INCOMPLETE AND ORTHOGONAL POLYNOMIALS

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The purpose of this survey is to reveal interrelationships among certain results on incomplete polynomials, the convergence of "ray sequences" of Pade approximants and the behavior of polynomials orthogonal on the whole real line $\mathbb{R}$. No attempt is made to be comprehensive in the discussion of these topics. But we shall highlight some recent theorems and emphasize techniques which form the common threads. We begin by posing three questions which have received separate attention in the literature.

1. Three Related (?) Problems.

(I) Incomplete Polynomials. In 1976, G. G. Lorentz [13] presented some basic results and raised several interesting questions concerning approximation by polynomials of the form

$$(1.1) \qquad P(x) = \sum_{k=s}^{n} a_k x^k , \quad s > 0 ,$$

which he calls incomplete polynomials. A survey of this topic appears in [14]. For our purposes, we shall measure "incompleteness" in the following relative sense.

DEFINITION 1.1. The polynomial $P(x)$ of (1.1) is said to be incomplete of type θ $(0 < \theta < 1)$ if $s/n \geq \theta$. The collection of all incomplete polynomials of type θ is denoted by I_θ.

For example, if π_m denotes the collection of all poly-

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nomials of degree at most m, then

$$I_{1/2} = \{x^m q_m(x) : q_m \in \pi_m \ , \ m \geq 1 \text{ arbitrary}\} \ .$$

For each θ, the collection I_θ contains polynomials of arbitrarily large degree and, while I_θ is not a linear space, it is closed under ordinary multiplication of polynomials.

Now, for each $P \in I_\theta$ $(0 < \theta < 1)$ with $P \not\equiv 0$, consider the sup norm $\|P\|_{[0,1]}$ of P on the interval $[0,1]$, and let

$$(1.2) \qquad \xi(P) := \min\{\xi \in [0,1] : |P(\xi)| = \|P\|_{[0,1]}\} \ .$$

Since $P(0) = 0$ and since $P \not\equiv 0$, then $0 < \xi(P) \leq 1$. A basic question one can ask is this: how close can $\xi(P)$ be to zero (as a function of θ), that is

Problem I. Find $\inf\{\xi(P) : P \in I_\theta \ , \ P \not\equiv 0\}$.

The solution to Problem I, which is related to the works of Lorentz [13], v. Golitschek [5] and Saff and Varga [22], [24], is described in §2.

(II) Padé Approximants. The subject of Padé approximants is a classical branch of the theories of continued fractions and rational interpolation that has significant applications to the analysis of numerical methods and to the study of critical point phenomena (cf. [19], [1], [2]). To define these approximants (relative to the point $z = 0$), consider a formal power series

(1.3) $$f(z) = \sum_{k=0}^{\infty} a_k z^k .$$

For each fixed pair of nonnegative integers (m,n), we determine $P_{m,n} \in \pi_m$ and $Q_{m,n} \in \pi_n$ with $Q_{m,n} \not\equiv 0$ such that

(1.4) $$Q_{m,n}(z)f(z) - P_{m,n}(z) = \sum_{k=m+n+1}^{\infty} c_k z^k .$$

Notice that (1.4) represents $m+n+1$ homogeneous linear equations in $m+n+2$ unknowns (the coefficients of $P_{m,n}$, $Q_{m,n}$). Hence (1.4) has nontrivial solutions (necessarily with $Q_{m,n} \not\equiv 0$). Although the polynomials $P_{m,n}$ and $Q_{m,n}$ satisfying (1.4) are not unique, the ratio $P_{m,n}/Q_{m,n}$ (in lowest terms) does, however, yield a unique rational function.

DEFINITION 1.2. The <u>Padé approximant</u> (PA) <u>of type</u> (m,n) for the power series (1.3) is the rational function

$$[m/n](z) := P_{m,n}(z)/Q_{m,n}(z) ,$$

where $P_{m,n} \in \pi_m$ and $Q_{m,n}(\not\equiv 0) \in \pi_n$ satisfy (1.4).

The PAs for (1.3) are typically displayed in a doubly infinite array known as the <u>Padé table</u>:

[0/0]	[1/0]	[2/0]	. . .
[0/1]	[1/1]	[2/1]	. . .
[0/2]	[1/2]	[2/2]	. . .
⋮	⋮	⋮	

Here the first row of the table consists of the $[m/0]$ - approximants which are simply the polynomial sections of f.

The convergence question for PAs can be stated as follows. Given the power series (1.3), what can be said about the convergence of sequences extracted from its Padé table such as rows, columns, diagonals, etc.?

In its full generality, the issue of convergence is a difficult matter. Indeed, there are "nice" functions $f(z)$ for which the spurious poles of the PAs misbehave and destroy convergence (cf. [20]). There are, however, convergence theorems which apply to special classes of functions $f(z)$. One such class consists of the Stieltjes functions

$$f(z) = \int_0^1 \frac{d\mu(t)}{1 + zt} , \tag{1.5}$$

where μ is a finite positive measure with $\mathrm{supp}(d\mu) = [0,1]$. For example, $z^{-1}\log(1+z)$ and $(1+z)^{-1/2}$ are easily seen to be of the form (1.5). From a classical result of Markoff [15], it is known that the diagonal sequences of PAs $[n+J/n](z)$, $J \geq -1$ fixed, $n = 1,2,\ldots$, of (1.5) (which have all their poles on the interval $(-\infty, -1)$) converge to $f(z)$ in the slit plane $\mathbb{C} - (-\infty, -1]$.

Our concern is with the behavior of "ray sequences" from the Padé table of Stieltjes functions, that is, sequences of the form $[m_i/n_i](z)$, where

$$m_i + n_i \to \infty \quad \text{and} \quad m_i/n_i \to \lambda(>1) . \tag{1.6}$$

We pose

Problem II. Describe the regions of uniform convergence (in the complex plane) for sequences $[m_i/n_i](z)$ of PAs of the Stieltjes function (1.5), where the pairs (m_i, n_i) satisfy (1.6).

Problem II was investigated in the dissertation of H. Stahl [26] and more recently by P. R. Graves-Morris [7]. These works are discussed in §3.

(III) Orthogonal Polynomials. In comparison to the theory of polynomials orthogonal on a finite interval, the behavior of polynomials orthogonal on the whole line $\mathbb{R}$ is less well understood. To be specific, for $\alpha > 0$ fixed, consider the exponential weight function

$$(1.7) \qquad w_\alpha(x) := \exp(-|x|^\alpha)$$

and let

$$(1.8) \qquad p_n(\alpha;x) = \gamma_n x^n + \ldots \in \pi_n \quad (\gamma_n = \gamma_n(\alpha) > 0) \ , \ n = 0,1,\ldots,$$

denote the polynomials which are normal and orthogonal on $\mathbb{R}$ with respect to w_α^2 , that is

$$(1.9) \qquad \int_{-\infty}^{\infty} w_\alpha^2(x) p_j(\alpha;x) p_k(\alpha;x) dx = \delta_{jk} \ .$$

For these polynomials we pose

Problem III. Determine the asymptotic behavior of the leading coefficients γ_n and the limiting distribution of the zeros of the polynomials $\{p_n(\alpha;x)\}_{n=1}^{\infty}$.

In the special case $\alpha = 2$, the $p_n(\alpha;x)$ are the classical Hermite polynomials and the answers to Problem III are well-known (cf. [27]). For arbitrary $\alpha > 0$, G. Freud conjectured [3] that

(1.10) $$\lim_{n\to\infty} n^{-1/\alpha}\,\frac{\gamma_{n-1}}{\gamma_n} = \frac{1}{2}\left(\frac{1}{\lambda_\alpha}\right)^{1/\alpha} ,$$

where

(1.11) $$\lambda_\alpha := \frac{\Gamma(\alpha)}{2^{\alpha-2}\{\Gamma(\frac{\alpha}{2})\}^2} .$$

He verified that (1.10) holds for $\alpha = 2,4,6$.

Based on the work of Nevai and Dehesa [18], Ullman [28], [29] has shown how the truth of Freud's conjecture leads to the "asymptotic contracted zero distribution" for the $p_n(\alpha;x)$. Specifically, he proved that, assuming the truth of this conjecture, we have for every $f \in C[-1,1]$,

(1.12) $$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{x_{k,n}}{X_n}\right) = \int_{-1}^{1} f(x)\,v(x)\,dx ,$$

where

(1.13) $$-X_n = x_{n,n} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n} =: X_n$$

are the zeros of $p_n(\alpha;x)$ and

(1.14) $$v(x) = v(\alpha;x) := \frac{\alpha}{\pi}\int_{|x|}^{1} \frac{y^{\alpha-1}}{\sqrt{y^2-x^2}}\,dy .$$

Although Freud's conjecture is, at this writing, still unresolved, the recent work of Mhaskar and Saff [16] provides answers to Problem III. This investigation is discussed in §4.

2. Results on Incomplete Polynomials

In describing the solution to Problem I of §1 we shall not take the most direct route. Rather, we shall do some sightseeing for the purpose of highlighting certain fundamental theorems concerning incomplete polynomials. We begin with Lorentz's original result [13] on "enforced convergence to zero."

THEOREM 2.1. *With the notation of Definition* 1.1, *let* θ $(0 < \theta < 1)$ *be fixed. If* $\{P_n\}$ *is a sequence of polynomials in* I_θ *such that* $\deg P_n \to \infty$ *and* $\|P_n\|_{[0,1]} \le M$ $\forall n$, *then*

$$\text{(2.1)} \qquad \lim_{n\to\infty} P_n(x) = 0 \quad \forall x \in [0,\theta^2) \ .$$

To see how this theorem is related to Problem I, let $P(\not\equiv 0) \in I_\theta$ and consider a point $\xi \in [0,1]$ for which

$$|P(\xi)| = \|P\|_{[0,1]} \ .$$

Now define the sequence

$$\text{(2.2)} \qquad q_n(x) := \left\{ \frac{P(x)}{\|P\|_{[0,1]}} \right\}^n , \quad n = 1,2,\ldots \ .$$

Then $\{q_n\} \subset I_\theta$, $\deg q_n \to \infty$ and $\|q_n\|_{[0,1]} = 1$ $\forall n$.

Since $\{q_n\}$ satisfies the hypotheses of Theorem 2.1, we have $q_n(x) \to 0$ in $[0,\theta^2)$. But $|q_n(\xi)| = 1$ $\forall n$, and hence $\xi \ge \theta^2$. This proves

COROLLARY 2.2. *If* $P \in I_\theta$, *then* $\|P\|_{[0,1]} = \|P\|_{[\theta^2,1]}$. *Moreover,*

$$\text{(2.3)} \qquad \inf\{\xi(P) : P \in I_\theta , \; P \not\equiv 0\} \geq \theta^2 ,$$

where $\xi(P)$ *is defined in* (1.2).

In general, $\|\cdot\|_K$ denotes the sup norm over the set K .

To obtain an upper estimate for the left-hand side of (2.3) we appeal to the "Weierstrass approximation property" of incomplete polynomials of type θ . The following result was obtained independently by Saff and Varga [24] and v. Golitschek [5] .

THEOREM 2.3. *Let* $F \in C[0,1]$ *with* $F \notin I_\theta$. *Then* F *is the uniform limit on* $[0,1]$ *of* I_θ*-polynomials* (θ *fixed*) *if and only if* $F(x) = 0$ *for* $0 \leq x \leq \theta^2$.

If, in Theorem 2.3, we let F be the "spike function" of Figure 2.1, where $0 < \varepsilon < 1 - \theta^2$, then it follows

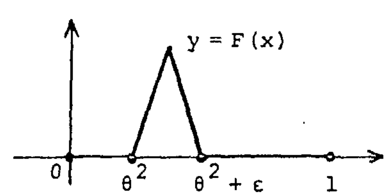

Figure 2.1

from Theorem 2.3 that there are incomplete polynomials of type θ which attain their maximum absolute value on $[0,1]$ at points arbitrarily close to θ^2 . Combining this fact with (2.3) of Corollary 2.2 we arrive at the solution to Problem I.

THEOREM 2.4. <u>For each</u> $0 < \theta < 1$,

$$\inf\{\xi(P) : P \in I_\theta , \quad P \not\equiv 0\} = \theta^2 \quad . \tag{2.4}$$

<u>Remark</u>. If $q(\not\equiv 0)$ is a lacunary polynomial of the form

$$q(x) = \sum_{i=0}^{k} b_i x^{\nu_i} \quad (0 < \nu_0 < \nu_1 < \dots < \nu_k \text{ integers}) , \tag{2.5}$$

then, since $q \in I_{\nu_0/\nu_k}$, inequality (2.3) implies

$$\xi(q) \geq (\nu_0/\nu_k)^2 \qquad (\nu_k \geq \nu_0 + k) \quad . \tag{2.6}$$

However, a much stronger result holds which we now describe.

DEFINITION 2.5. The polynomial $q(x)$ of (2.5) is said to be a <u>lacunary incomplete polynomial of type</u> θ $(0 < \theta < 1)$ if $\nu_0/(\nu_0 + k) \geq \theta$. The collection of all such polynomials is denoted by LI_θ .

Since $I_\theta \subset LI_\theta$, the following result of Saff and Varga [25] provides an extension of Theorem 2.4.

THEOREM 2.6. <u>For each</u> $0 < \theta < 1$,

$$\inf\{\xi(q) : q \in LI_\theta , \quad q \not\equiv 0\} = \theta^2 \quad . \tag{2.7}$$

The proof of this theorem is a straightforward consequence of Theorem 2.4 and the following lemma which appears in [25].

LEMMA 2.7. <u>Suppose the weight function</u> $w(x) \in C[0,1]$ <u>satisfies</u> $w(0) = 0$ <u>and</u> $w(x) > 0$ <u>for</u> $x \in (0,1]$. <u>For each</u> $k \geq 1$, <u>let</u>

(2.8) $$P_k^*(x) = P_k^*(w;x) = x^k - \sum_{i=0}^{k-1} c_i^* x^i$$

be the unique extremal polynomial for the Chebyshev problem

(2.9) $$\inf\{\|w(x)(x^k - \sum_{i=0}^{k-1} c_i x^i)\|_{[0,1]} : (c_0,\ldots,c_{k-1}) \in \mathbb{R}^k\} \quad ,$$

and set

(2.10) $$\xi_k^* := \min\{x \in (0,1] : |w(x)P_k^*(x)| = \|wP_k^*\|_{[0,1]}\} \quad .$$

If $p(x)$ is any real lacunary polynomial of the form

(2.11) $$p(x) = \sum_{i=0}^{k} b_i x^{\mu_i} \quad ,$$

then

(2.12) $$|p(x)| \le \frac{\|wp\|_{[0,1]}}{\|wP_k^*\|_{[0,1]}} |P_k^*(x)| \quad , \forall\ 0 \le x \le \xi_k^* \quad .$$

Consequently, if $\xi \in (0,1]$ satisfies $|w(\xi)p(\xi)| = \|wp\|_{[0,1]}$, where $p \not\equiv 0$ is of the form (2.11), then

(2.13) $$\xi_k^* \le \xi \quad .$$

To deduce Theorem 2.6, suppose the polynomial q of (2.5) is real and belongs to LI_θ . With $w(x) = x^{\nu_0}$ in Lemma 2.7, inequalities (2.13) and (2.3) imply

$$\xi(q) \ge \xi_k^* \ge \left(\frac{\nu_0}{\nu_0 + k}\right)^2 \ge \theta^2 \quad .$$

Thus $\xi(q) \ge \theta^2$ for all real $q \in LI_\theta$ and hence, by symmetry, for arbitrary $q \in LI_\theta$. Theorem 2.6 now follows from

Theorem 2.4 and the fact that $I_\theta \subset LI_\theta$.

Since the heart of the solution to Problem I is Theorem 2.1, we now describe an extension of this result which will also be useful in our discussion of Problem II. It is natural to suspect, in Theorem 2.1, that the convergence to zero in (2.1) takes place in some region of the complex plane containing the interval $[0,\theta^2)$. This is true, and can be described as follows. For fixed θ $(0 < \theta < 1)$, let $w = \phi(z)$ map the exterior of the interval $[\theta^2,1]$ in the z-plane conformally onto the exterior of the unit circle $|w| = 1$ so that $\phi(\infty) = \infty$. The inverse of this mapping is simply a modified Joukowski transformation, namely

$$(2.14) \qquad z = \phi^{-1}(w) = \frac{1+\theta^2}{2} + \frac{(1-\theta^2)}{2}\left\{\frac{w+w^{-1}}{2}\right\}, \quad |w| > 1 \quad .$$

Next, set

$$(2.15) \quad G(z;\theta) := \begin{cases} |\phi(z)| \left|\dfrac{(1-\theta)\phi(z) + (1+\theta)}{(1+\theta)\phi(z) + (1-\theta)}\right|^{\theta}, & z \in \mathbb{C} - [\theta^2,1] \ , \\ 1 \ , & z \in [\theta^2,1] \quad . \end{cases}$$

Then $G(z;\theta)$, so defined, is continuous on $\mathbb{C}$ and $G(0;\theta) = 0$. Let $\Lambda(\theta)$ denote the level curve

$$(2.16) \qquad \Lambda(\theta) := \{z \in \mathbb{C} : G(z;\theta) = 1\} \quad ,$$

and $\Lambda^{\circ}(\theta)$ denote its interior:

$$(2.17) \qquad \Lambda^{\circ}(\theta) := \{z \in \mathbb{C} : G(z;\theta) < 1\} \quad .$$

As illustrated in Figure 2.2, the interval $[0,\theta^2)$ is

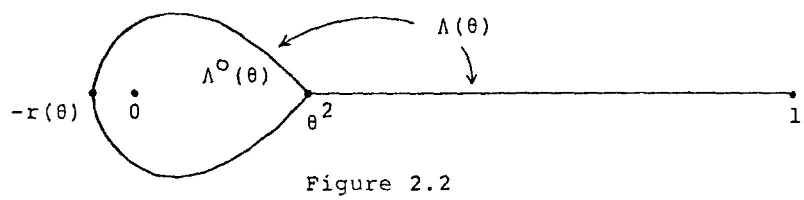

Figure 2.2

contained in $\Lambda^o(\theta)$ and $\Lambda(\theta)$ has the shape of a <u>tennis racket</u> with handle the segment $[\theta^2,1]$. The importance of this curve is revealed in

THEOREM 2.8. *For any fixed* θ $(0 < \theta < 1)$, *let* $\{P_n\}$ *be a sequence of polynomials in* I_θ *such that* $d_n := \deg P_n \to \infty$ *and*

(2.18) $$\limsup_{n\to\infty} \|P_n\|_{[0,1]}^{1/d_n} \le 1 .$$

Then

(2.19) $$\lim_{n\to\infty} P_n(z) = 0 \quad \forall z \in \Lambda^o(\theta) .$$

More precisely, if $K \subset \Lambda^o(\theta)$ *is closed, then*

(2.20) $$\limsup_{n\to\infty} \|P_n\|_K^{1/d_n} \le \max_{z\in K} G(z;\theta) < 1 .$$

Furthermore, $\Lambda^o(\theta)$ *is the largest open set for which* (2.19), *in general, holds.*

The first portion of Theorem 2.8 was proved, using different techniques, by Kemperman and Lorentz [9] and Saff and Varga [22]. In the former work, the maximum principle is used to show that if $p(x) = x^s q_k(x)$, $q_k \in \pi_k$, $s > 0$, $n := s + k$, then

$$(2.21)\qquad |p(z)| \le \|p\|_{[0,1]} [G(z;s/n)]^n , \quad \forall\, z \in \mathbb{C} ,$$

from which (2.20) of Theorem 2.8 easily follows. On the other hand, the results of [22], which include the sharpness assertion of Theorem 2.8, are derived from a study of the asymptotic behavior of certain Jacobi polynomials $P_n^{(\alpha,\beta_n)}(x)$, where $\beta_n/n \to$ const. An analysis of these polynomials also leads to the solution of the following electrostatics problem which, in turn, gives a physical interpretation of the Lorentz tennis racket $\Lambda(\theta)$.

Suppose that on the interval $[0,1]$, a fixed charge of amount θ $(0 < \theta < 1)$ is placed at $x = 0$, and a continuous charge of amount $1-\theta$ is placed on $[0,1]$ allowing it to reach equilibrium, the only constraint being that the charges remain confined to the interval $[0,1]$. For the logarithmic potential and its corresponding force field, the problem is to describe the distribution of the continuous charge.

As shown by Saff, Ullman and Varga [21] and also by Stahl [26] (cf. §3), the continuous charge of amount $1-\theta$ lies entirely in the interval $[\theta^2,1]$ and has point density

$$(2.22)\qquad d\nu_\theta(x) = \frac{1}{\pi x}\sqrt{\frac{x-\theta^2}{1-x}}\,dx .$$

Moreover, the tennis racket $\Lambda(\theta)$ is a level curve of the potential

$$(2.23)\qquad P_\theta(z) := \theta \log|z| + \int_{\theta^2}^{1} \log|z-x|\,d\nu_\theta(x) ;$$

(2.24) $\Lambda(\theta) = \left\{ z \in \mathbb{C} : P_\theta(z) = \log\left[\frac{(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}}{4}\right]\right\}$.

The constant

(2.25) $$\Delta(\theta) := (1+\theta)^{1+\theta}(1-\theta)^{1-\theta}/4 \quad ,$$

which appears in (2.24), is the <u>capacity</u> (transfinite diameter) of the compact set

(2.26) $$\overline{\Lambda}(\theta) := \Lambda(\theta) \cup \Lambda^{o}(\theta) = \{z \in \mathbb{C} : G(z;\theta) \leq 1\} \quad .$$

It plays the same role in the theory of approximation by incomplete polynomials of type θ on the <u>segment</u> $[0,1]$ as it does in the theory of ordinary (unconstrained) polynomial approximation on the set $\overline{\Lambda}(\theta)$. To be more precise, for each pair (s,k) of nonnegative integers, consider the Chebyshev extremal problem

(2.27) $$E_{s,k} := \inf\{ \|x^s(x^k - q_{k-1}(x))\|_{[0,1]} : q_{k-1} \in \pi_{k-1}\} \quad q_{-1} := 0,$$

and let $Q^*_{s,k}(x) = x^s(x^k - q^*_{k-1}(x))$ satisfy

(2.28) $$E_{s,k} = \|Q^*_{s,k}\|_{[0,1]} \quad .$$

The <u>constrained Chebyshev polynomials</u> $T^*_{s,k} := Q^*_{s,k}/E_{s,k}$ were studied in [23] where the next two theorems are established.

THEOREM 2.9. <u>Let</u> (s_i,k_i) <u>be a sequence of nonnegative integer pairs such that</u>

(2.29) $$n_i := s_i + k_i \to \infty \quad \underline{\text{and}} \quad s_i/n_i \to \theta \quad (0 < \theta < 1) \ .$$

<u>Then, with the notation of</u> (2.25) <u>and</u> (2.27),

$$\lim_{i\to\infty} [E_{s_i,k_i}]^{1/n_i} = \Delta(\theta) \quad . \tag{2.30}$$

THEOREM 2.10. Let the pairs (s_i,k_i) satisfy (2.29) and let $\{p_i\}$ be a sequence of polynomials of the form

$$p_i(x) = x^{s_i} \prod_{j=1}^{k_i} (x - x_{i,j}) \quad , \tag{2.31}$$

where the points $x_{i,j}$, $1 \le j \le k_i$, $i = 1,2,\ldots,$ all lie in some fixed finite interval $[a,b]$ containing $[\theta^2,1]$. If

$$\limsup_{i\to\infty} \|p_i\|_{[0,1]}^{1/n_i} \le \Delta(\theta) \qquad (n_i := s_i + k_i) \quad , \tag{2.32}$$

then

$$\lim_{i\to\infty} |p_i(z)|^{1/n_i} = \Delta(\theta)G(z;\theta) \quad , \forall\, z \in \mathbb{C} - [a,b] \quad , \tag{2.33}$$

the convergence being uniform on any compact subset of $\mathbb{C} - [a,b]$.

In [23, Prop. 9], the last theorem is established for sequences of extremal polynomials $Q^*_{s_i,k_i}$ (cf. (2.28)); however, a similar argument utilizing inequality (2.21) gives the more general statement of Theorem 2.10.

It is important to note that Theorems 2.9 and 2.10 yield the sharpness assertion of Theorem 2.8. Indeed, let

$$P_{n_i} = T^*_{s_i,k_i} = Q^*_{s_i,k_i}/E_{s_i,k_i} \qquad (n_i := s_i + k_i) \quad ,$$

where $s_i/n_i \downarrow \theta$. Then $\{P_{n_i}\} \subset I_\theta$, $\|P_{n_i}\|_{[0,1]} = 1 \;\; \forall\, i$ and all the nontrivial zeros of P_{n_i} lie on $[\theta^2,1]$. Hence, from (2.30) and (2.33), we have

$$\lim_{i \to \infty} |P_{n_i}(z)|^{1/n_i} = G(z;\theta) > 1 \qquad \forall\, z \in \mathbb{C} - \bar{\Lambda}(\theta) , \tag{2.34}$$

which shows that the P_{n_i} actually <u>diverge</u> exterior to $\bar{\Lambda}(\theta)$.

We conclude our discussion of incomplete polynomials by referencing some related works. For polynomials in z having a zero on the unit circle, analogs of the theorems on incomplete polynomials were obtained by Lachance, Saff and Varga [12]. They study extremal problems of the form

$$\inf \{ \| (z-1)^s (z^k - q_{k-1}(z)) \|_{|z| \le 1} : q_{k-1} \in \pi_{k-1} \} .$$

Results for real polynomials vanishing at <u>both</u> endpoints of an interval are derived by these same authors in [11]. More recently, v. Golitschek and Lorentz [6] and Lachance and Saff [10] investigated extensions of the theory to polynomials having a proportion of zeros lying on an <u>interval</u> rather than concentrated at a single point. For example, in [6] appears the following elegant result.

THEOREM 2.11. *Let* $\{T_n\}$ *be a sequence of trigonometric polynomials of respective degrees* $\le n$ *which converges uniformly on the unit circle* U *to the function* f . *If* N_n *is the number of zeros of* T_n *on* U , *then*

$$m\{ t : f(t) = 0 \} \ge 2 \limsup_{n \to \infty} \frac{N_n}{n} , \tag{2.35}$$

where m *denotes the Lebesgue measure on* U . *The constant* 2 *in* (2.35) *is best possible*.

A sample of the results in [10] is

THEOREM 2.12. *Let* θ $(0 < \theta < 1)$ *be fixed and let* $-r(\theta)$ *denote the unique point where the level curve* $\Lambda(\theta)$ *intersects the negative real axis* (cf. Figure 2.2). *If* $p_n \in \pi_n$ *has (at least)* θn *zeros in the interval* $[0,a]$ *, where*

$$(2.36) \qquad 0 \le a \le \lambda(\theta) := \frac{r(\theta)}{1+r(\theta)} \quad ,$$

then

$$(2.37) \qquad \|p_n\|_{[0,1]} = \|p_n\|_{[\theta^2,1]} \quad .$$

More precisely, if $p_n \not\equiv 0$,

$$(2.38) \qquad |p_n(x)| < \|p_n\|_{[0,1]} \quad , \quad \forall\, -r(\theta) + a(1+r(\theta)) < x < \theta^2 \quad .$$

The result is best possible in the sense that if $a > \lambda(\theta)$, *then* (2.37), *in general, does not hold.*

3. Ray Sequences of Padé Approximants for Stieltjes Functions

In [7], P. R. Graves-Morris investigated the convergence of certain ray sequences of PAs for Stieltjes functions of the form

$$(3.1) \qquad f(z) = \int_0^1 \frac{d\alpha(t)}{1+zt} \quad , \quad z \in \mathbb{C} - (-\infty,-1] \quad ,$$

where the distribution function $\alpha : [0,1] \to \mathbb{R}$ is nondecreasing and has infinitely many points of increase. His main result is stated as

THEOREM 3.1. Let $f(z)$ be a function of the form (3.1) and let (m_i, n_i) be a sequence of nonnegative integer pairs satisfying

$$\text{(3.2)} \qquad m_i + n_i \to \infty \quad \text{and} \quad m_i/n_i \to \lambda (>1) \quad \text{as} \quad i \to \infty .$$

Then, the sequence $[m_i/n_i](z)$ of PAs to $f(z)$ converges to $f(z)$ inside the pierced heart-shaped domain H_λ (see Figure 3.1) defined by

$$\text{(3.3)} \quad H_\lambda := \{z \in \mathbb{C} : 2^\lambda |t^+ z - z - 2| < |t^+ - 1|^\lambda |t^+ + 1| |z|^{(\lambda+1)/2}\} ,$$

where

$$t^+ := \frac{\lambda + 1 + z + \sqrt{(\lambda-1)^2(z+1)^2 + 4\lambda(z+1)}}{\lambda z} .$$

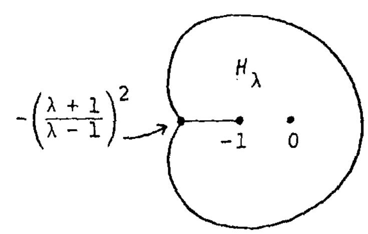

Figure 3.1

In the limiting case when $\lambda = +\infty$, the region H_λ reduces to the unit disk $|z| < 1$ which is the familiar region of convergence for the sections of the power series of f, i.e. the PAs $[m/0](z)$. As $\lambda \to 1^+$, H_λ tends to the slit plane $\mathbb{C} - (-\infty, -1]$ which, as mentioned in §1, is the region of convergence for the diagonal approximants $[n + J/n](z)$, $J \geq -1$ fixed, $n = 1, 2, \dots$.

In his proof of Theorem 3.1, Graves-Morris studies the asymptotic behavior of the Jacobi polynomials $P_n^{(0,\gamma n)}(z)$ $(\gamma > 0$ fixed$)$, a subject already mentioned in our discussion of incomplete polynomials (cf. [22]). Thus it is not surprising that there is a connection between the regions H_λ and $\Lambda^o(\theta)$. In fact, we shall see that, via a simple transformation, "the heart of Graves-Morris is the tennis racket of Lorentz." This is made precise in

THEOREM 3.2. Let $f(z)$ be a Stieltjes function of the form (3.1), where $\mathrm{supp}(d\alpha) = [0,1]$ and

$$\frac{1}{\sqrt{x(1-x)}} \log \alpha' \in L^1[0,1] \quad . \tag{3.4}$$

For each θ $(0 < \theta < 1)$, let

$$H(\theta) := \{z \in \mathbb{C} : -1/z \in \mathbb{C}^* - \bar{\Lambda}(\theta)\} \ , \quad \mathbb{C}^* := \mathbb{C} \cup \{\infty\} \ , \tag{3.5}$$

$$H^e(\theta) := \{z \in \mathbb{C} : -1/z \in \Lambda^o(\theta)\} \ , \tag{3.6}$$

where $\Lambda^o(\theta)$ and $\bar{\Lambda}(\theta)$ are defined, respectively, in (2.17) and (2.26) (see Figure 3.2). If (m_i, n_i) is a sequence of nonnegative integer pairs satisfying (3.2), then

$$[m_i/n_i](z) \to f(z) \quad , \ \forall\, z \in H\left(\frac{\lambda - 1}{\lambda + 1}\right) \ , \tag{3.7}$$

$$[m_i/n_i](z) \to \infty \quad \forall\, z \in H^e\left(\frac{\lambda - 1}{\lambda + 1}\right) - (-\infty, -(\lambda+1)^2/(\lambda-1)^2) \ . \tag{3.8}$$

The convergence in (3.7) is uniform and geometric on every compact set in $H((\lambda - 1)/(\lambda + 1))$.

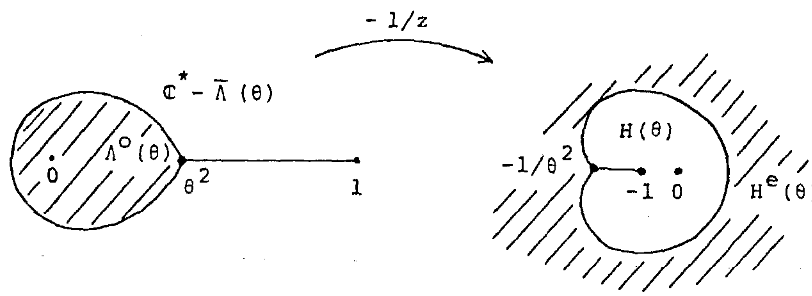

Figure 3.2

To sketch the proof of Theorem 3.2, we first observe that from the equations (1.4), the Padé denominator $Q_{m,n}(z)$ for a power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

can be represented as the determinant (cf. [1])

$$(3.9) \qquad Q_{m,n}(z) = \mathit{const.} \begin{vmatrix} a_{m-n+1} & a_{m-n+2} & \cdots & a_{m+1} \\ \vdots & \vdots & & \vdots \\ a_m & a_{m+1} & \cdots & a_{m+n} \\ z^n & z^{n-1} & \cdots & 1 \end{vmatrix},$$

where $a_k := 0$ if $k < 0$. This is reminiscent of the representation (cf. [27]) for polynomials p_n which are orthogonal with respect to a distribution $d\mu(t)$ on $[0,1]$:

$$(3.10) \qquad p_n(z) = \text{const.} \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & z & \cdots & z^n \end{vmatrix},$$

where the c_i are the moments

$$c_i := \int_0^1 t^i d\mu(t) \quad . \tag{3.11}$$

If

$$f(z) = \int_0^1 \frac{d\alpha(t)}{1+zt} = \sum_{k=0}^{\infty} \left(\int_0^1 t^k d\alpha(t) \right) (-z)^k \quad , \tag{3.12}$$

then, on replacing z by $-1/z$ in (3.9) and multiplying by z^n, we obtain for $m = n-1$ a determinant which is identical to (3.10). Hence the polynomials $\{z^n Q_{n-1,n}(-1/z)\}_{n=1}^{\infty}$ are orthogonal with respect to $d\alpha(t)$ on $[0,1]$. By shifting the subscript, it likewise follows that $\{z^n Q_{n,n}(-1/z)\}_{n=0}^{\infty}$ are orthogonal with respect to $t\,d\alpha(t)$ and, in general, $\{z^n Q_{n+J,n}(-1/z)\}_{n=0}^{\infty}$ are orthogonal with respect to $t^{J+1} d\alpha(t)$ on $[0,1]$. For each $J = -1,0,1,\ldots,$ $n = 1,2,\ldots,$ we can therefore write

$$z^n Q_{n+J,n}(-1/z) = p_{n,J+1}(z) \quad , \tag{3.13}$$

where $p_{n,J+1}$ denotes the polynomial of degree n which is <u>normal</u> and orthogonal with respect to $t^{J+1} d\alpha(t)$ on $[0,1]$. We remark that since all the zeros of $p_{n,J+1}(z)$ lie on $(0,1)$, then, from (3.13), all the poles of the PA $[n+J/n](z)$ for (3.12) lie on $(-\infty,-1)$ for $J \geq -1$, $n = 1,2,\ldots$.

Next, we appeal to the error formula (cf. [2, Part II, p.129])

$$\int_0^1 \frac{d\alpha(t)}{1+zt} - [m/n](z) = \frac{(-z)^{m+n+1}}{\{(-z)^n p_{n,J+1}(-\frac{1}{z})\}^2} \int_0^1 \frac{p_{n,J+1}^2(t)\, t^{J+1} d\alpha(t)}{1+tz} \quad , \tag{3.14}$$

where $J := m - n \geq -1$. For simplicity, suppose $m = \lambda n - 1$ with $\lambda > 1$. (We assume here and below that, whenever necessary, parameters are integer-valued.) Because of the normalization

$$(3.15) \qquad \int_0^1 p_{n,J+1}^2(t)\,t^{J+1}\,d\alpha(t) = 1 \quad ,$$

the behavior of the error (3.14) is determined by the behavior of the expression

$$(3.16) \qquad \frac{(-z)^{m+n+1}}{\{(-z)^n p_{n,J+1}(-\frac{1}{z})\}^2} = \frac{(-z)^{(\lambda+1)n}}{\{(-z)^n p_{n,(\lambda-1)n}(-\frac{1}{z})\}^2} = \frac{(-1)^{(\lambda+1)n}}{\{q_n(-\frac{1}{z})\}^2} \quad ,$$

where

$$(3.17) \qquad q_n(z) := z^{(\lambda-1)n/2}\,p_{n,(\lambda-1)n}(z) \quad .$$

Observe that the q_n are <u>incomplete polynomials</u> of type

$$\theta = \frac{(\lambda-1)n/2}{[(\lambda-1)n/2] + n} = \frac{\lambda-1}{\lambda+1} \quad .$$

Moreover, from (3.15), we have

$$\int_0^1 q_n^2(t)\,d\alpha(t) = 1 \quad .$$

Now, the assumption of (3.4) implies (cf. [17, p.157]) that the L^2-norm and the L^∞-norm of the polynomials q_n are asymptotically equivalent in the sense that

$$(3.18)\qquad \lim_{n\to\infty} \|q_n\|_{[0,1]}^{1/d_n} = 1 \quad , \quad d_n := \deg q_n = (\lambda+1)n/2 \quad .$$

Hence, by Theorem 2.8, we have

$$(3.19)\qquad q_n(z) \to 0 \qquad \forall\, z \in \Lambda^{\circ}(\theta) = \Lambda^{\circ}\left(\frac{\lambda-1}{\lambda+1}\right) \quad ,$$

and so, from (3.14) and (3.16), we deduce (3.8).

To prove (3.7), we can use the method of [7] or appeal to Theorem 2.10 and the fact that for the polynomials $p_{n,(\lambda-1)n}(x) = \hat{\gamma}_n x^n + \ldots \ (\hat{\gamma}_n > 0)$, we have

$$(3.20)\qquad \lim_{n\to\infty} \left(\frac{1}{\hat{\gamma}_n}\right)^{1/d_n} = \Delta(\theta) = \Delta\left(\frac{\lambda-1}{\lambda+1}\right) \qquad \text{(cf. (2.25))} \quad ,$$

where $d_n = (\lambda+1)n/2$. Either technique yields

$$(3.21)\qquad q_n(z) \to \infty \quad \forall\, z \in \mathbb{C} - \bar{\Lambda}(\theta) \quad ,$$

from which (3.7) follows. □

What has previously gone unrecognized in the literature concerning incomplete polynomials is the important (unpublished) work of H. Stahl. His dissertation [26] in 1976 (the same year Lorentz introduced the study of incomplete polynomials) concerns the convergence of ray sequences of PAs (about $z = \infty$) for functions of the form

$$(3.22)\qquad f(z) = \int \frac{d\mu(t)}{t-z} \quad ,$$

where μ is <u>any</u> finite positive measure with compact support E . Stahl's method, which is similar to Frostman's "minimum energy" approach [8], can be described as follows.

Let $E \subset \mathbb{R}$ be compact with $\mathrm{cap}(E) > 0$ and consider the collection $\mathcal{W}(E)$ of positive unit measures on E. For fixed θ $(0 \le \theta < 1)$ and $\mu \in \mathcal{W}(E)$, set

$$(3.23) \qquad p_\theta(z;\mu) := \theta \log|z|^{-1} + (1-\theta)\int \log|z-x|^{-1} d\mu(x) \quad .$$

Then, for the energy integral

$$(3.24) \qquad I_\theta(\mu) := (1-\theta)^2 \iint \log|x-y|^{-1} d\mu(x)\, d\mu(y) + 2(1-\theta)\theta \int \log|x|^{-1} d\mu(x) \quad ,$$

Stahl shows that there exists a unique measure $\mu_\theta \in \mathcal{W}(E)$ such that

$$(3.25) \qquad I_\theta(\mu_\theta) = \inf\{I_\theta(\mu) : \mu \in \mathcal{W}(E)\} \quad ,$$

and an associated constant σ_θ satisfying

$$(3.26) \qquad p_\theta(x;\mu_\theta) \le \sigma_\theta \quad \forall\, x \in \mathrm{supp}(d\mu_\theta) \quad ,$$

$$(3.27) \qquad p_\theta(x;\mu_\theta) \ge \sigma_\theta \quad \forall\, x \in E - K \quad , \quad \mathrm{cap}(K) = 0 \quad .$$

If $\theta = 0$, then $d\mu_0$ is the classical <u>equilibrium distribution</u> and σ_0 is the <u>Robin's constant</u> $V(E)$ satisfying

$$\exp(-V(E)) = \mathrm{cap}(E) \quad .$$

In the special case when $E = [0,1]$ and θ $(0 < \theta < 1)$ is arbitrary, an analysis of the electrostatics problem described in §2 shows that $d\mu_\theta(x)$ is the distribution (2.22), $(d\mu_\theta = d\nu_\theta/(1-\theta))$, $\mathrm{supp}(d\mu_\theta) = [\theta^2,1]$ and $\sigma_\theta = -\log \Delta(\theta)$.

For an arbitrary compact set E, the interior of the "generalized tennis racket" is the set

$$\Lambda_E^{\circ}(\theta) := \{z \in \mathbb{C} : p_\theta(z;\mu_\theta) > \sigma_\theta\} \quad , \tag{3.28}$$

which plays an essential role in Stahl's convergence theorems. Because of their detailed and comprehensive nature, we shall mention only the following sample of his results.

THEOREM 3.3. *Let* $f(z)$ *be a function of the form* (3.22), *where* $E := \mathrm{supp}(d\mu) \subset \mathbb{R}$ *is compact,* E *has no isolated points and* $\mathrm{cap}(E) > 0$. *Let* I *denote the smallest closed interval containing* E. *If* (m_i, n_i) *is a sequence of non-negative integer pairs satisfying*

$$m_i + n_i \to \infty \quad \text{and} \quad m_i/n_i \to \lambda\,(<\infty) \quad , \tag{3.29}$$

then the sequence of PAs $[m_i/n_i]_\infty(z)$ *about* $z = \infty$ *for* $f(z)$ *converges to* $f(z)$ *in the region*

$$\mathbb{C}^* - (\mathbb{R} \cup \overline{\Lambda_E^{\circ}(\theta)}) \quad , \quad \theta = \left|\frac{\lambda - 1}{\lambda + 1}\right| \quad . \tag{3.30}$$

Furthermore, if $E \subset [0,+\infty)$ *or* $E \subset (-\infty, 0]$, *then convergence holds on the larger set*

$$\mathbb{C}^* - (I \cup \overline{\Lambda_E^{\circ}(\theta)}) \quad , \quad \theta = \left|\frac{\lambda - 1}{\lambda + 1}\right| \quad . \tag{3.31}$$

Stahl's work also includes *divergence* results which, in effect, generalize Theorem 2.8. For example, with μ and E as above, set

$$a(\delta) := \inf\{\mu([x - \delta, x + \delta]) : x \in E\} \qquad (\delta > 0) \quad , \tag{3.32}$$

and assume that a finite constant τ exists such that

$$\liminf_{\delta \to 0} \frac{a(\delta)}{\delta^{\tau}} > 0 \quad . \tag{3.33}$$

If, in Theorem 3.3, $E \subset [0,+\infty)$ and $\lambda > 1$, then Stahl shows that

$$[m_i/n_i]_{\infty}(z) \to \infty \ \forall z \in \Lambda_E^{o}(\theta) \ , \quad \theta = \frac{\lambda - 1}{\lambda + 1} \quad . \tag{3.34}$$

4. Polynomials with Exponential Weights on $\mathbb{R}$

In [16], Mhaskar and Saff study extremal problems of the following form. For $\alpha > 0$ and $\infty \geq r > 0$, set

$$E_{n,r}(\alpha) := \inf_{q_{n-1} \in \pi_{n-1}} \|\exp(-|x|^{\alpha})(x^n - q_{n-1}(x))\|_{L_r} \quad , \tag{4.1}$$

where, as usual,

$$\|\cdot\|_{L_r} := \left(\int_{-\infty}^{\infty} |\cdot|^r\right)^{1/r} \quad , \quad \|\cdot\|_{L_\infty} := \text{sup norm on } \mathbb{R} \quad .$$

We let $T_{n,r}(\alpha;x) = x^n + \ldots \in \pi_n$ be an extremal polynomial for (4.1):

$$E_{n,r}(\alpha) = \|\exp(-|x|^{\alpha})T_{n,r}(\alpha;x)\|_{L_r} \quad , \tag{4.2}$$

and pose

Problem III'. For fixed α and r, determine the asymptotic behavior (as $n \to \infty$) of $E_{n,r}(\alpha)$ and the limiting distribution of the zeros of the polynomials $\{T_{n,r}(\alpha;x)\}_{n=1}^{\infty}$.

Notice that Problem III of §1 is a special case of Problem III' since, for $r = 2$, we have

$$(4.3)\qquad T_{n,2}(\alpha;x) = \frac{1}{\gamma_n(\alpha)} p_n(\alpha;x) \quad , \quad E_{n,2}(\alpha) = \frac{1}{\gamma_n(\alpha)} \quad ,$$

where, as in (1.8), the polynomials $p_n(\alpha;x) = \gamma_n(\alpha)x^n + \ldots \in \pi_n$ are orthonormal with respect to $\exp(-2|x|^\alpha)$ on $\mathbb{R}$.

A starting point for solving Problem III' is the observation that expressions of the form $\exp(-|x|^\alpha)P_n(x)$, $P_n \in \pi_n$, $\alpha > 0$, are analogous to incomplete polynomials in the sense that they vanish at the "endpoints" $\pm\infty$ of the interval $\mathbb{R}$. Thus we embark on a parallel course of study by seeking an answer to the following L_∞-problem.

<u>Problem</u> I'. If $P_n (\not\equiv 0) \in \pi_n$ and $\xi \in \mathbb{R}$ satisfies

$$(4.4)\qquad \exp(-|\xi|^\alpha)|P_n(\xi)| = \|\exp(-|x|^\alpha)P_n(x)\|_{L_\infty} \qquad (\alpha > 0) \quad ,$$

then how *large* can $|\xi|$ be (as a function of α and n)?

For the special case $\alpha = 2$, G. Freud [4] established a related result which we state as

THEOREM 4.1. *If* $P_n \in \pi_n$, $\varepsilon > 0$ *and*

$$(4.5)\qquad e^{-x^2}|P_n(x)| \le M \qquad \forall\ |x| \le (1+\varepsilon)\sqrt{n} \quad ,$$

then

$$(4.6)\qquad e^{-x^2}|P_n(x)| \le Mc(\varepsilon) \qquad \forall\ x \in \mathbb{R} \quad ,$$

where $c(\varepsilon)$ *is some positive constant (independent of* n).

Freud's result, which suggests that the L_∞-norm of $e^{-x^2}P_n(x)$ lives on the interval $[-\sqrt{n},\sqrt{n}]$, is derived from properties of the Hermite polynomials. A better analysis, similar to the proof of inequality (2.21) for incomplete polynomials, proceeds as follows.

Introduce a parameter $a(>0)$ and assume that

$$e^{-x^2}|P_n(x)| \le M \qquad \forall\, x \in [-a,a] \quad , \tag{4.7}$$

or, equivalently,

$$e^{-a^2x^2}|P_n(ax)| \le M \qquad \forall\, x \in [-1,1] \quad . \tag{4.8}$$

Our goal is to find a "smallest choice" for a so that inequality (4.8) implies

$$e^{-a^2x^2}|P_n(ax)| \le M \qquad \forall\, x \in \mathbb{R} \quad . \tag{4.9}$$

For this purpose, let

$$w = \psi(z) = z + \sqrt{z^2 - 1} \tag{4.10}$$

denote the mapping of the segment $[-1,1]$ in the z-plane onto the exterior of the unit circle $|w| = 1$. Then, since $P_n \in \pi_n$, the function

$$F(z) := \frac{\exp(-a^2z^2)P_n(az)}{\exp(-a^2z\sqrt{z^2-1})[\psi(z)]^n} \quad , \tag{4.11}$$

is analytic in $\mathbb{C} - [-1,1]$, even at $z = \infty$ (the essential singularities as well as the poles cancel). Moreover, as $z \to [-1,1]$, we have

(4.12) $$|\exp(-a^2 z\sqrt{z^2-1})| \to 1 \quad \text{and} \quad |\psi(z)| \to 1 \quad .$$

Thus, from (4.8),

$$\limsup_{z \to [-1,1]} |F(z)| \le M \quad ,$$

and so, by the maximum principle,

(4.13) $$|F(z)| \le M \quad \forall\, z \in \mathbb{C} \quad .$$

In particular, this gives

(4.14) $$e^{-a^2x^2}|P_n(ax)| \le Mg_a(x) \quad \forall\, x \in \mathbb{R} \quad ,$$

where

(4.15) $$g_a(x) := |\exp(-a^2x\sqrt{x^2-1})|\,|\psi(x)|^n \quad .$$

Thus, for (4.9) to hold, we seek choices for the parameter a which ensure that

(4.16) $$g_a(x) < 1 \quad \forall\, x \in (-\infty,-1) \cup (1,+\infty) \quad .$$

Since $g_a(x) = 1 \;\; \forall\, x \in [-1,1]$ (cf. (4.12)), this will be true if $g_a'(x) > 0 \;\; \forall\, x < -1$ and $g_a'(x) < 0 \;\; \forall\, x > 1$.

Now, for $x > 1$, equations (4.10) and (4.15) yield

(4.17) $$\sqrt{x^2-1}\;\frac{g_a'(x)}{g_a(x)} = n - a^2(2x^2-1) \quad .$$

As $x \to 1^+$, the right-hand side of (4.17) approaches $n - a^2$. Thus, $g_a(x)$ decreases at $x = 1$ only when $n - a^2 \le 0$ or $a \ge \sqrt{n}$. In fact, it can be readily verified that the choice $a = \sqrt{n}$ yields (4.16) which, in turn, implies (4.9). We thus arrive at the following improvement of Theorem 4.1.

THEOREM 4.2. *If* $P_n \in \pi_n$ *and*

$$e^{-x^2}|P_n(x)| \le M \qquad \forall\ |x| \le \sqrt{n} \quad , \tag{4.18}$$

then

$$e^{-x^2}|P_n(x)| \le M \qquad \forall\ x \in \mathbb{R} \quad . \tag{4.19}$$

More generally,

$$e^{-|z|^2}|P_n(z)| \le M\left[G\left(2; \frac{z}{\sqrt{n}}\right)\right]^n \qquad \forall\ z \in \mathbb{C} \quad , \tag{4.20}$$

where

$$G(2;z) := \left|\psi(z)\exp\left(z^2 - |z|^2 - z\sqrt{z^2-1}\right)\right| \quad . \tag{4.21}$$

For arbitrary $\alpha > 0$, we can give a similar argument based on the properties of the potential

$$L(\alpha;z) := \int_{-1}^{1} \log|z-t|\,v(\alpha;t)\,dt \quad , \tag{4.22}$$

where $v(\alpha;t)$ is the Ullman distribution defined in (1.14). As shown in [16], $L(\alpha;z)$ is continuous on $\mathbb{C}$ and satisfies

$$L(\alpha;x) = \frac{|x|^\alpha}{\lambda_\alpha} - \log 2 - \frac{1}{\alpha} \quad , \qquad \forall\ x \in [-1,1] \quad , \tag{4.23}$$

where λ_α is the constant of (1.11) arising in the Freud conjecture. The generalization of Theorem 4.2 is

THEOREM 4.3 *If* $P_n \in \pi_n$ *and*

$$\exp(-|x|^{\alpha})|P_n(x)| \le M, \quad \forall\ |x| \le (n/\lambda_{\alpha})^{1/\alpha} \quad (\alpha > 0) \tag{4.24}$$

then

$$\exp(-|x|^{\alpha})|P_n(x)| \le M \qquad \forall\ x \in \mathbb{R} . \tag{4.25}$$

More generally,

$$\exp(-|z|^{\alpha})|P_n(z)| \le M\left[G\left(\alpha; \frac{z}{n^{1/\alpha}}\right)\right]^n, \qquad \forall\ z \in \mathbb{C}, \tag{4.26}$$

where

$$G(\alpha;z) := \exp\{L(\alpha;\lambda_{\alpha}^{1/\alpha}z) + \log 2 + \frac{1}{\alpha} - |z|^{\alpha}\} . \tag{4.27}$$

If, in Theorem 4.3, we take $P_n(z) = T_{n,\infty}(\alpha;z)$ with $M = E_{n,\infty}(\alpha)$ then, on letting $z \to \infty$ in (4.26), we obtain the lower estimates

$$n^{-1/\alpha}[E_{n,\infty}(\alpha)]^{1/n} \ge \frac{1}{2}\left(\frac{1}{e\lambda_{\alpha}}\right)^{1/\alpha}, \quad n = 1,2,\ldots . \tag{4.28}$$

As shown in [16], these estimates are sharp in a limiting sense:

THEOREM 4.4. *For each fixed* $\alpha > 0$,

$$\lim_{n\to\infty} n^{-1/\alpha}[E_{n,\infty}(\alpha)]^{1/n} = \frac{1}{2}\left(\frac{1}{e\lambda_{\alpha}}\right)^{1/\alpha} . \tag{4.29}$$

Returning to Theorem 4.3, let $\Omega(\alpha)$ denote the level curve

(4.30) $$\Omega(\alpha) := \{z \in \mathbb{C} : G(\alpha;z) = 1\}$$

and let $\Omega^e(\alpha)$ be its exterior:

(4.31) $$\Omega^e(\alpha) := \{z \in \mathbb{C} : G(\alpha;z) < 1\} .$$

As sketched in Figure 4.1, the curve $\Omega(\alpha)$ is symmetric about the real axis and contains the segment $[-\lambda_\alpha^{-1/\alpha}, \lambda_\alpha^{-1/\alpha}]$.

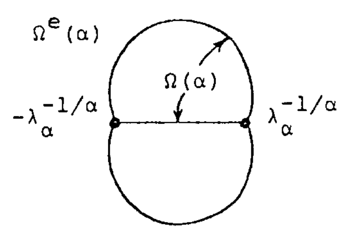

Figure 4.1

It plays the role of an "exponential tennis racket" as revealed in the following analog of Theorem 2.8.

THEOREM 4.5. *For fixed* $\alpha > 0$, *let* $\{B_n\}_{n=1}^{\infty}$ *be a sequence of functions of the form*

(4.32) $$B_n(z) = \exp(-n|z|^\alpha)p_n(z) , \quad p_n \in \pi_n ,$$

such that

(4.33) $$\limsup_{n \to \infty} \|B_n\|_{L_\infty}^{1/n} \le 1 .$$

Then

(4.34) $$\lim_{n\to\infty} B_n(z) = 0 \qquad \forall\, z \in \Omega^e(\alpha) \quad .$$

Furthermore, $\Omega^e(\alpha)$ *is the largest open set for which* (4.34), *in general, holds.*

We remark that, from Theorem 4.3, the norm in (4.33) can be replaced by the sup norm over the interval $[-\lambda_\alpha^{-1/\alpha}, \lambda_\alpha^{-1/\alpha}]$.

To further dramatize the analogy with incomplete polynomials, we give

DEFINITION 4.6. A function of the form (4.32), where $p_n \in \pi_n$, $n > 0$, is called an *exponentially weighted polynomial of type* α . The collection of all such functions (with $n > 0$ arbitrary) is denoted by EP_α .

If $B(\not\equiv 0) \in EP_\alpha$, we set

(4.35) $$\eta(B) := \max\{|\eta| : \eta \in \mathbb{R} ,\ |B(\eta)| = \|B\|_{L_\infty}\} \quad .$$

Then the following theorem of [16] furnishes a mate to the "θ^2-result" of Theorem 2.4.

THEOREM 4.7. *For each* $\alpha > 0$,

(4.36) $$\sup\{\eta(B) : B \in EP_\alpha ,\ B \not\equiv 0\} = \lambda_\alpha^{-1/\alpha} \quad ,$$

where λ_α *is defined in* (1.11).

The proof of Theorem 4.7 given in [16] utilizes a result (see Theorem 4.9 below) on the distribution of the zeros of the extremal polynomials $T_{n,\infty}(\alpha;x)$. Unlike the proof of Theorem 2.4 presented in §2, no appeal is made to the "Weierstrass approximation property" of exponentially weighted polynomials. In fact, there remains the following

Open Problem. Suppose $F \in C(\mathbb{R})$ and $\alpha > 0$ is fixed. Is it true that F is the uniform limit on $\mathbb{R}$ of EP_α functions if $F(x) = 0 \quad \forall\, |x| \geq \lambda_\alpha^{-1/\alpha}$?

Thus far, our discussion has focused on the L_∞-norm. For the other norms, we can appeal to the Nikolskii-type inequalities of [16] which show that, in an n-th root sense, the L_∞ and L_r-norms for exponentially weighted polynomials are asymptotically equivalent. Thus, as a consequence of Theorem 4.4, we obtain

THEOREM 4.8. *For each fixed* $\alpha > 0$ *and* $r > 0$,

$$\lim_{n\to\infty} n^{-1/\alpha}[E_{n,r}(\alpha)]^{1/n} = \frac{1}{2}\left(\frac{1}{e\lambda_\alpha}\right)^{1/\alpha} . \tag{4.37}$$

For the special case $r = 2$, it follows from (4.3) and (4.37) that

$$\lim_{n\to\infty} \frac{n^{-1/\alpha}}{[\gamma_n(\alpha)]^{1/n}} = \frac{1}{2}\left(\frac{1}{e\lambda_\alpha}\right)^{1/\alpha} , \tag{4.38}$$

which implies that Freud's conjecture of (1.10) is true in a Cesàro summability sense.

We conclude our discussion with a theorem on the distribution of the (necessarily real) zeros of the extremal polyfollowing result is established in [16].

THEOREM 4.9. Let $\alpha > 0$ and $r > 0$ be fixed. For any interval $[c,d]$ and for each $n = 1,2,\ldots$, let $N_{n,r}([c,d])$ denote the number of zeros of the normalized polynomials $T_{n,r}(\alpha;\ (n/\lambda_\alpha)^{1/\alpha}x)$ which lie in $[c,d]$. Then

$$\lim_{n\to\infty} n^{-1}N_{n,r}([c,d]) = \int_c^d v(\alpha;t)\,dt \quad , \tag{4.39}$$

where $v(\alpha;t)$ is the unit measure with support $[-1,1]$ defined in (1.14).

It is important to note that, in the special case when $r = 2$, the equations (4.3) and (4.39) yield the "contracted zero distribution" for the orthonormal polynomials $p_n(\alpha;x)$ of (1.8). Namely, (independent of the validity of Freud's conjecture) the result (1.12) of Ullman holds with X_n replaced by $(n/\lambda_\alpha)^{1/\alpha}$.

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