SINGULARITIES OF FUNCTIONS DETERMINED BY THE POLES OF PADÉ APPROXIMANTS

J. Karlsson
Department of Mathematics
University of Umeå
S-901 87 Umeå
Sweden

E. B. Saff †
Center for Mathematical Services
University of South Florida
Tampa, Florida 33620
USA

1. Introduction.

Given a (formal) power series $f(z) = \sum a_n z^n$, the classical Padé approximant of type $[n/\nu]$ is a rational function $P_{n,\nu}/Q_{n,\nu}$, where $P_{n,\nu}, Q_{n,\nu} (\neq 0)$ are polynomials of respective degrees at most n and ν, and such that the power series of $Q_{n,\nu}f-P_{n,\nu}$ starts with terms of degree $\geq n+\nu+1$. When qualitative properties of $f(z)$ are known (such as the existence and number of poles, branch points, etc.), a fundamental question in the study of Padé approximants is whether the poles of these approximants tend to the singularities of $f(z)$. Some classical results in this direction include the theorem of Montessus de Ballore on meromorphic functions [9](cf. [13]) and results for Stieltjes series [11], [7].

Much less is known, however, concerning the inverse problem. Here the essential question is the following. Suppose that $f(z)$ is a formal power series and that the poles of some sequence of its Padé approximants converge to a set L. Does it follow that f (or some continuation of f) is singular on L? A related problem is whether the function f is actually analytic off L.

The purpose of this paper is to survey known results concerning the inverse problem and to present some proofs of theorems previously announced [6] by the authors. In sections 2 and 3 we consider rational interpolants with fixed denominator degree. For the special case of Padé approximants of type $[n/1]$ our result in §2 concerns the validity of what physicists call the Domb-Sykes method [3]. In §3 we study rational interpolation in more general triangular schemes, and in §4 we discuss diagonal sequences of Padé approximants.

2. Padé Approximants with Fixed Denominator Degree

The earliest result in the inverse direction is the following theorem due to Fabry [2, p. 377], which can be regarded as a refinement of the ratio test.

Theorem 2.1. (Fabry) *Suppose that* $f(z) = \sum_0^\infty a_n z^n$ *is a (formal) power series for which*

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \alpha .$$

†The research of this author was supported, in part, by the National Science Foundation.

Then $f(z)$ is analytic in the disk $|z|<|\alpha|$ and α is a singularity of $f(z)$.

Of course the conclusion regarding the radius of convergence is evident, so the significant part is that α is actually a singularity of $f(z)$. Now it is easy to see, directly from the definition, that a_n/a_{n+1} is the pole of the Padé approximant of type $[n/1]$ to $f(z)$. Thus Fabry's theorem can be reformulated as a result concerning the second row of the Padé table: *If the poles of the* $[n/1]$ *Padé approximants for* f *converge to* α, *then* α *is a singularity of* f.

A substantial generalization of Theorem 2.1 to other rows of the Padé table was recently proved by Vavilov, Lopez, and Prohorov [12]. They established

Theorem 2.2. *Suppose that* f *is analytic at the origin, and that the denominators* $Q_{n,\nu}$ *(suitably normalized) of the Padé approximants of type* $[n/\nu]$, $\nu>0$ *fixed,* $n=0,1,2,\ldots$, *converge to a polynomial* Q *of degree* ν. *If* Q *has a single zero* α *of largest modulus and this zero is simple, then* f *is meromorphic with precisely* $\nu-1$ *poles (at the smaller zeros of* Q*) in the disk* $|z|<|\alpha|$. *Furthermore,* f *has a singularity at* α.

If in Fabry's theorem (or Theorem 2.2), additional information is given regarding the *degree of convergence* of the poles of Padé approximants, can we then describe the precise nature of the singularity at α? A fundamental theorem in this regard was obtained by Kovačeva [8] who studied the case of geometric convergence of the poles. She proved the following result which is the converse of the Montessus de Ballore theorem:

Theorem 2.3. *Suppose* f *is analytic at the origin and there exists a polynomial* Q *of degree* ν, *with* $Q(0)\neq 0$, *such that the denominators* $Q_{n,\nu}$ *(suitably normalized) of the Padé approximants of type* $[n/\nu]$, $n=0,1,\ldots$, *satisfy* $\|Q_{n,\nu}-Q\| = O(R^{-n})$, $R>1$, *where* $\|\cdot\|$ *represents any of the equivalent norms on the* $(\nu+1)$*-dimensional space of polynomial coefficients. Then* f *is meromorphic with precisely* ν *poles (at the zeros of* Q*) in the disk* $|z|<R|\alpha_\nu|$, *where* $|\alpha_\nu|$ *is the maximum modulus of the zeros of* Q.

Note that, unlike Theorem 2.2, no restriction is made here on the number of limiting poles of largest modulus. We also remark that by generalizing the Hadamard theory, Kovačeva extended Theorem 2.3 to the case of Newton series. In §3 we shall return to this result and deduce a similar theorem for even more general interpolation schemes.

Regarding slower than geometric convergence, there is a technique related to algebraic singularities which is known as the Domb-Sykes method [3]. This method is based on the observation (letting the singular point $\alpha=1$ for convenience) that if $f(z) = \sum_{0}^{\infty} a_n z^n$ is of the form

(2.1) $f(z) = C(1-z)^{\lambda} + h(z),\ \lambda \notin \mathbb{N}$, or $f(z) = C(1-z)^{\lambda}\log(1-z) + h(z),\ \lambda \in \mathbb{N}$,†

where $h(z)$ is analytic on $|z| \leq R$, $R>1$, then

$$a_{n+1}/a_n = 1 - (1+\lambda)/(n+1) + O(R^{-n}) \ . \tag{2.2}$$

Graphically, (2.2) implies that when the successive ratios (the reciprocals of the poles of the Padé approximants of type [n/1]) are plotted against $1/(n+1)$, then they asymptotically lie on a straight line. In the inverse direction, we can easily prove that *if (2.2) holds for a function* f, *then* f *must be of the form (2.1), where* $h(z)$ *is analytic in* $|z|<R$. This is a special case of our

Theorem 2.4. Suppose that f is analytic at the origin and $\nu>0$ is fixed. Let $\left\{\alpha_{n,\nu}^{(k)}\right\}_{k=1}^{\nu}$ denote the zeros of the denominator $Q_{n,\nu}$ of the Padé approximant of type $[n/\nu]$ for f, where

$$|\alpha_{n,\nu}^{(0)}| \leq |\alpha_{n,\nu}^{(1)}| \leq \cdots \leq |\alpha_{n,\nu}^{(\nu)}| \ .$$

Suppose there exist ν nonzero complex numbers $\left\{\alpha^{(k)}\right\}_{k=1}^{\nu}$ such that

$$\left|\alpha^{(\nu-1)}/\alpha^{(\nu)}\right| < 1/R, \quad R>1 \ , \tag{2.3}$$

$$\lim_{n\to\infty} \alpha_{n,\nu}^{(k)} = \alpha^{(k)}, \quad k = 1,2,\ldots,\nu-1, \tag{2.4}$$

and

$$\frac{1}{\alpha_{n,\nu}^{(\nu)}} = \frac{1}{\alpha^{(\nu)}}\left[1 - \frac{1+\lambda}{n+\nu}\right] + O(R^{-n}) \ , \ \text{as} \ n\to\infty. \tag{2.5}$$

Then, if $\lambda \notin \mathbb{N}$, the function f must be of the form

$$f(z) = \frac{C(z-\alpha^{(\nu)})^{\lambda} + h(z)}{\prod_{k=1}^{\nu-1}(z-\alpha^{(k)})^2} \ , \tag{2.6}$$

---

† $\mathbb{N}$ denotes the set of nonnegative integers.

where h *is analytic in* $|z| < R|\alpha^{(\nu)}|$ *and* $h(\alpha^{(k)}) = -C(\alpha^{(k)}-\alpha^{(\nu)})^{\lambda}$, *for* k=1, ..., ν-1. *If* $\lambda \in \mathbb{N}$, *then*

$$(2.7) \qquad f(z) = \frac{C(z-\alpha^{(\nu)})^{\lambda}\log(z-\alpha^{(\nu)}) + h(z)}{\prod_{k=1}^{\nu-1} (z-\alpha^{(k)})^2},$$

where h *is analytic in* $|z| < R|\alpha^{(\nu)}|$ *and* $h(\alpha^{(k)}) = -C(\alpha^{(k)}-\alpha^{(\nu)}) \log(\alpha^{(k)}-\alpha^{(\nu)})$, *for* k=1, ..., ν-1.†

We remark that when $\lambda=-1$, the convergence in (2.5) is geometric, and Theorem 2.4 reduces to a special case of Theorem 2.3. In fact, if λ is any negative integer, then Theorem 2.4 states that f is meromorphic in $|z| < R|\alpha^{(\nu)}|$, with poles at the $\alpha^{(k)}$, k=1, ..., ν, and this more general situation is not covered by Theorem 2.3.

Proof of Theorem 2.4. Normalizing the $Q_{n,\nu}(z)$ so that they are monic, it follows from (2.4) and (2.5) that

$$\lim_{n\to\infty} Q_{n,\nu}(z) = \prod_{k=1}^{\nu} (z-\alpha^{(k)}) \equiv Q(z), \quad \forall\, z.$$

Further, from (2.3), the polynomial Q has a single zero of largest modulus, namely at $\alpha^{(\nu)}$, and this zero is simple. Hence Theorem 2.2 implies that in the disk $|z| < |\alpha^{(\nu)}|$, the function f is meromorphic with precisely ν-1 poles (at the $\alpha^{(k)}$, k=1, ..., ν-1). Now set

$$\hat{Q}_{n,\nu}(z) \equiv \prod_{k=1}^{\nu-1} (z-\alpha^{(k)}_{n,\nu}), \qquad \hat{Q}(z) \equiv \prod_{k=1}^{\nu-1} (z-\alpha^{(k)}).$$

From the Padé conditions, we have

$$(z-\alpha^{(\nu)}_{n,\nu})\hat{Q}_{n,\nu}(z)\hat{Q}(z)f(z)-\hat{Q}(z)P_{n,\nu}(z) = O(z^{n+\nu+1}), \quad \text{as} \quad z\to 0,$$

and since $\hat{Q}f$ is analytic in $|z| < |\alpha^{(\nu)}|$, Hermite's formula implies that for any $0 < \sigma < |\alpha^{(\nu)}|$

$$(2.8) \qquad \limsup_{n\to\infty} \left[\max_{|z|\le\sigma} \left|(z-\alpha^{(\nu)}_{n,\nu})\hat{Q}_{n,\nu}\hat{Q}f - \hat{Q}P_{n,\nu}\right|\right]^{1/n} \le \sigma/|\alpha^{(\nu)}|.$$

†More precisely, for any λ, the conditions on h are meant to indicate that f has poles in $\alpha^{(1)},\ldots,\alpha^{(\nu-1)}$ with corresponding multiplicity (not twice the multiplicity).

Since $(\hat{Q}f)(\alpha^{(k)})\neq 0$ and $(\hat{Q}P_{n,\nu})(\alpha^{(k)})=0$ for $k<\nu$, it easily follows from (2.8) and (2.3) that, on any compact set $K\subset\mathbb{C}$,

$$(2.9)\qquad \max_{z\in K}|\hat{Q}_{n,\nu}(z)-\hat{Q}(z)| = O(R^{-n}), \quad \text{as } n\to\infty.$$

Next, for notational convenience, if g is analytic at $z=0$, we let $I_n(g)$ be the coefficient of z^n in the Maclaurin expansion for g. Further, we set $\alpha\equiv\alpha^{(\nu)}$. Then from the Padé conditions and the fact that the degree of $\hat{Q}P_{n,\nu}$ is at most $n+\nu-1$, we have

$$I_{n+\nu}\left[\left(\frac{z}{\alpha}-1\right)\hat{Q}^2 f\right] = I_{n+\nu}\left[\left\{\left(\frac{z}{\alpha}-1\right)\hat{Q}-\left(\frac{z}{\alpha_{n,\nu}^{(\nu)}}-1\right)\hat{Q}_{n,\nu}\right\}\hat{Q}f\right].$$

Using (2.5), this last equation can be written in the form

$$(2.10)\qquad I_{n+\nu}\left[\left(\frac{z}{\alpha}-1\right)\hat{Q}^2 f\right] = I_{n+\nu}\left[\left(\frac{z}{\alpha}-1\right)\left(\hat{Q}-\hat{Q}_{n,\nu}\right)\hat{Q}f\right] - I_{n+\nu}\left[zO(R^{-n})\hat{Q}_{n,\nu}\hat{Q}f\right] +$$

$$\frac{(1+\lambda)}{(n+\nu)\alpha} I_{n+\nu}\left[z\left(\hat{Q}_{n,\nu}-\hat{Q}\right)\hat{Q}f\right] + \frac{(1+\lambda)}{(n+\nu)\alpha} I_{n+\nu}\left[z\hat{Q}^2 f\right].$$

Now the first, second, and third terms on the right-hand side of (2.10) are each $O(\rho^{-n})$ for every $1<\rho<R|\alpha|$. Hence on multiplying (2.10) by $z^{n+\nu}$ and summing, we find that

$$F(z) \equiv \int_0^z \hat{Q}^2(t)f(t)dt$$

satisfies the following differential equation

$$(2.11)\qquad (z-\alpha)F'(z) = (1+\lambda)F(z) + G(z),$$

where G is analytic in $|z|<R|\alpha|$. Solving (2.11) by the usual method gives

$$F(z) = (z-\alpha)^{1+\lambda}\int_0^z \frac{G(t)dt}{(t-\alpha)^{2+\lambda}}.$$

Thus F is analytic in $|z|<R|\alpha|$ except for a branch cut emanating from α,

and so the same is true for

$$(\hat{Q}^2 f)(z) = \frac{d}{dz}\left[(z-\alpha)^{1+\lambda}\int_0^z \frac{G(t)dt}{(t-\alpha)^{2+\lambda}}\right] .$$

Next, let $G(z) = \sum_0^\infty c_k(z-\alpha)^k$ be the Taylor expansion for G about α (which converges for $|z-\alpha| < (R-1)|\alpha|$) and select a point z_0 so that $|z_0| < |\alpha|$ and $|z_0-\alpha| < (R-1)|\alpha|$. Then, on consistently choosing the same branch of the logarithm (say with a radial cut from α to infinity) we can integrate term-by-term to obtain, for z off the cut and $|z-\alpha| < (R-1)|\alpha|$,

$$(\hat{Q}^2 f)(z) = \frac{d}{dz}\left[C_1(z-\alpha)^{1+\lambda} + (z-\alpha)^{1+\lambda}\int_{z_0}^z \frac{G(t)dt}{(t-\alpha)^{2+\lambda}}\right]$$

$$= C_2(z-\alpha)^\lambda + \frac{d}{dz}\left[(z-\alpha)^{1+\lambda}\sum_{k=0}^\infty c_k\int_{z_0}^z (t-\alpha)^{k-\lambda-2}dt\right] .$$

If $\lambda \notin \mathbb{N}$, this gives

$$(\hat{Q}^2 f)(z) = C_2(z-\alpha)^\lambda + \frac{d}{dz}\left[\sum_{k=0}^\infty c_k\frac{(z-\alpha)^k}{(k-\lambda-1)} - C_3(z-\alpha)^{1+\lambda}\right]$$

$$= C(z-\alpha)^\lambda + h(z),$$

where h is analytic in $|z-\alpha| < (R-1)|\alpha|$, and hence in $|z| < R|\alpha|$. If $\lambda \in \mathbb{N}$, then the integration gives rise to the logarithmic term of (2.7). Thus f must be of the form (2.6) or (2.7). □

We remark that Theorem 2.4 has a converse (a direct theorem) which is fairly straightforward to prove.

3. Generalized Taylor Series.

In this section we consider rational interpolation in general triangular schemes and deduce an inverse result related to Theorem 2.3. As in the setting of the second author's generalization [10] of the theorem of Montessus de Ballore, we let E be a closed bounded point set in the z-plane whose complement K (with respect to extended plane) is connected and regular in the sense that K possesses a Green's function $G(z)$ with pole at infinity. For $\sigma>1$, we let Γ_σ denote generically the level curve

$$\Gamma_\sigma : G(z) = \log \sigma,$$

and we denote by E_σ the interior of Γ_σ.

Next, we consider a triangular scheme of interpolation points

$$(3.1)\qquad \begin{array}{l} \beta_0^{(0)} \\ \beta_0^{(1)},\ \beta_1^{(1)} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \beta_0^{(n)},\ \beta_1^{(n)},\ \ldots,\ \beta_n^{(n)} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

(not necessarily distinct in any row) which lie on E.† Setting

$$(3.2)\qquad w_n(z) \equiv \prod_{k=0}^{n} (z-\beta_k^{(n)}) \quad , \quad n = 0,1,2,\ldots,$$

we assume that

$$(3.3)\qquad \lim_{n\to\infty} |w_n(z)|^{1/n} = \Delta \exp G(z) ,$$

uniformly in z on each closed bounded subset of K, where Δ is the transfinite diameter [14, §4.4] of E. We remark that condition (3.3) is equivalent to

$$(3.4)\qquad \limsup_{n\to\infty} [\max |w_n(z)| : z\varepsilon E]^{1/n} \leq \Delta \qquad \text{(cf. [14,§7.4])}.$$

While the assumption of (3.3) is sufficient for proving a generalization of the (direct) theorem of Montessus de Ballore, the study of the inverse problem for rational interpolation in the points (3.1) requires much more refined properties, which we now state.

For functions f analytic in the points (3.1), we let I_n denote the divided difference operator in the points $\beta_0^{(n)}, \beta_1^{(n)}, \ldots, \beta_n^{(n)}$, that is

$$(3.5)\qquad I_n(f) = f[\beta_0^{(n)}, \ldots, \beta_n^{(n)}] = \frac{1}{2\pi i}\int_C \frac{f(z)}{w_n(z)}\, dz \ , \quad n = 0, 1, \ldots,$$

†With slight modifications in the subsequent discussion, it suffices to assume, more generally, that no limit points of (3.1) lie exterior to E.

where the contour C is suitably chosen so as to enclose all the points $\{\beta_k^{(n)}\}_{k=0}^n$. We remark that if $L_n(z)$ is the unique polynomial of degree at most n which interpolates f in the points $\{\beta_k^{(n)}\}_{k=0}^n$, then $I_n(f)$ is simply the coefficient of z^n in the expansion of $L_n(z)$.

Now for each $n = 0, 1, \ldots,$ it is easy to see that there exists a unique monic polynomial $P_n(z)$ of degree n, such that

$$I_j(P_n) = \delta_{j,n}, \quad \text{for all } j = 0, 1, \ldots \quad . \tag{3.6}$$

We shall refer to these polynomials $P_n(z)$ as <u>basis polynomials</u> for the scheme (3.1). They can be generated via the recurrence formula

$$P_n(z) = z^n - \sum_{k=0}^{n-1} I_k(z^n)P_k(z), \quad P_0(z) \equiv 1 .$$

The divided difference operators together with the associated basis polynomials give rise to a <u>Generalized Taylor Series</u> (GTS) for f, namely

$$f(z) \sim \sum_{n=0}^{\infty} I_n(f)P_n(z) . \tag{3.7}$$

A discussion of the algebraic properties of GTS can be found in Gelfond [4]. For our purposes we require that these series represent f on E as described in

<u>Definition</u> 3.1. The scheme (3.1) is said to have the <u>Walsh-Hadamard property</u> with respect to E if condition (3.3) holds and if, for every function f analytic on E, the series (3.7) converges to f uniformly on E.

The essential feature of such schemes is given in

<u>Lemma</u> 3.2. <u>If (3.1) has the Walsh-Hadamard property with respect to E and f is any function analytic on E, then</u>

$$\limsup_{n\to\infty} |I_n(f)|^{1/n} = 1/\Delta\rho(f) , \tag{3.8}$$

<u>where</u>

$$\rho(f) \equiv \sup\left\{\sigma: f \text{ is analytic in } E_\sigma\right\} .$$

<u>Moreover, the GTS for f converges to f uniformly on compact subsets of</u> $E_{\rho(f)}$.

Proof. We first show that the basis polynomials $P_n(z)$ satisfy

(3.9) $$\lim_{n\to\infty} |P_n(z)|^{1/n} = \Delta \exp G(z) \quad ,$$

uniformly on each closed bounded subset of K. For this purpose, let $\mu>1$ and select a point t on Γ_μ. Then for the function $g(z) = 1/(t-z)$, it is easy to verify that

$$I_n(g) = 1/w_n(t) \text{ , for all } n\geq 0 \text{ .}$$

Hence, from the Walsh-Hadamard property, the series

$$\sum_0^\infty P_n(z)/w_n(t)$$

converges uniformly for z on E. Since the terms of this series are uniformly bounded on E, condition (3.3) implies that

$$\limsup_{n\to\infty} \left[\max |P_n(z)| : z\varepsilon E\right]^{1/n} \leq \lim_{n\to\infty} |w_n(t)|^{1/n} = \Delta\mu \text{ .}$$

On letting μ tend to 1 in the last inequality, we have

$$\limsup_{n\to\infty} \left[\max |P_n(z)| : z\varepsilon E\right]^{1/n} \leq \Delta \quad ,$$

from which (3.9) follows.

Now let $f(z)$ be analytic on E. Then it follows immediately from (3.3) and (3.5) that

(3.10) $$\limsup_{n\to\infty} |I_n(f)|^{1/n} \leq 1/\Delta\rho(f) \text{ .}$$

Assume, to the contrary, that strict inequality holds in (3.10). Then it is easy to verify, from (3.9), that the GTS of f converges uniformly on compact subsets of E_λ, for some $\lambda > \rho(f)$. Since this series gives an analytic continuation of $f(z)$, we reach a contradiction to the definition of the number $\rho(f)$. □

We now consider rational interpolation in the scheme (3.1). For f analytic in these points and (n,ν) a given pair of nonnegative integers, we let $r_{n,\nu}$ be the unique rational function of the form

$$r_{n,\nu} = p_{n,\nu}/q_{n,\nu}, \quad \deg p_{n,\nu} \leq n, \quad \deg q_{n,\nu} \leq \nu, \quad q_{n,\nu} \neq 0,$$

such that

(3.11) $$q_{n,\nu}(z)f(z) - p_{n,\nu}(z) = 0 \quad \text{for} \quad z = \beta_0^{(n+\nu)}, \beta_1^{(n+\nu)}, \ldots, \beta_{n+\nu}^{(n+\nu)} .$$

(In case of repeated points β, equation (3.11) is to be interpreted in the Hermite (derivative) sense.) Our main result is

Theorem 3.3. *Suppose f is analytic on E and the scheme (3.1) satisfies the Walsh-Hadamard property with respect to E. Let $\nu>0$ be fixed and suppose there exists a polynomial q of the form*

(3.12) $$q(z) = \prod_{k=1}^{\nu} (z-\alpha_k), \qquad \alpha_k \notin E \quad \text{for} \quad 1\leq k\leq\nu ,$$

such that the (denominator) polynomials $q_{n,\nu}$ (suitably normalized) of (3.11) satisfy

(3.13) $$||q_{n,\nu} - q|| = O(R^{-n}) , \quad R>1 , \quad n = 0, 1, \ldots,$$

where $||\cdot||$ represents any of the equivalent norms on the $(\nu+1)$-dimensional space of polynomial coefficients. Then, either

(i) f is meromorphic with at most $\nu-1$ poles in the whole plane, with the poles of f in zeros of q, or

(ii) f is meromorphic with precisely ν poles in $E_{R\sigma^*}$, where

(3.14) $$\sigma^* \equiv \max_{k=1}^{\nu} \{\sigma_k : \alpha_k \varepsilon \Gamma_{\sigma_k}\} ,$$

and these ν poles of f are the zeros, α_k, of q.

Proof. Suppose first that f is not meromorphic with at most $\nu-1$ poles in the whole plane. With the notation of Definition 3.1, we then need to show that

(3.15) $$\rho(qf) \geq R\sigma^* ,$$

and that f actually has poles in the ν zeros of q (with corresponding multiplicities).

Let $\tilde{q}(z)$ be the monic polynomial whose zeros are the poles of f in $E_{\rho(qf)}$ (if no such poles exist, we set $\tilde{q}(z) \equiv 1$). Then $\tilde{q}$ divides q, and

(3.16) $$\rho(qf) = \rho(\tilde{q}f) = \rho(\tilde{q}qf) .$$

We claim that $\tilde{q}(z) \equiv q(z)$. Suppose this is not the case. Then $\deg \tilde{q} \le \nu-1$, and the interpolation conditions (3.11) imply that $I_{n+\nu}(\tilde{q}q_{n,\nu}f) = 0$. Hence we have

$$(3.17) \qquad I_{n+\nu}(q\tilde{q}f) = I_{n+\nu}[(q-q_{n,\nu})\tilde{q}f], \quad n = 0, 1, \ldots .$$

Next, select any point $z_o \varepsilon E$ and write

$$q(z)-q_{n,\nu}(z) = \sum_{k=o}^{\nu} b_k^{(n)}(z-z_o)^k.$$

From (3.13) we have $|b_k^{(n)}| = O(R^{-n})$ as $n\to\infty$, and furthermore, by Lemma 3.2,

$$\limsup_{n\to\infty} |I_{n+\nu}[(z-z_o)^k\tilde{q}f]|^{1/n} = \limsup_{n\to\infty} |I_{n+\nu}(\tilde{q}f)|^{1/n} > 0, \qquad k=0,\ldots,\nu ,$$

where the positivity assertion follows from our assumption that f is not meromorphic with at most $\nu-1$ poles in the whole plane. Therefore, from (3.17), there holds

$$(3.18) \qquad \limsup_{n\to\infty} |I_{n+\nu}(q\tilde{q}f)|^{1/n} \le \frac{1}{R} \limsup_{n\to\infty} |I_{n+\nu}(\tilde{q}f)|^{1/n} < \limsup_{n\to\infty} |I_{n+\nu}(\tilde{q}f)|^{1/n} ,$$

and so $\rho(q\tilde{q}f) > \rho(\tilde{q}f)$, which contradicts (3.16). Thus, as claimed, $\tilde{q}(z) \equiv q(z)$, which means that all the zeros of q lie in $E_{\rho(qf)}$ and that f has actual poles in these ν zeros. To establish (3.15), we simply repeat the above argument with $\tilde{q}$ replaced by $\hat{q}$, where $\hat{q}$ is the monic polynomial whose zeros are the poles of f in E_{σ^*} (compare (3.18)).

Finally, if f is meromorphic with at most $\nu-1$ poles in the whole plane, then as is easily seen from the proof of the Montessus de Ballore theorem, these poles must be limit points of zeros of the $q_{n,\nu}$, and hence must lie in the zeros of q. □

We now mention some examples of schemes (3.1) which satisfy the Walsh-Hadamard (WH) property.

Example 1. Let E be the disk $|z| \le \tau$, $\tau>0$, and let all the $\beta_k^{(n)}$ be zero. Then $I_n(f) = f^{(n)}(0)/n!$, $P_n(z) = z^n$, and the GTS is simply the Maclaurin series for f. Hence, for a function f analytic at $z=0$, Theorem 3.3 applies to the Padé approximants of f. Notice, however, that Theorem 2.3 is somewhat stronger in this case since for possibility (i) it implies that f is rational with at most $\nu-1$ poles.

Example 2. Suppose that the scheme (3.1) is independent of n, that is, $\beta_k^{(n)} = \beta_k$ for all n. Then we have

$$P_n(z) = w_{n-1}(z) = \prod_{k=0}^{n-1} (z-\beta_k) ,$$

and the GTS is just the Newton series for f. Hence, if condition (3.3) holds, the Newton scheme has the WH property. In this case, as in the first example, the result of Kovačeva [8] is somewhat stronger than Theorem 3.3. However, her results do not apply to the examples which follow.

Example 3. Let E be the unit disk $|z| \leq 1$, and let the scheme (3.1) consist of the roots of unity, i.e., $w_n(z) = z^{n+1} - 1$, $n=0,1,\ldots$. If $f(z) = \sum_0^\infty a_k z^k$ is analytic on E, then it is easy to verify that

$$I_n(f) = \frac{1}{n+1} \sum_{i=0}^{n} \lambda_{n+1}^{i} f(\lambda_{n+1}^{i}) = a_n + a_{2n+1} + a_{3n+2} + a_{4n+3} + \cdots ,$$

where λ_{n+1} is a primitive $(n+1)$st root of unity. As shown by Ching and Chui [1] , the basis polynomials $P_n(z)$ are given by

$$P_n(z) = \sum_{k|n} \mu(n/k) z^k, \qquad n = 0, 1, \ldots,$$

where $k|n$ means k is a factor of n, and μ is the Möbius function defined by

$$\mu(j) = \begin{cases} 1 , & \text{if } j = 1 \\ (-1)^k , & \text{if } j = \Pi \text{ (} k \text{ distinct primes)} \\ 0 , & \text{if } p^2|j \text{ for some } p>1 . \end{cases}$$

It is readily shown (cf. [1]) that the GTS gives maximal convergence to f, so this interpolation scheme has the WH property.

Example 4. Suppose the points β are the same in each fixed row of (3.1), that is, $w_n(z) = \prod_{k=o}^{n} (z-\beta^{(n)})$, $n=0, 1, \ldots$. In this case, $I_n(f) = f^{(n)}(\beta^{(n)})/n!$, and the basis polynomials $P_n(z)$ are the Gontcharoff polynomials (cf. [4]). As discussed in the next example, if $E : |z| \leq \tau$, $\tau>0$, and the points $\beta^{(n)}$ tend to zero sufficiently fast, then these points have the WH property.

Example 5. For the scheme (3.1), suppose that the points $\beta_k^{(n)}$ tend to zero as $n \to \infty$ $(0 \leq k \leq n)$. Let Δ_n be the smallest convex polygon containing the points $\beta_0^{(n)}, \ldots, \beta_n^{(n)}$, and let d_n be the diameter of the smallest circle which encloses the polygons Δ_{n-1} and Δ_n. If $\Sigma d_n < \infty$, then, as shown by Gelfond [4, p. 47], the scheme (3.1) has the WH property with respect to any disk $E : |z| \le \tau, \tau>0$.

4. Inverse Theorems for Diagonal Approximants.

With the notation of the previous section, we shall prove

Theorem 4.1. *Let E be a compact point set (not a single point) whose complement K is simply connected, and suppose the interpolation scheme (3.1) satisfies the WH property with respect to E. Let f be analytic on E and for $n=1,2,\ldots,$ let $q_{n,n-1}$ and $q_{n,n}$ denote, respectively, the denominator polynomials (defined in (3.11)) for the rational interpolants $r_{n,n-1}$ and $r_{n,n}$. If all the zeros of these polynomials tend to infinity with n, i.e.,*

$$(4.1) \qquad \min\{|\zeta| : q_{n,n-1}(\zeta) = 0 \quad \text{or} \quad q_{n,n}(\zeta) = 0\} \to \infty \quad \text{as} \quad n\to\infty,$$

then f must be entire.

Proof. Since f is analytic on the closed set E, there exists a $\lambda>1$ such that f is analytic on $\overline{E}_\lambda$. Thus for $z \in \Gamma_\sigma$, $1<\sigma<\lambda$, Hermite's formula gives

$$(4.2) \qquad f(z) - \frac{p_{n,n}(z)}{q_{n,n}(z)} = \frac{1}{2\pi i}\int_{\Gamma_\lambda} \frac{w_{2n}(z)q_{n,n}(t)f(t)dt}{w_{2n}(t)q_{n,n}(z)(t-z)} .$$

Now note that (4.1) implies

$$\limsup_{n\to\infty} [\max\{|q_{n,n}(t)/q_{n,n}(z)| ; t\in\Gamma_\lambda, z\in\Gamma_\sigma\}]^{1/n} \le 1 ,$$

and so, on estimating the integral in (4.2) and using (3.3), we obtain

$$(4.3) \qquad \limsup_{n\to\infty}\left[\max_{z\in\Gamma_\sigma}\left|f(z) - \frac{p_{n,n}(z)}{q_{n,n}(z)}\right|\right]^{1/n} \le \left(\frac{\sigma}{\lambda}\right)^2 < 1, \qquad 1<\sigma<\lambda .$$

Next, let $L_{2n}(z)$ be the unique polynomial of degree $\le 2n$ which interpolates $f(z)$ in the points $\{\beta_k^{(2n)}\}_{k=0}^{2n}$, and let $L_{2n-1}(z)$ be the polynomial of degree $\le 2n-1$ which interpolates $f(z)$ in the points $\{\beta_k^{(2n)}\}_{k=0}^{2n-1}$. Let $\rho>1$. Then, for n sufficiently large and $1<\tau<\rho$, we have

(4.4) $$\frac{p_{n,n}(z)}{q_{n,n}(z)} - L_{2n}(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{w_{2n}(z)}{w_{2n}(t)}\frac{p_{n,n}(t)}{q_{n,n}(t)}\frac{dt}{t-z} \quad , \quad z\varepsilon\Gamma_\tau \quad .$$

Set

$$R_n \equiv \min\{\sigma : \zeta\varepsilon\Gamma_\sigma \quad \text{and} \quad q_{n,n}(\zeta) = 0\} ,$$

so that, from (4.1), $R_n \to \infty$ as $n\to\infty$. Since, from (4.3), the sequence $p_{n,n}/q_{n,n}$ is uniformly bounded on E, it follows from a lemma of Walsh [14 , p. 250] that for $R_n > \rho$

$$\left| \frac{p_{n,n}(t)}{q_{n,n}(t)} \right| \leq A \left[\frac{R_n\rho - 1}{R_n - \rho}\right]^n \quad , \quad \text{for all} \quad t\varepsilon\Gamma_\rho ,$$

where A is a constant independent of n. Using this estimate in (4.4) we obtain for $z\varepsilon\Gamma_\tau$

(4.5) $$\limsup_{n\to\infty} \left| \frac{p_{n,n}(z)}{q_{n,n}(z)} - L_{2n}(z) \right|^{1/n} \leq \left(\frac{\tau^2}{\rho^2}\right)\rho = \frac{\tau^2}{\rho} \quad .$$

Estimating the difference $p_{n,n}/q_{n,n} - L_{2n-1}$ in a similar way gives

(4.6) $$\limsup_{n\to\infty} \left| \frac{p_{n,n}(z)}{q_{n,n}(z)} - L_{2n-1}(z) \right|^{1/n} \leq \frac{\tau^2}{\rho} , \quad z\varepsilon\Gamma_\tau \quad .$$

Thus, on combining (4.5) and (4.6), we have

$$\limsup_{n\to\infty} \left| I_{2n}(f) \right|^{1/n} = \limsup_{n\to\infty} \left| \frac{L_{2n}(z) - L_{2n-1}(z)}{\hat{w}_{2n}(z)} \right|^{1/n} \leq \frac{\tau^2}{\rho(\Delta\tau)^2} = \frac{1}{\rho\Delta^2} \quad ,$$

where $\hat{w}_{2n}(z) \equiv w_{2n}(z)/(z-\beta_{2n}^{(2n)})$. On letting $\rho\to\infty$, we therefore obtain

(4.7) $$\lim_{n\to\infty} \left| I_{2n}(f) \right|^{1/2n} = 0 \quad .$$

Finally, if the above argument is repeated for the sequence $p_{n,n-1}/q_{n,n-1}$, we also get

(4.8) $$\lim_{n\to\infty} \left| I_{2n-1}(f) \right|^{1/(2n-1)} = 0 ,$$

and so, by the WH property, f must be entire. □

<u>Remark</u> 1. With the conditions of Theorem 4.1, it follows immediately from inequality (4.3)(with $\lambda=\infty$) that the sequence of rational interpolants $\{r_{n,n}\}$ converges faster than geometrically to f on any compact set in the plane; the same being true for the sequence $\{r_{n,n-1}\}$.

<u>Remark</u> 2. Theorem 4.1 remains valid if the hypothesis concerning the two sequences $\{q_{n,n-1}\}$, $\{q_{n,n}\}$ is replaced by the same assumption for any two diagonal sequences $\{q_{n,n+k}\}$, $\{q_{n,n+j}\}$, where k and j have opposite parity.

<u>Remark</u> 3. There are several instances when it suffices to know the behavior of only one diagonal sequence in Theorem 4.1. This is especially the case when there is a relationship between the even and odd numbered rows of the scheme (3.1). For example, a slight modification in the proof of Theorem 4.1 shows that *if* $w_{2n-1}(z)$ *divides* $w_{2n}(z)$ *for all* n *large, then the assumption that the zeros of the sequence* $\{q_{n,n}\}$ *tend to infinity implies that* f *is entire.* This is certainly the case if the scheme (3.1) is Newton and, in particular, if the interpolants are Padé approximants. In the Padé case, even more can be said concerning the inverse problem. In this regard, we mention a recent result of Gončar and Lungu [5]:

<u>Theorem</u> 4.2. <u>Let</u> f <u>be analytic at infinity and, for</u> $n=0,1,\ldots,$ <u>let</u> $\tilde{Q}_{n,n}$ <u>be the denominator of the</u> $[n/n]$ <u>Padé approximant to</u> f <u>(interpolating at infinity), where</u> $\tilde{Q}_{n,n}$ <u>is assumed to be monic of degree</u> n. <u>Let</u> L <u>be the set of limit points of the zeros of the</u> $\tilde{Q}_{n,n}$, <u>and suppose that</u> L <u>is regular in the sense that its complement has a Green's function with pole at infinity. If</u>

$$\lim_{n\to\infty} [\max |\tilde{Q}_{n,n}(z)| \; ; \; z\in L]^{1/n} = \Delta(L) ,$$

<u>where</u> $\Delta(L)$ <u>is the transfinite diameter of</u> L, <u>then</u> f <u>is analytic off</u> L.

References

1. C-H. Ching and C. K. Chui, Mean boundary value problems and Riemann series, *J. Approx. Theory* 10(1974), 324-336.

2. P. Dienes, The Taylor Series, Dover, New York, 1957.

3. D. S. Gaunt and A. J. Guttman, Asymptotic analysis of coefficients. In: Phase Transitions and Critical Phenomena III, (Domb, Green, eds.) Academic Press, New York, 1974, pp. 181-243.

4. A. O. Gelfond, Differenzenrechnung, VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

5. A. A. Gončar and K.N. Lungu, Poljuci diagonalnih approksimatsij Padé i analitičeskoe prodolženie funktsij, *Mat. Sb.* 111 (153)(1980), 279-292.

6. J. Karlsson and E. B. Saff, Singularities of analytic functions determined by Padé approximants, *Notices of Amer. Math. Soc.* 26(1979), p. A-584.

7. J. Karlsson and B. von Sydow, The convergence of Padé approximants to series of Stieltjes, *Ark. Mat.* 14(1976), 43-53.

8. R. K. Kovačeva, Obobščennie approksimatsii Padé i meromorfnoe prodolženie funktsij, *Mat. Sb.* 109 (151)(1979), 365-377.

9. R. de Montessus de Ballore, Sur les fractions continues algébriques, *Bull. Soc. Math. France* 301(1902), 26-32.

10. E. B. Saff, An extension of Montessus de Ballore's theorem on the convergence of interpolating rational functions, *J. Approx. Theory* 6(1972), 63-67.

11. T. J. Stieltjes, Recherches sur les fractions continues, *Ann. Fac. Sci. Univ. Toulouse* 8(1894), 1-122.

12. V. V. Vavilov, G. Lopez, and V. A. Prohorov, Ob odnoj obpatnoj zadače dlja strok tablitsi Padé, *Mat. Sb.* 110 (152)(1979), 117-127.

13. H. Wallin, Rational interpolation to meromorphic functions, *these Proceedings*.

14. J. L. Walsh, Interpolation and Approximation by Rational Functions in the Complex Domain, 5th ed., Colloq. Publ. Vol. 20, Amer. Math. Soc., Providence, 1969.