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A NOTE ON THE SHARPNESS OF J. L. WALSH'S THEOREM AND ITS EXTENSIONS FOR INTERPOLATION IN THE ROOTS OF UNITY

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§ 1. Introduction and statements of new results

Let A_ϱ denote the collection of functions analytic in $|z|<\varrho$ and having a singularity on the circle $|z|=\varrho$, where it is assumed that $1<\varrho<\infty$. Next, for each positive integer n, let $p_{n-1}(z; f)$ denote the Lagrange polynomial interpolant, of degree at most $n-1$, of $f(z)\in A_\varrho$ in the n-th roots of unity, i.e.,

$$p_{n-1}(\omega; f) = f(\omega) \tag{1.1}$$

where ω is any n-th root of unity, and let

$$P_{n-1}(z;f) := \sum_{k=0}^{n-1} a_k z^k \tag{1.2}$$

be the $(n-1)$-st partial sum of $f(z)=\sum_{k=0}^{\infty} a_k z^k$. Letting

$$D_\tau := \{z\in \mathbf{C}\colon\ |z| < \tau\}, \tag{1.3}$$

then a beautiful result of J. L. Walsh [2, p. 153] can be stated as

THEOREM A. *For each $f(z)\in A_\varrho$, the interpolating polynomials of* (1.1) *and* (1.2) *satisfy*

$$\lim_{n\to\infty} \{p_{n-1}(z; f)-P_{n-1}(z; f)\} = 0, \quad \textit{for all} \quad z\in D_{\varrho^2}. \tag{1.4}$$

Moreover, the result of (1.4) *is best possible in the sense that there is some $\hat f(z)\in A_\varrho$ and some $\hat z$ with $|\hat z|=\varrho^2$ for which the sequence $\{p_{n-1}(\hat z; \hat f)-P_{n-1}(\hat z; \hat f)\}_{n=1}^{\infty}$ does not tend to zero as $n\to\infty$.*

Note that in Theorem A, *no* sharpness assertions are made for *arbitrary* functions $f(z)\in A_\varrho$; in particular, no statement is made on the behavior of the sequence

$$\{p_{n-1}(z; f)-P_{n-1}(z; f)\}_{n=1}^{\infty} \tag{1.5}$$

in $|z|>\varrho^2$. One of the aims of this note is to in fact address this behavior in $|z|>\varrho^2$. As a special case of Theorem 1 below, we prove that, for *any* $f(z)\in A_\varrho$, the sequence in (1.5) can be bounded in at most *one* point in $|z|>\varrho^2$. This fact is of special interest in the case when $f(z)$ in A_ϱ is also continuous in the disk $|z|\le\varrho$; for such functions, it has been shown in [1, Thm. 2] that (1.4) is valid for *all* $|z|\le\varrho^2$.

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For our own purposes below, we need a recent extension of Theorem A. For additional notation, set

$$P_{n-1,j}(z;f) := \sum_{k=0}^{n-1} a_{k+jn} z^k, \quad j=0,1,\ldots. \tag{1.6}$$

Then, the following result of Cavaretta, Sharma, and Varga [1, Thm. 1], which gives Theorem A as the special case $l=1$, can be stated as

THEOREM B. *For each $f(z)\in A_\varrho$, and for each positive integer l, there holds*

$$\lim_{n\to\infty}\left\{p_{n-1}(z;f)-\sum_{j=0}^{l-1}P_{n-1,j}(z;f)\right\}=0, \quad \textit{for all} \quad z\in D_{\varrho^{l+1}}, \tag{1.7}$$

the convergence being uniform and geometric on any closed subset of $D_{\varrho^{l+1}}$. Moreover, the result of (1.7) *is best possible in the sense that there is some $\tilde f(z)\in A_\varrho$ and some $\tilde z$ with $|\tilde z|=\varrho^{l+1}$ for which the sequence*

$$\left\{p_{n-1}(z;f)-\sum_{j=0}^{l-1}P_{n-1,j}(z;f)\right\}_{n=1}^{\infty} \tag{1.8}$$

with $z=\tilde z$ and $f=\tilde f$, does not tend to zero as $n\to\infty$.

Our first new result is

THEOREM 1. *For each $f(z)\in A_\varrho$, and for each positive integer l, the sequence* (1.8) *can be bounded in at most l distinct points in $|z|>\varrho^{l+1}$. This result is sharp, in the sense that, given any l distinct points $\{\eta_k\}_{k=1}^{l}$ in the annulus $\varrho^{l+1}<|z|<\varrho^{l+2}$, there is an $\hat f(z)\in A_\varrho$ for which*

$$\lim_{n\to\infty}\left\{p_{n-1}(\eta_k;\hat f)-\sum_{j=0}^{l-1}P_{n-1,j}(\eta_k;\hat f)\right\}=0, \quad k=1,2,\ldots,l. \tag{1.9}$$

There is an extension of Theorem 1 which we can also state. Note, of course, that Theorem A involves only the Lagrange interpolation of f in the n-th roots of unity. For r a fixed positive integer, Theorem B can be extended using Hermite interpolation. For notation, let $h_{rn-1}(z;f)$ denote the Hermite polynomial interpolant, of degree at most $rn-1$, to $f, f', \ldots, f^{(r-1)}$ in the n-th roots of unity, i.e.,

$$h_{rn-1}^{(j)}(\omega;f)=f^{(j)}(\omega), \quad j=0,1,\ldots,r-1, \tag{1.10}$$

where again ω is any n-th root of unity. If $f(z)=\sum_{j=0}^{\infty}a_jz^j$, we set

$$H_{rn-1,0}(z;f) := \sum_{k=0}^{rn-1} a_k z^k, \tag{1.11}$$

and we set

$$H_{rn-1,j}(z;f) := \hat\beta_j(z^n)\sum_{k=0}^{n-1} a_{k+n(r+j-1)}z^k, \quad j=1,2,\ldots, \tag{1.12}$$

where

$$\hat\beta_j(z) := \sum_{k=0}^{r-1}\binom{r+j-1}{k}(z-1)^k, \quad j=1,2,\ldots. \tag{1.13}$$

Then, the following result of Cavaretta, Sharma, and Varga [1, Thm. 3], which gives Theorem B as the special case $r=1$, can be stated as

THEOREM C. *For each $f(z)\in A_\varrho$, and for each pair of positive integers r and l, there holds*

$$\lim_{n\to\infty}\left\{h_{rn-1}(z; f)-\sum_{j=0}^{l-1} H_{rn-1,j}(z; f)\right\}=0, \quad \text{for all} \quad z\in D_{\varrho^{1+(l/r)}}, \tag{1.14}$$

the convergence being uniform and geometric for any closed subset of $D_{\varrho^{1+(l/r)}}$. Moreover, the result of (1.14) is best possible in the sense that there is some $\hat{f}(z)\in A_\varrho$ and some $\hat{z}$ with $|\hat{z}|=\varrho^{1+(l/r)}$ for which the sequence

$$\left\{h_{rn-1}(z; f)-\sum_{j=0}^{l-1} H_{rn-1,j}(z; f)\right\}_{n=1}^{\infty}, \tag{1.15}$$

with $z=\hat{z}$ and $f=\hat{f}$, does not tend to zero as $n\to\infty$.

Our second new result, which sharpens Theorem C and gives Theorem 1 as the special case $r=1$, can be stated as

THEOREM 2. *For each $f(z)\in A_\varrho$, and for each pair of positive integers r and l, the sequence* (1.15) *can be bounded in at most $r+l-1$ distinct points in $|z|>\varrho^{1+(l/r)}$. This result is sharp, in the sense that, given any $r+l-1$ distinct points $\{\eta_k\}_{k=1}^{r+l-1}$ in the annulus $\varrho^{1+(l/r)}<|z|<\min\left\{\varrho^{l+2};\ \varrho^{1+\frac{l}{r-1}}\right\}$, there is an $\tilde{f}(z)\in A_\varrho$ for which*

$$\lim_{n\to\infty}\left\{h_{rn-1}(\eta_k; \tilde{f})-\sum_{j=0}^{l-1} H_{rn-1,j}(\eta_k; \tilde{f})\right\}=0, \quad k=1,2,\dots,r+l-1. \tag{1.16}$$

Since the proof of Theorem 2 is completely analogous to the proof of Theorem 1, we shall give only the proof of Theorem 1.

§ 2. Proof of Theorem 1

To establish the first part of Theorem 1, consider any (fixed $f\in A_\varrho$, consider any fixed positive integer l, and suppose that there are $(l+1)$ distinct points $\{y_k\}_{k=1}^{l+1}$ in $|z|>\varrho^{l+1}$ for which

$$\left|p_{n-1}(y_k; f)-\sum_{j=0}^{l-1} P_{n-1,j}(y_k; f)\right|\leqq M, \quad \forall n\geqq 1, \ \forall 1\leqq k\leqq l+1. \tag{2.1}$$

If $f(z):=\sum_{j=0}^{\infty} a_j z^j$, then the hypothesis that f is analytic in $|z|<\varrho$ with a singularity on $|z|=\varrho$ gives us that

$$\overline{\lim_{n\to\infty}}\, |a_n|^{1/n}=\frac{1}{\varrho}. \tag{2.2}$$

Thus, for any $\varepsilon>0$ with $1<\varrho-\varepsilon$ and with

$$(\varrho-\varepsilon)^{l+2}>\varrho^{l+1}, \tag{2.3}$$

there is an $n_0(\varepsilon)$ for which

$$|a_n| \leqq \frac{1}{(\varrho-\varepsilon)^n}, \quad \forall n \geqq n_0(\varepsilon). \tag{2.4}$$

Next, since all the points $\{y_k\}_{k=1}^{l+1}$ lie in $|z|>\varrho^{l+1}$, then

$$\varrho^{l+1}<\sigma_1:=\min_{1\leqq k\leqq l+1}|y_k| \leqq \max_{1\leqq k\leqq l+1}|y_k|=:\sigma_2, \tag{2.5}$$

and we choose the least positive integer m for which

$$\sigma_2<\varrho^{m+1}, \quad (\text{where } l<m). \tag{2.6}$$

Applying Theorem B (with l chosen as m), we have that the sequence $\left\{p_{n-1}(z;f)-\sum_{j=0}^{m-1}P_{n-1,j}(z;f)\right\}_{n=1}^{\infty}$ converges to zero for all $z\in D_{\varrho^{m+1}}$. In particular, as the points $\{y_k\}_1^{l+1}$ all lie in $D_{\varrho^{m+1}}$ from (2.5) and (2.6), then there exists a constant M_1 such that

$$\left|p_{n-1}(y_k;f)-\sum_{j=0}^{m-1}P_{n-1,j}(y_k;f)\right| \leqq M_1, \quad \forall n\geqq 1, \quad \forall 1\leqq k\leqq l+1. \tag{2.7}$$

Using the hypothesis of (2.1), this in turn implies that

$$\left|\sum_{j=l}^{m-1}P_{n-1,j}(y_k;f)\right| \leqq M_2, \quad \forall n\geqq 1, \quad \forall 1\leqq k\leqq l+1. \tag{2.8}$$

Recalling from (1.6) the definition of $P_{n-1,j}(z;f)$, then it follows from (2.4) that

$$|P_{n-1,j}(z;f)| \leqq \sum_{k=0}^{n-1}\frac{|z|^k}{(\varrho-\varepsilon)^{k+jn}}=\frac{1}{(\varrho-\varepsilon)^{jn}}\sum_{k=0}^{n-1}\left(\frac{|z|}{\varrho-\varepsilon}\right)^k, \quad \forall n\geqq n_0(\varepsilon).$$

Thus,

$$|P_{n-1,j}(z;f)| \leqq \frac{n|z|^n}{(\varrho-\varepsilon)^{(j+1)n}}, \quad \forall n\geqq n_0(\varepsilon), \quad \forall |z|>\varrho, \quad \forall j\geqq 1. \tag{2.9}$$

This can be used as follows. From (2.9), we see that, if $l+1\leqq m-1$, then

$$\left|\sum_{j=l+1}^{m-1}P_{n-1,j}(z;f)\right| \leqq \frac{(m-l-1)n|z|^n}{(\varrho-\varepsilon)^{(l+2)n}}, \quad \forall n\geqq n_0(\varepsilon), \quad \forall |z|>\varrho. \tag{2.10}$$

Hence, from (2.8) and (2.10),

$$|P_{n-1,l}(y_k;f)| \leqq M_2+\frac{(m-l-1)n|y_k|^n}{(\varrho-\varepsilon)^{(l+2)n}}, \quad \forall n\geqq n_0(\varepsilon), \quad \forall 1\leqq k\leqq l+1. \tag{2.11}$$

Now, because of (2.11), it further follows that

$$|y_k^l P_{n,l}(y_k; f) - P_{n-1,l}(y_k; f)| \leqq M_3 + \frac{M_4 n |y_k|^n}{(\varrho-\varepsilon)^{(l+2)n}}, \tag{2.12}$$

for all $n \geqq n_0(\varepsilon)$, all $1 \leqq k \leqq l+1$. Next, because of the definition of $P_{n-1,j}(z; f)$, it can be verified that

$$z^l P_{n,l}(z; f) - P_{n-1,l}(z; f) = \sum_{j=n}^{l+n} a_{ln+j} z^j - \sum_{j=0}^{l-1} a_{ln+j} z^j. \tag{2.13}$$

Obviously, the last term in (2.13) is bounded, independent of n, in the points $\{y_k\}_{k=1}^{l+1}$, whence from (2.12) and (2.13),

$$\left|\sum_{j=n}^{l+n} a_{ln+j} y_k^j\right| \leqq M_5 + \frac{M_4 n |y_k|^n}{(\varrho-\varepsilon)^{(l+2)n}}. \tag{2.14}$$

On dividing through by $|y_k|^n$ in (2.14), we obtain

$$\left|\sum_{j=0}^{l} a_{n(l+1)+j} y_k^j\right| \leqq \frac{M_5}{|y_k|^n} + \frac{M_4 n}{(\varrho-\varepsilon)^{(l+2)n}}, \tag{2.15}$$

and so, from the definition of σ_1 in (2.5), there follows

$$\left|\sum_{j=0}^{l} a_{n(l+1)+j} y_k^j\right| \leqq \frac{M_5}{\sigma_1^n} + \frac{M_4 n}{(\varrho-\varepsilon)^{(l+2)n}}, \tag{2.16}$$

for all $n \geqq n_0(\varepsilon)$, all $1 \leqq k \leqq l+1$. If, for convenience, we set

$$\tau := \max\left\{\frac{1}{\sigma_1}; \frac{1}{(\varrho-\varepsilon)^{l+2}}\right\}, \tag{2.17}$$

then it follows from (2.3) and (2.5) that

$$\tau < \frac{1}{\varrho^{l+1}}. \tag{2.18}$$

Next, we write a system of $(l+1)$ linear equations in the "unknowns" $a_{(l+1)n+j}$, i.e.,

$$\sum_{j=0}^{l} y_k^j a_{(l+1)n+j} =: f_{k,n}, \quad k = 1, 2, \ldots, l+1 \tag{2.19}$$

where, from (2.16) and (2.17),

$$|f_{k,n}| \leqq M_6 n \tau^n, \quad \forall n \geqq n_0(\varepsilon), \quad \forall 1 \leqq k \leqq l+1. \tag{2.20}$$

In matrix notation, we can write the system of equations (2.19) as

$$\begin{bmatrix} 1 & y_1 & \cdots & y_1^l \\ 1 & y_2 & \cdots & y_2^l \\ \vdots & & & \vdots \\ 1 & y_{l+1} & \cdots & y_{l+1}^l \end{bmatrix} \cdot \begin{bmatrix} a_{(l+1)n} \\ a_{(l+1)n+1} \\ \vdots \\ a_{(l+1)n+l} \end{bmatrix} = \begin{bmatrix} f_{1,n} \\ f_{2,n} \\ \vdots \\ f_{l+1,n} \end{bmatrix}. \tag{2.21}$$

The coefficient matrix, Δ, in (2.21) is a Vandermonde matrix, and, as the points $\{y_k\}_{k=1}^{l+1}$ are *distinct* by hypothesis, then Δ is nonsingular. Using Cramer's rule, it is easy to see from (2.20) and the fact that the $\{y_k\}_1^{l+1}$ are fixed distinct points, that

$$|a_{(l+1)n+j}| \leqq M_7 n\tau^n, \quad \forall n \geqq n_0(\varepsilon), \quad \forall 0 \leqq j \leqq l. \tag{2.22}$$

However, (2.22) implies that

$$\varlimsup_{n\to\infty} |a_n|^{1/n} \leqq \tau^{1/(l+1)} < \frac{1}{\varrho}, \tag{2.23}$$

the last inequality coming from (2.18). As this contradicts (2.2), then there can be at most l distinct points $\{\eta_k\}_{k=1}^{l}$ in $|z|>\varrho^{l+1}$ for which the sequence (1.8) is bounded, completing thefirst part of the proof.

To establish the second part of Theorem 1, let $w_l(z)$ be any monic polynomial of degree l with precisely l distinct zeros in the annulus $\varrho^{l+1}<|z|<\varrho^{l+2}$, i.e.,

$$w_l(z) = \prod_{k=1}^{l}(z-\eta_k) =: \sum_{j=0}^{l} \beta_j z^j, \tag{2.24}$$

where

$$\varrho^{l+1} < |\eta_k| < \varrho^{l+2} \quad \text{for} \quad k=1,2,\ldots,l. \tag{2.25}$$

Consider then the particular function

$$\hat{f}(z) := \frac{w_l(z)}{\varrho^{l+1}-z^{l+1}}. \tag{2.26}$$

Clearly, $\hat{f}\in A_\varrho$, and $\hat{f}$ has $l+1$ poles on $|z|=\varrho$. We now show that with these definitions, (1.9) of Theorem 1 is satisfied. From Theorem B, we know that

$$\lim_{n\to\infty}\left\{p_{n-1}(z;\hat{f}) - \sum_{j=0}^{l} P_{n-1,j}(z;\hat{f})\right\} = 0, \quad \forall z\in D_{\varrho^{l+2}}. \tag{2.27}$$

We claim that

$$\lim_{n\to\infty} P_{n-1,l}(\eta_k;\hat{f}) = 0, \quad \forall 1 \leqq k \leqq l. \tag{2.28}$$

To establish (2.28), write $\hat{f}(z) := \sum_{k=0}^{\infty} \hat{a}_k z^k$. It follows from (2.24) and (2.26) that

$$\hat{a}_{m(l+1)+j} = \frac{\beta_j}{\varrho^{(m+1)(l+1)}}, \quad \forall 0 \leqq j \leqq l, \quad \forall m \geqq 0. \tag{2.29}$$

Next, by definition,

$$P_{n-1,l}(z;\hat{f}) = \sum_{k=0}^{n-1} \hat{a}_{ln+k} z^k, \tag{2.30}$$

and we consider the case when n is a multiple of $(l+1)$, i.e., $n=(l+1)s$. On regrouping terms in (2.30) for such n, $P_{n-1,l}(z:\hat{f})$ can be expressed as

$$P_{s(l+1)-1,l}(z;\hat{f}) = \sum_{k=0}^{s-1} z^{k(l+1)} \sum_{j=0}^{l} \hat{a}_{(l+1)[sl+k]+j} z^j. \tag{2.31}$$

But, the inner sum of (2.31) can be seen from (2.29) and (2.24) to be

$$\sum_{j=0}^{l} \hat{a}_{(l+1)[sl+k]+j} z^j = \frac{w_l(z)}{\varrho^{(l+1)[sl+k+1]}}. \tag{2.32}$$

Since $w_l(\eta_k)=0$ by definition, it follows from (2.31) that

$$P_{s(l+1)-1,l}(\eta_k;\hat{f}) = 0, \quad \forall 1 \leqq k \leqq l, \quad \forall s \geqq 1. \tag{2.33}$$

Having just considered the case when n is a multiple of $(l+1)$, we now suppose that $n=s(l+1)+t$, where $1 \leqq t \leqq l$. On similarly regrouping the terms in (2.30) and using the fact that $w_l(\eta_k)=0$, it can be shown that

$$P_{s(l+1)+t-1,l}(\eta_k;\hat{f}) = \sum_{j=0}^{t-1} \hat{a}_{sl(l+1)+lt+j}\eta_k^j. \tag{2.34}$$

Since the $\{\eta_k\}_{k=1}^{l}$ are fixed, and t does not exceed l, then, as $|\hat{a}_n| \to 0$ as $n \to \infty$ from (2.29), we have from (2.33) and (2.34) that

$$\lim_{n\to\infty} P_{n-1,l}(\eta_k;\hat{f}) = 0, \quad \forall 1 \leqq k \leqq l, \tag{2.35}$$

as claimed in (2.28). Thus, with (2.27) and the first part of Theorem 1, the sequence

$$\left\{p_{n-1}(z;\hat{f}) - \sum_{j=0}^{l-1} P_{n-1,j}(z;\hat{f})\right\}_{n=1}^{\infty} \tag{2.36}$$

is convergent (to zero), only in the points $\{\eta_k\}_{k=1}^{l}$ and unbounded for all other points in $\{z \in \mathbf{C}: |z| > \varrho^{l+1}\}$.

Added in proof. (April 14, 1983) The second part of Theorem 1 remains valid if any l distinct points $\{\eta_k\}_{k=1}^{l}$ are arbitrarily chosen in $|z| > \varrho^{l+1}$, with a similar improvement holding for Theorem 2. This has been shown by the author and, more generally by T. Hermann, "Some remarks on an extension of a Theorem of Walsh", *J. Approx. Th.* (to appear).

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