L^\infty(\mathbb{R})},$$

then

$$(2) \qquad |\xi| \leq a_n(\alpha) := (n/\lambda_\alpha)^{1/\alpha},$$

where

$$(3) \qquad \lambda_\alpha := \frac{\Gamma(\alpha)}{2^{\alpha-2}\left\{\Gamma(\frac{\alpha}{2})\right\}^2}.$$

Moreover, inequality (2) *is best possible in the sense that there exists a sequence of polynomials* $p_n \in \mathcal{P}_n$, $n = 1,2,\ldots,$ *and corresponding extreme points* ξ_n *such*

ISBN 0-12-174580-5

<u>that</u> $\xi_n/a_n(\alpha) \to 1$ <u>as</u> $n \to \infty$.

In further analogy with the known results for incomplete polynomials (cf. [3], [4]) we obtained in [2] the asymptotic behavior (as $n \to \infty$) of the error

$$\varepsilon_{n,p}(\alpha) := \min\{\|\exp(-|x|^\alpha)(x^n - p_{n-1}(x))\|_{L^p(\mathbb{R})} : p_{n-1} \in \mathcal{P}_{n-1}\}$$

$$= \|\exp(-|x|^\alpha) T_{n,p}(\alpha;x)\|_{L^p(\mathbb{R})} , \quad 0 < p \le \infty ,$$

and proved a "contracted zero distribution" theorem for the extremal polynomials $T_{n,p}(\alpha;x)$. In the important special case when $p = 2$, the latter result yields the zero distribution for polynomials which are normal and orthogonal on $\mathbb{R}$ with respect to the weight function $\exp(-2|x|^\alpha)$.

For general weight functions, explicit estimates such as (2) seem difficult to obtain. There is, however, another important class of weight functions which our methods handle quite easily, namely the <u>Laguerre weights</u> $t^s e^{-t}$ on $[0,+\infty)$. The aim of the present paper is to state these new results and to outline the method of proof.

THEOREM 2. <u>Let</u> $m \ge 0$ <u>be an integer</u>, $s \ge 0$, $n := s + m > 0$, <u>and</u> $p_m \in \mathcal{P}_m$ <u>with</u> $p_m(x) \not\equiv 0$. <u>If</u> $\xi \in [0,+\infty)$ <u>satisfies</u>

$$|\xi^s e^{-\xi} p_m(\xi)| = \|t^s e^{-t} p_m(t)\|_{L^\infty[0,+\infty)} , \tag{4}$$

<u>then</u>

$$n\left(1 - \sqrt{1 - (s/n)^2}\right) \le \xi \le n\left(1 + \sqrt{1 - (s/n)^2}\right) . \tag{5}$$

We remark that in the special case when $s = 0$, the result of Theorem 2 was obtained by Saff and Varga [5].

To state the next result we need the following notation. For each $p (0 < p \leq \infty)$, $s \geq 0$, and integer $m \geq 1$, set

$$(6) \quad E_{s,m,p} := \min\{\|t^s e^{-t}(t^m - q_{m-1}(t))\|_{L^p[0,+\infty)} : q_{m-1} \in \mathcal{P}_{m-1}\},$$

and let $\mathcal{T}_{s,m,p}(t) = t^m + \ldots \in \mathcal{P}_m$ satisfy

$$(7) \qquad E_{s,m,p} = \|t^s e^{-t} \mathcal{T}_{s,m,p}(t)\|_{L^p[0,+\infty)} .$$

Then we have

THEOREM 3. Let θ $(0 \leq \theta < 1)$ be fixed and suppose $\{s_i\}$ is a sequence of nonnegative real numbers and $\{m_i\}$ is a sequence of nonnegative integers such that $n_i := s_i + m_i > 0$ for each i, $n_i \to \infty$ and

$$(8) \qquad s_i/n_i \to \theta \quad \text{as} \quad i \to \infty .$$

Then

A. For each p $(0 < p \leq \infty)$, the minimal error defined in (6) satisfies

$$(9) \quad \lim_{i \to \infty} n_i^{-1} E_{s_i,m_i,p}^{1/n_i} = \left\{ \frac{(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}}{4e^2} \right\}^{1/2} .$$

B. All the zeros of the extremal polynomials $\mathcal{T}_{s_i,m_i,p}$ are real and, if $p \geq 1$, simple. For $p = \infty$, the largest zero ζ_{s_i,m_i} and smallest zero ξ_{s_i,m_i} of $\mathcal{T}_{s_i,m_i,\infty}$ satisfy

(10) $$\lim_{i\to\infty} \zeta_{s_i,m_i}/n_i = 1+\sqrt{1-\theta^2}\ ,\ \lim_{i\to\infty} \xi_{s_i,m_i}/n_i = 1-\sqrt{1-\theta^2}\ .$$

C. *For each* p $(0 < p \le \infty)$ *and interval* $[c,d] \subset [0,+\infty)$, *let* $N_{i,p}(c,d)$ *denote the number of zeros of the normalized polynomial* $T_{s_i,m_i,p}(n_i t)$ *which lie in* $[c,d]$. *Then*

(11) $$\lim_{i\to\infty} \frac{N_{i,p}(c,d)}{m_i} = \int_c^d d\mu_\theta(x) \quad ,$$

where

$$d\mu_\theta(x) := \frac{1}{(1-\theta)\pi} \frac{\sqrt{(b-x)(x-a)}}{x}\, dx \quad ,$$

$$a := 1 - \sqrt{1-\theta^2} \quad , \quad b := 1 + \sqrt{1-\theta^2} \quad ,$$

and $\mathrm{supp}(d\mu_\theta) = [a,b]$.

The basic idea of the proofs of Theorems 2 and 3 is to consider the following problem from potential theory:

For fixed θ , $0 \le \theta < 1$, *and each interval* $[a,b] \subset [0,+\infty)$, *find a function*

$$g(x) = g(x;a,b) \in L^1[a,b] \quad ,$$

and a constant $F(a,b)$ *such that*

(i) $$\int_a^b g(x)\,dx = 1-\theta \quad ,$$

(ii) $$\int_a^b \log|t-x|\,g(x)\,dx = t - \theta \log t + F(a,b) \quad .$$

A formal solution to this problem can be obtained by matching the Fourier coefficients of the appropriate functions. This yields

$$(12)\qquad F(a,b) = (1-\theta)\log\left(\frac{b-a}{4}\right) + 2\theta\log\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right) - \frac{a+b}{2} .$$

The next step in the proof is to maximize the right-hand side of (12) for $(a,b) \in [0,+\infty) \times [0,+\infty)$. From elementary calculus, this occurs when

$$a = 1 - \sqrt{1-\theta^2} \quad , \quad b = 1 + \sqrt{1-\theta^2} .$$

With this choice for $[a,b]$ the techniques of [2] can be used to show that, under the hypotheses of Theorem 3,

$$(13)\qquad \lim_{i\to\infty} n_i^{-1} E_{s_i,m_i,p}^{1/n_i} = \exp(F(a,b))$$

$$= \left\{\frac{(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}}{4e^2}\right\}^{1/2} ,$$

and, moreover, that the associated distribution $g(x)dx$ is nonnegative and gives the asymptotic zero distribution for the $T_{s_i,m_i,p}$ as stated in Theorem 3(C).

We remark that an alternative method to derive (13) is to first let $p = 2$ and use the explicit formulas [6] for Laguerre polynomials to verify (13) in this special case. Then, the Nikolskii-type inequalities of [2] yield (13) for all $p > 0$.

References

1. G. G. Lorentz, Approximation by incomplete polynomials (problems and results), Padé and Rational Approxima-

tions: Theory and Applications (Edited by E. B. Saff and R. S. Varga), Academic Press, New York (1977), 289-302.

2. H. N. Mhaskar and E. B. Saff, Extremal problems for polynomials with exponential weights, to appear in Trans. Amer. Math. Soc.

3. E. B. Saff, J. L. Ullman, and R. S. Varga, Incomplete polynomials: an electrostatics approach, Approximation Theory III (Edited by E. W. Cheney), Academic Press, New York (1980), 769-782.

4. E. B. Saff and R. S. Varga, On incomplete polynomials, Numerische Methoden der Approximationstheorie, Band 4 (L. Collatz, G. Meinardus, and H. Werner, eds.), International Series of Numerical Mathematics 42, Birkhäuser, Basel-Stuttgart (1978), 281-298.

5. E. B. Saff and R. S. Varga, On incomplete polynomials, II, Pacific J. Math., 92 (1981), 161-172.

6. G. Szegö, Orthogonal Polynomials (3rd edition), Amer. Math. Soc. Coll. Publ., 23, Amer. Math. Soc., Providence (1967).

H. N. Mhaskar
Department of Mathematics
California State University, Los Angeles
Los Angeles, CA 90032

E. B. Saff†
Center for Mathematical Services
University of South Florida
Tampa, FL 33620

†Research supported in part by the National Science Foundation.