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BOUNDS FOR ALGEBRAIC POLYNOMIALS WITH ZEROS IN AN INTERVAL

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1. INTRODUCTION. In 1976, G.G. Lorentz [3, 4] initiated the study of polynomials of the form

$$(1.1) \qquad p_n(x) = x^s \sum_{k=0}^{m} a_k x^k, \qquad s + m = n, \quad s > 0 .$$

For these so-called *incomplete polynomials*, the following theorem is now well-known.

THEOREM A ([2,5]). Let $p_n(x)$ be an arbitrary (real or complex) polynomial of degree at most n, not identically zero, and assume $p_n(x)$ has a zero of order (at least) $s =: \theta n > 0$ at $x = 0$. If $|p_n(x)| \le M$ for $\theta^2 \le x \le 1$, then

$$(1.2) \qquad |p_n(x)| < M \quad \text{for} \quad -r_o(\theta) < x < \theta^2 ,$$

where $r_o(\theta)$ is a positive constant defined in (2.5) below. In particular, (1.2) implies

$$(1.3) \qquad \|p_n\|_{[0,1]} = \|p_n\|_{[\theta^2,1]} ,$$

where for any compact set B, $\|p_n\|_B := \max\{|p_n(z)| : z \in B\}$. Furthermore this result is best possible in the sense that the inequality (1.2) does not in general hold on any open interval properly containing $(-r_o(\theta), \theta^2)$.

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In this note, as in a recent paper [1] of M. v. Golitschek and G.G. Lorentz concerning trigonometric polynomials, we relax the assumption that $p_n(x)$ has an s order zero at $x = 0$ and instead only require that $p_n(x)$ have s zeros in an interval $[0,a]$. We shall prove

THEOREM 1. Let $p_n(x)$ be an arbitrary (real or complex) polynomial of degree at most n, not identically zero, and assume $p_n(x)$ has (at least) $s := \theta n > 0$ zeros in an interval $[0,a]$, where

$$(1.4) \qquad 0 \le a \le \lambda_o(\theta) := \frac{r_o(\theta)}{1+r_o(\theta)} .$$

If $|p_n(x)| \le M$ for $\theta^2 \le x \le 1$, then

$$(1.5) \qquad |p_n(x)| < M \quad \text{for} \quad -r_o(\theta) + a(1+r_o(\theta)) < x < \theta^2.$$

In particular, (1.5) implies

$$(1.6) \qquad \|p_n\|_{[0,1]} = \|p_n\|_{[\theta^2,1]} .$$

Moreover, this result is best possible in the following senses:

(i) the inequality (1.5) does not in general hold on any open interval properly containing $(-r_o(\theta)+a(1+r_o(\theta)),\theta^2)$, and

(ii) if $a > \lambda_o(\theta)$, then, in general, equation (1.6) is no longer true.

We defer the proof of Theorem 1 to §3. In certain special cases the numbers $r_o(\theta)$ and $\lambda_o(\theta)$ are known explicitly. In particular,

$$r_o(1/2) = 1/8, \quad r_o(2/3) = 1/4, \quad \text{and} \quad r_o(1) = 1,$$

which gives

$$\lambda_o(1/2) = 1/9, \quad \lambda_o(2/3) = 1/5, \quad \text{and} \quad \lambda_o(1) = 1/2.$$

More generally, numerical approximations are available. In Table 1.1 we display $r_o(\theta)$ and $\lambda_o(\theta)$ for θ in increments of 0.1.

θ	$r_0(\theta)$	$\lambda_0(\theta)$
.0	.000 000	.000 000
.1	.004 413	.004 394
.2	.017 914	.017 598
.3	.041 323	.039 683
.4	.076 170	.070 778
.5	.125 000	.111 111
.6	.191 973	.161 055
.7	.284 154	.221 277
.8	.414 613	.293 093
.9	.612 224	.397 739
1.0	1.000 000	.500 000

Table 1.1

The outline of this note is as follows. In §2 we give some necessary notation and review some known results relating to incomplete polynomials. In §3 we state and prove the remainder of our new results.

2. NOTATION AND KNOWN RESULTS. For each θ, with $0 \le \theta < 1$, define for $z \in \mathbf{C} \setminus [\theta^2,1]$

$$\phi(z) = \phi(z;\theta) := \frac{2z - (1+\theta^2) + 2\sqrt{(z-\theta^2)(z-1)}}{1-\theta^2}, \tag{2.1}$$

where $\sqrt{(z-\theta^2)(z-1)}$ has branch cut $[\theta^2,1]$ and behaves like z as z tends to infinity. The function $w = \phi(z)$ maps the exterior of the interval $[\theta^2,1]$ in the z-plane onto the exterior of the circle $|w| = 1$ in the w-plane. Next, for $0 \le \theta < 1$, put

$$G(z;\theta) := \begin{cases} 1, & z \in [\theta^2,1] \\ |\phi(z)| \left|\dfrac{\phi(z) - \phi(0)}{1 - \phi(0)\phi(z)}\right|^{\theta}, & z \in \mathbf{C}\setminus [\theta^2,1]. \end{cases} \tag{2.2}$$

In the case where $\theta = 1$, we extend the definition of $G(z;\theta)$ continuously by setting $G(z;1) := |z|$. We now state without proof the principle theorem for incomplete polynomials.

THEOREM B. ([2,5,6]). Let $p_n(x)$ be as in Theorem A. If $|p_n(x)| \leq M$ for $\theta^2 \leq x \leq 1$, then

$$|p_n(z)| \leq M\{G(z;\theta)\}^n \tag{2.3}$$

for all complex z. Moreover, this result is best possible in the sense that, for each $0 \leq \theta \leq 1$, there exists an infinite sequence of incomplete polynomials

$$\{P_k(x) = x^{s_k} \sum_{i=0}^{m_k} a_{ki}x^i : s_k \geq \theta(s_k+m_k)\}_{k=1}^{\infty}, \quad s_k + m_k \to \infty \text{ as } k \to \infty,$$

satisfying $\|P_k\|_{[\theta^2,1]} = 1$, $k = 1,2,\ldots$, and for which

$$\lim_{k\to\infty} |P_k(z)|^{1/n_k} = G(z;\theta), \quad n_k := s_k + m_k, \tag{2.4}$$

for each $z \in \mathbf{C} \setminus [\theta^2,1]$.

In [5] it was shown that for fixed θ $(0<\theta\leq 1)$, the function $G(t;\theta)$, considered as a function of the real variable t, is decreasing from infinity to zero on $(-\infty,0)$ and is increasing to 1 on $(0,\theta^2)$. Consequently, there exists a unique real number $r_o(\theta)$, with $r_o(\theta) > 0$, that satisfies

$$G(-r_o(\theta);\theta) = 1. \tag{2.5}$$

Since $G(t;\theta) < 1$ for $-r_o(\theta) < t < \theta^2$, Theorem A follows immediately from (2.3). Furthermore, Theorem B allows us to give a more precise estimate than (1.2).

THEOREM C. Let $p_n(x)$ be as in Theorem A, and let $[\alpha,\beta] \subset (-r_o(\theta),\theta^2)$. If $|p_n(x)| \leq M$ for $\theta^2 \leq x \leq 1$, then

$$|p_n(x)| \leq M\rho^n \tag{2.6}$$

for $\alpha \leq x \leq \beta$, where $\rho := \max\{G(\alpha;\theta),G(\beta;\theta)\} < 1$.

Theorem B also provides an extension of Theorem A to the complex plane. If we define, for $0 < \theta \le 1$,

$$\Omega(\theta) := \{z \in \mathbf{C}: \; G(z;\theta) < 1\}, \tag{2.7}$$

then under the assumptions of Theorem A, (1.2) will hold for all complex x in $\Omega(\theta)$.

We conclude our summary of results for incomplete polynomials with a theorem concerning sequences of such polynomials.

THEOREM D ([2,5]). Let $0 < \theta \le 1$ be fixed and suppose $\{Q_k(x)\}_{k=1}^{\infty}$ is a sequence of incomplete polynomials of the form

$$Q_k(x) = x^{s_k} \sum_{i=0}^{m_k} a_{ki}x^i, \quad s_k \ge \theta(s_k+m_k), \quad k = 1,2,\ldots, \tag{2.8}$$

where $s_k + m_k \to \infty$ as $k \to \infty$. If

$$\limsup_{k\to\infty} \|Q_k\|_{[\theta^2,1]}^{1/(s_k+m_k)} \le 1, \tag{2.9}$$

then

$$Q_k(z) \to 0 \quad \text{as} \quad k \to \infty, \tag{2.10}$$

for each z in $\Omega(\theta)$. In particular, (2.10) holds for $-r_o(\theta) < x < \theta^2$.

3. NEW RESULTS. In §2 it was shown that many familiar results for incomplete polynomials follow immediately if a suitably sharp upper bound for the polynomials can be determined. We use a similar approach in

THEOREM 2. Let $p_n(x)$ be a polynomial of degree at most n of the form

$$p_n(x) = q(x) \prod_{j=1}^{s} (x-x_j), \qquad q \in \pi_{n-s}, \tag{3.1}$$

where $s =: \theta n > 0$ and $0 \le x_j \le a < \theta^2$ for each $j = 1,\ldots,s$. If $|p_n(x)| \le M$ for $\theta^2 \le x \le 1$, then

$$(3.2)\qquad |p_n(z)| \le M\{G(z;\theta)\}^n \prod_{j=1}^{s}\left|\frac{\theta^2}{z}\cdot\frac{z-x_j}{\theta^2-x_j}\right|, \quad \forall z \ne 0 \ \text{ in } \ \mathbb{C},$$

and

$$(3.3)\qquad |p_n(z)| \le M\{G(\tfrac{z-a}{1-a};\ \theta)\}^n \prod_{j=1}^{s}\left|\frac{1-a}{z-a}\cdot\frac{z-x_j}{1-x_j}\right|, \quad \forall z \ne a \ \text{ in } \ \mathbb{C}.$$

As usual, the symbol π_k denotes the collection of all complex polynomials of degree at most k.

PROOF OF THEOREM 2. With $p_n(x)$ as in (3.1) define the two functions

$$(3.4)\qquad \Psi(x) := \prod_{j=1}^{s}\left(\frac{x}{x-x_j}\right) \quad \text{and} \quad \Phi(x) := \prod_{j=1}^{s}\left(\frac{x-a}{x-x_j}\right).$$

It is easy to verify that $\Psi(x)$ is nonincreasing for $x \in (a,\infty)$, while $\Phi(x)$ is nondecreasing on this interval. Consequently, for $\theta^2 \le x \le 1$,

$$(3.5)\qquad 0 < \Psi(x) \le \Psi(\theta^2) \quad \text{and} \quad 0 < \Phi(x) \le \Phi(1).$$

Next, we introduce the two constrained polynomials

$$(3.6)\qquad P(x) := \Psi(x)\cdot p_n(x) = x^s q(x),$$

and

$$(3.7)\qquad Q(x) := \Phi(x)\cdot p_n(x) = (x-a)^s q(x).$$

Since $P(x)$ is an incomplete polynomial and since $|P(x)| \le M\Psi(\theta^2)$ for $\theta^2 \le x \le 1$, we can apply Theorem B of §2 to obtain

$$(3.8)\qquad |P(z)| \le M\Psi(\theta^2)\{G(z;\theta)\}^n$$

for all complex z. Recalling the definitions of $P(x)$ and $\Psi(x)$, it is a simple matter to obtain (3.2).

Likewise, the constrained polynomial $Q(x)$ satisfies $|Q(x)| \le M\Phi(1)$ for $\theta^2 \le x \le 1$. After a simple linear transformation we again apply Theorem B to obtain

$$(3.9)\qquad |Q(z)| \le M\Phi(1)\{G(\tfrac{z-a}{1-a};\theta)\}^n$$

for all complex z. The inequality (3.3) now follows immediately from the definitions of $Q(x)$ and $\Phi(x)$. □

At first glance the upper bounds given by (3.2) and (3.3) appear to be rather crude, especially when compared to the asymptotically sharp result of Theorem B. However, these growth estimates will enable us to prove a more precise version of Theorem 1, from which Theorem 1 will follow as a corollary.

THEOREM 3. Let $p_n \in \pi_n$ be a polynomial of the form (3.1), where $s =: \theta n > 0$ and

$$(3.10)\qquad 0 \le x_j \le a < \lambda_o(\theta), \qquad j=1,\dots,s,$$

with $\lambda_o(\theta)$ defined as in (1.4). If

$$(3.11)\qquad [\alpha,\beta] \subset (-r_o(\theta)+a(1+r_o(\theta)),\theta^2),$$

and if $|p_n(x)| \le M$ for $\theta^2 \le x \le 1$, then

$$(3.12)\qquad |p_n(x)| \le M\rho^n \quad \text{for} \quad \alpha \le x \le \beta,$$

where

$$(3.13)\qquad \rho := \max\{G(\tfrac{\alpha-a}{1-a};\theta),\ G(\tfrac{-a}{1-a};\theta),\ G(a;\theta),\ G(\beta;\theta)\} < 1 .$$

The proof requires the following property of $G(t;\theta)$.

LEMMA. For each fixed θ, with $0 < \theta < 1$, the function $G(t;\theta)/|t|^\theta$, where t is a real variable, has a removable discontinuity at $t = 0$ and is a decreasing function on the interval $(-\infty,\theta^2)$.

Because the proof of this lemma involves a straight-forward but lengthy computation based on the definition in (2.2), we shall omit the details.

PROOF OF THEOREM 3. First, assume that $[0,a] \subset [\alpha,\beta]$. For each $j = 1,\dots,s$, we have

$$\left|\frac{\theta^2}{x}\cdot\frac{x-x_j}{\theta^2-x_j}\right|\le 1, \quad\text{for}\quad a\le x\le\theta^2 ,$$

and so inequality (3.2) implies that

$$|p_n(x)|\le M\{G(x;\theta)\}^n, \quad\text{for}\quad a\le x\le\theta^2 . \tag{3.14}$$

Since, as mentioned in §2, the function $G(t;\theta)$ is increasing from zero to one on the interval $0\le t\le\theta^2$, then (3.14) yields

$$|p_n(x)|\le M\{G(\beta;\theta)\}^n, \quad\text{for}\quad a\le x\le\beta. \tag{3.15}$$

On the other hand, for $\alpha\le x<a$,

$$\left|\frac{1-a}{x-a}\cdot\frac{x-x_j}{1-x_j}\right|\le\left|\frac{a-\alpha}{x-a}\right| , \quad j=1,\dots,s,$$

and so inequality (3.3) of Theorem 2 implies

$$|p_n(x)|\le M(a-\alpha)^s\left\{\frac{G\left(\frac{x-a}{1-a};\theta\right)}{|x-a|^\theta}\right\}^n, \quad\text{for}\quad \alpha\le x<a . \tag{3.16}$$

Taking $x=\alpha$ in the right-hand side of (3.16) and applying the above lemma, we find

$$|p_n(x)|\le M\left\{G\left(\frac{\alpha-a}{1-a};\theta\right)\right\}^n, \quad\text{for}\quad \alpha\le x<a . \tag{3.17}$$

It now follows, by combining (3.15) and (3.17), that

$$|p_n(x)|\le M\rho_o^n , \quad\text{for}\quad \alpha\le x\le\beta ,$$

where

$$\rho_o := \max\left\{G\left(\frac{\alpha-a}{1-a};\theta\right),\ G(\beta;\theta)\right\} .$$

Notice that from the hypotheses (3.10) and (3.11) the constant ρ_o is strictly less than one.

If $[0,a]\not\subset[\alpha,\beta]$, then we let $\alpha' := \min\{0,\alpha\}$ and $\beta' := \max\{a,\beta\}$. From the lemma we obtain

$$1/r_o(\theta)^\theta = G(-r_o(\theta);\theta)/r_o(\theta)^\theta \geq G(\theta^2;\theta)/\theta^{2\theta} = 1/\theta^{2\theta} ,$$

which implies $\theta^2 \geq r_o(\theta) > \lambda_o(\theta)$. Hence

$$[0,a] \subset [\alpha',\beta'] \subset (-r_o(\theta)+a(1+r_o(\theta)),\theta^2), \ [\alpha,\beta] \subset [\alpha',\beta'] ,$$

and so the first part of the proof applies. $\square$

PROOF OF THEOREM 1. If $0 \leq a < \lambda_o(\theta)$, then (3.12) of Theorem 3 immediately implies (1.5). Now assume that $a = \lambda_o(\theta)$, so that $-r_o(\theta) + a(1+r_o(\theta)) = 0$. Then as in the proof of Theorem 3, inequality (3.3) and the lemma of Theorem 3 imply

$$|p_n(x)| < Ma^s\left\{\frac{G(\frac{x-a}{1-a};\theta)}{|x-a|^\theta}\right\}^n \leq Ma^s\left\{\frac{G(\frac{0-a}{1-a};\theta)}{|0-a|^\theta}\right\}^n = M, \text{ for } 0 < x < a = \lambda_o(\theta).$$

Similarly, from inequality (3.2), we find

$$|p_n(x)| \leq M\{G(x;\theta)\}^n < M , \text{ for } a \leq x < \theta^2 .$$

Hence (1.5) holds for $a = \lambda_o(\theta)$.

To verify the sharpness statements, we first consider assertion (i) of Theorem 1. The value θ^2 cannot be replaced by any larger number $\theta^2 + \varepsilon$, $\varepsilon > 0$, since it is known [5] that there exist sequences of incomplete polynomials, satisfying the hypotheses of Theorem 1, which attain their absolute maximum on $[0,1]$ at points arbitrarily close to θ^2. To show that the point $-r_o(\theta) + a(1+r_o(\theta))$ cannot, in general, be decreased, we make use of the incomplete polynomials $P_k(x)$ satisfying (2.4). For each $0 < \theta \leq 1$ and $k \geq 1$, set

$$\hat{p}_k(x) := P_k\left(\frac{x-a}{1-a}\right) . \tag{3.18}$$

It is easy to see that $\|\hat{p}_k\|_{[\theta^2,1]} \leq 1$ and that $\hat{p}_k(x)$ has a zero of order (at least) s_k at $x = a$, where $s_k \geq \theta n_k > 0$. Furthermore, for $x < -r_o(\theta) + a(1+r_o(\theta))$, we have

$$|\hat{p}_k(x)|^{1/n_k} = |P_k(t)|^{1/n_k}, \text{ where } t := \left(\frac{x-a}{1-a}\right) < -r_o(\theta).$$

Consequently, together with (2.4) and (2.5), we have for such x,

$$(3.19)\qquad \lim_{k\to\infty} |\hat{p}_k(x)|^{1/n_k} = \lim_{k\to\infty} |P_k(t)|^{1/n_k} = G(t;\theta) > 1,$$

which completes the proof of assertion (i).

As for statement (ii), if $\lambda_o(\theta) < a \leq \theta^2$, then the same sequence defined in (3.18) satisfies $\|\hat{p}_k\|_{[\theta^2,1]} \leq 1$ and

$$(3.20)\qquad \|\hat{p}_k\|_{[0,1]}^{1/n_k} \geq |\hat{p}_k(0)|^{1/n_k} = |P_k(-r_o(\theta)-\varepsilon)|^{1/n_k},\ k=1,2,\ldots,$$

for some $\varepsilon > 0$. But this last fact, with (2.4), implies

$$(3.21)\qquad \liminf_{k\to\infty} \|\hat{p}_k\|_{[0,1]}^{1/n_k} \geq G(-r_o(\theta)-\varepsilon;\theta) > 1 . \qquad \square$$

As an immediate consequence of Theorem 3 we have

THEOREM 4. Let $0 < \theta \leq 1$ and let $\{p_k(x)\}_{k=1}^{\infty}$ denote an infinite sequence of polynomials of increasing degrees n_k, respectively, having at least $s_k \geq \theta n_k$ zeros in the interval $[0,a]$, where $0 \leq a < \lambda_o(\theta)$. If

$$(3.22)\qquad \limsup_{k\to\infty} \|p_k\|_{[\theta^2,1]}^{1/n_k} \leq 1 ,$$

then

$$(3.23)\qquad p_k(x) \to 0 \quad \text{for} \quad -r_o(\theta) + a(1+r_o(\theta)) < x < \theta^2 , \text{ as } k \to \infty .$$

The convergence in (3.23) is uniform and geometric on any closed subinterval $[\alpha,\beta] \subset (-r_o(\theta)+a(1+r_o(\theta)),\theta^2)$.

We finally remark that, using Theorem 2, both Theorem 3 and Theorem 4 can be extended to a region of the complex plane which contains the real interval $-r_o(\theta) + a(1+r_o(\theta)) < x < \theta^2$.

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