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AN EXTENSION TO RATIONAL FUNCTIONS OF A THEOREM OF J. L. WALSH ON DIFFERENCES OF INTERPOLATING POLYNOMIALS (*)

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Résumé. — *Dans cet article, un théorème de J. L. Walsh, sur les différences de polynômes d'interpolation en les racines de l'unité et en l'origine, est étendu aux différences de fonctions rationnelles d'interpolation en des ensembles de points plus généraux.*

Abstract. — *In this paper, a theorem of J. L. Walsh, on differences of polynomials interpolating in the roots of unity and in the origin, is extended to differences of rational functions interpolating in more general point sets.*

1. INTRODUCTION

Our main purpose is to generalize, to the rational case, a well-known and beautiful result of J. L. Walsh on the convergence of differences of interpolating polynomials. To state this result, we first introduce some needed notation.

Let A_ρ denote the set of functions $f(z)$ analytic in the disk $|z| < \rho$, where we assume that $1 < \rho < \infty$. With π_m denoting the set of all complex polynomials of degree at most m, let $p_n(z;f) \in \pi_n$ be the Lagrange polynomial interpolant of $f(z) \in A_\rho$ in the $(n+1)$-st roots of unity, i.e.,

$$p_n(\omega;f) = f(\omega), \quad \forall\omega \text{ such that } \omega^{n+1} = 1, \tag{1.1}$$

for each nonnegative integer n. Writing $f(z) = \sum_{j=0}^{\infty} a_j z^j$ for $|z| < \rho$, we let

$$P_n(z;f) := \sum_{j=0}^{n} a_j z^j$$

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be the associated n-th partial sum of f, so that

$$P_n(z;f) - f(z) = \mathcal{O}(z^{n+1}), \quad \text{as} \quad z \to 0. \tag{1.2}$$

Letting

$$D_\tau := \{ z \in \mathbb{C} : |z| < \tau \} \quad \text{and} \quad \overline{D}_\tau := \{ z \in \mathbb{C} : |z| \leqslant \tau \}, \tag{1.3}$$

then this particular result of Walsh [7; 8 p. 153] can be stated as

THEOREM A : *If $f \in A_\rho$, then the interpolating polynomials $p_n(z)$ of* (1.1) *and $P_n(z)$ of* (1.2) *satisfy*

$$\lim_{n\to\infty} [p_n(z;f) - P_n(z;f)] = 0, \quad \forall |z| < \rho^2, \tag{1.4}$$

the convergence being uniform and geometric on any closed subset of D_{ρ^2}. More precisely, on any closed subset $\mathcal{H}$ of any $\overline{D}_\tau$ with $\rho \leqslant \tau \leqslant \infty$, there holds

$$\limsup_{n\to\infty} \{ \max_{z\in\mathcal{H}} | p_n(z;f) - P_n(z;f) | \}^{1/n} \leqslant \frac{\tau}{\rho^2}. \tag{1.5}$$

Furthermore, the result of (1.4) *is best possible in the sense that there is some $\hat{f}(z) \in A_\rho$ and some $\hat{z}$ with $|\hat{z}| = \rho^2$ for which $p_n(\hat{z};\hat{f}) - P_n(\hat{z};\hat{f})$ does not tend to zero as $n \to \infty$.*

In a recent paper, Cavaretta, Sharma, and Varga [2] give several generalizations of Theorem A for the case of polynomial interpolation. Our present goal is to extend some of these results to differences of *rational* functions which interpolate a *meromorphic* function. Although our main result (*cf.* Theorem 2.1) deals with more general interpolation schemes and their associated geometries, we first state, for purposes of illustration, our extension of Theorem A where the interpolation points are again the roots of unity and the origin.

For notation, for each nonnegative integer ν and for each ρ, with $1 < \rho < \infty$, let $M_\rho(\nu)$ denote the set of functions $F(z)$ which are meromorphic with precisely ν poles (counting multiplicity) in the disk D_ρ, and which are analytic at $z = 0$ and on $|z| = 1$. Given $F \in M_\rho(\nu)$, consider the rational interpolant

$$S_{n,\nu}(z;F) = S_{n,\nu}(z) = U_{n,\nu}(z)/V_{n,\nu}(z), \quad \text{with} \quad U_{n,\nu} \in \pi_n, V_{n,\nu} \in \pi_\nu, \tag{1.6}$$

of type (n, ν) of $F(z)$ which, in analogy with (1.1), is to satisfy

$$S_{n,\nu}(\omega) = F(\omega), \quad \forall\omega \text{ such that } \omega^{n+\nu+1} = 1. \tag{1.7}$$

Similarly, consider the *Padé* rational interpolant (*cf.* Baker [1], Perron [4])

$$R_{n,\nu}(z\,;F) = R_{n,\nu}(z) = P_{n,\nu}(z)/Q_{n,\nu}(z)\,, \quad \text{with} \quad P_{n,\nu} \in \pi_n, Q_{n,\nu} \in \pi_\nu\,, \tag{1.8}$$

of type (n, ν) of $F(z)$ which, in analogy with (1.2), is to satisfy

$$R_{n,\nu}(z) - F(z) = \mathcal{O}(z^{n+\nu+1})\,, \quad \text{as} \quad z \to 0\,. \tag{1.9}$$

(We assume here and throughout that the denominator polynomials $V_{n,\nu}(z)$, $Q_{n,\nu}(z)$ of (1.6) and (1.8) are normalized so as to be *monic.*)

It is important to note that the existence and uniqueness of the rational interpolants $S_{n,\nu}(z)$ and $R_{n,\nu}(z)$ of (1.7) and (1.9) are, for all n large, guaranteed by a theorem of Montessus de Ballore [3] and its generalization due to Saff [5]; this latter result is stated in § 2 as Theorem B.

With the above notation, we shall prove in § 3 the result of

THEOREM 1.1 : *If $F \in M_\rho(\nu)$, and if $\{ \alpha_j \}_{j=1}^{\nu}$ are the ν poles of F in D_ρ (listed according to multiplicities), then the rational interpolants $S_{n,\nu}$ of (1.7) and $R_{n,\nu}$ of (1.9) satisfy*

$$\lim_{n\to\infty} [S_{n,\nu}(z\,;F) - R_{n,\nu}(z\,;F)] = 0\,, \quad \forall z \in D_{\rho^2} \setminus \bigcup_{j=1}^{\nu} \{ \alpha_j \}\,, \tag{1.10}$$

the convergence being uniform and geometric on any closed subset of

$$D_{\rho^2} \setminus \bigcup_{j=1}^{\nu} \{ \alpha_j \}\,.$$

More precisely, on any closed subset $\mathcal{H}$ of any $\overline{D}_\tau \setminus \bigcup_{j=1}^{\nu} \{ \alpha_j \}$ with $\rho \leqslant \tau < \infty$, there holds

$$\limsup_{n\to\infty} \{ \max_{z\in\mathcal{H}} | S_{n,\nu}(z\,;F) - R_{n,\nu}(z\,;F) | \}^{1/n} \leqslant \frac{\tau}{\rho^2}\,. \tag{1.11}$$

The result of (1.10) is best possible in the sense that, for any $\nu \geqslant 0$, and for any ρ with $1 < \rho < \infty$, there is an $\hat{F}_\nu \in M_\rho(\nu)$ such that

$$\limsup_{n\to\infty} \{ \min_{|z|=\rho^2} | S_{n,\nu}(z\,;\hat{F}_\nu) - R_{n,\nu}(z\,;\hat{F}_\nu) | \} > 0\,. \tag{1.12}$$

We remark that the special case $\nu = 0$ of Theorem 1.1 reduces to Walsh's Theorem A. We further note that the sharpness result (1.12) of theorem 1.1 generalizes the corresponding result for $\nu = 0$ of Cavaretta, Sharma, and Varga [2].

Concerning the behavior of the (monic) denominator polynomials of the rational interpolants $S_{n,\nu}(z; F)$ and $R_{n,\nu}(z; F)$ of Theorem 1.1, it is known from Saff's Theorem B (*cf.* § 2) that

$$\lim_{n\to\infty} V_{n,\nu}(z) = \lim_{n\to\infty} Q_{n,\nu}(z) = B(z) := \prod_{i=1}^{\nu} (z - \alpha_i), \quad \forall z \in \mathbb{C},$$

and, moreover, as a special case of (2.22), that on each compact set $\mathcal{K} \subset \mathbb{C}$,

$$\limsup_{n\to\infty} \{ \max_{z\in\mathcal{K}} | V_{n,\nu}(z) - B(z) | \}^{1/n} \leqslant [\max_{i=1,\ldots,\nu} (1, | \alpha_i |)]/\rho, \tag{1.13}$$

and

$$\limsup_{n\to\infty} \{ \max_{z\in\mathcal{K}} | Q_{n,\nu}(z) - B(z) | \}^{1/n} \leqslant [\max_{i=1,\ldots,\nu} | \alpha_i |]/\rho. \tag{1.14}$$

Clearly, (1.13) and (1.14) together imply

$$\limsup_{n\to\infty} \{ \max_{z\in\mathcal{K}} | V_{n,\nu}(z) - Q_{n,\nu}(z) | \}^{1/n} \leqslant [\max_{i=1,\ldots,\nu} (1, | \alpha_i |)]/\rho. \tag{1.15}$$

But, as a special case of Corollary 2.4, we can improve (1.15) by means of

COROLLARY 1.2 : *With the assumptions of Theorem* 1.1, *there holds on every compact set* $\mathcal{K} \subset \mathbb{C}$

$$\limsup_{n\to\infty} \{ \max_{z\in\mathcal{K}} | V_{n,\nu}(z) - Q_{n,\nu}(z) | \}^{1/n} \leqslant \frac{1}{\rho}. \tag{1.16}$$

The outline of the present paper is as follows. In § 2, we state and prove our main results for general interpolation schemes, and in § 3 we consider some specific applications of these results.

2. MAIN RESULTS

Our aim to extend Theorem A in two directions. Firts, we wish to consider triangular interpolation schemes that are associated with planar sets more general than that of the disk. Second, we shall replace polynomial interpolation to analytic functions by certain types of rational interpolation to meromorphic functions.

For these purposes, let E be a closed bounded point set in the z-plane whose complement K (with respect to the extended plane) is connected and regular in the sense that K possesses a Green's function $G(z)$ with pole at infinity (*cf.* [8, p. 65]). Let Γ_σ, for $\sigma > 1$, denote generically, the locus

$$\Gamma_\sigma := \{ z \in \mathbb{C} : G(z) = \log \sigma \}, \tag{2.1}$$

and denote by E_σ the interior of Γ_σ.

Next, for each nonnegative integer ν, and for each ρ, with $1 < \rho < \infty$, let $M(E_\rho; \nu)$ denote the set of functions $F(z)$ which are analytic on E and meromorphic with precisely ν poles (counting multiplicity) in the open set E_ρ. For $F \in M(E_\rho; \nu)$, we consider rational interpolation in the two triangular schemes

$$\begin{array}{lcl} \beta_1^{(0)} & & \tilde{\beta}_1^{(0)} \\ \beta_1^{(1)}, \beta_2^{(1)} & & \tilde{\beta}_1^{(1)}, \tilde{\beta}_2^{(1)} \\ \cdots\cdots & (2.2) & \cdots\cdots \qquad (2.3) \\ \beta_1^{(n)}, \beta_2^{(n)}, \ldots, \beta_{n+1}^{(n)} & & \tilde{\beta}_1^{(n)}, \tilde{\beta}_2^{(n)}, \ldots, \tilde{\beta}_{n+1}^{(n)} \\ \cdots\cdots\cdots\cdots & & \cdots\cdots\cdots\cdots \end{array}$$

where we assume that no limit points of the tableaux in (2.2) or (2.3) lie exterior to E. To be specific, we let $r_{n,\nu}(z)$ be the rational function of the form

$$r_{n,\nu}(z; F) = r_{n,\nu}(z) = \frac{p_{n,\nu}(z)}{q_{n,\nu}(z)}, \ p_{n,\nu} \in \pi_n, \ q_{n,\nu} \in \pi_\nu, \ q_{n,\nu} \text{ monic}, \tag{2.4}$$

which interpolates $F(z)$ in the $n + \nu + 1$ points $\{ \beta_i^{(n+\nu)} \}_{i=1}^{n+\nu+1}$, i.e.,

$$r_{n,\nu}(\beta_i^{(n+\nu)}) = F(\beta_i^{(n+\nu)}), \ i = 1, 2, \ldots, n + \nu + 1, \tag{2.5}$$

and we let $\tilde{r}_{n,\nu}(z)$ be the rational function of the form

$$\tilde{r}_{n,\nu}(z; F) = \tilde{r}_{n,\nu}(z) = \frac{\tilde{p}_{n,\nu}(z)}{\tilde{q}_{n,\nu}(z)}, \ \tilde{p}_{n,\nu} \in \pi_n, \ \tilde{q}_{n,\nu} \in \pi_\nu, \ \tilde{q}_{n,\nu} \text{ monic}, \tag{2.6}$$

which interpolates $F(z)$ in the $n + \nu + 1$ points $\{ \tilde{\beta}_i^{(n+\nu)} \}_{i=1}^{n+\nu+1}$, i.e.,

$$\tilde{r}_{n,\nu}(\tilde{\beta}_i^{(n+\nu)}) = F(\tilde{\beta}_i^{(n+\nu)}), \ i = 1, 2, \ldots, n + \nu + 1. \tag{2.7}$$

In the tableaux (2.2) and (2.3), we do not require that the entries in any particular row consist of distinct points. In the case of repeated points, interpolation in (2.5) or (2.7) is understood to be taken in the Hermite (derivative) sense.

Unlike polynomial interpolation, the existence of the above rational interpolants is by no means assured without further assumptions on the behaviors of the triangular schemes. Also, to establish a theorem (analogous to Theorem A) which asserts that the difference $\tilde{r}_{n,\nu}(z) - r_{n,\nu}(z)$ tends to zero in some « large » region, we need to assume that the tableaux (2.2) and (2.3) are, in some sense, « close » to one another.

To specify these assumptions, set

$$w_n(z) := \prod_{j=1}^{n+1} (z - \beta_j^{(n)}), \ \tilde{w}_n(z) := \prod_{j=1}^{n+1} (z - \tilde{\beta}_j^{(n)}), \ w_{-1}(z) = \tilde{w}_{-1}(z) := 1. \tag{2.8}$$

Concerning the triangular scheme (2.2), we suppose that

$$\lim_{n\to\infty} | w_n(z) |^{1/n} = \Delta \exp G(z), \tag{2.9}$$

uniformly in z on each closed bounded subset of K, where Δ is the *transfinite diameter* (or capacity) [8, § 4.4] of E. We remark that the existence of *some* triangular scheme $\{ \beta_j^{(n)} \}$ for E for which (2.9) holds, is well-known: for example, on defining the tableau $\{ \beta_j^{(n)} \}$ to consist of the *Fekete points* for E, then (2.9) holds (*cf.* [8, p. 172]). Next, since each $w_j(z)$ in (2.8) is monic of precise degree $j + 1$, there are unique constants $\gamma_j(n)$, $0 \leqslant j \leqslant n$, such that

$$\tilde{w}_n(z) = w_n(z) + \sum_{j=0}^{n} \gamma_j(n)\, w_{j-1}(z), \quad \forall n \geqslant 1, \tag{2.10}$$

For ρ fixed, we assume (as in Cavaretta, Sharma, and Varga [2, § 10]) that there exists a constant λ, with $-\infty \leqslant \lambda < 1$, such that

$$\limsup_{n\to\infty} \left\{ \sum_{j=0}^{n} | \gamma_j(n) | (\Delta\rho)^j \right\}^{1/n} \leqslant \Delta\rho^{\lambda} (< \Delta\rho), \tag{2.11}$$

where Δ is the transfinite diameter of E. With the above assumptions, we can show that, for each $F \in M(E_\rho; \nu)$ and for each n sufficiently large, the rational interpolants $r_{n,\nu}(z; F)$ and $\tilde{r}_{n,\nu}(z; F)$ of $F(z)$ in (2.5) and (2.7) do indeed exist and are unique. Our main result is

THEOREM 2.1 : *Let ρ be fixed with $1 < \rho < \infty$, and suppose that the tableaux* (2.2) *and* (2.3) *have no limit points exterior to E and satisfy the conditions* (2.9) *and* (2.11). *If $F \in M(E_\rho; \nu)$, $\nu \geqslant 0$, and if $\{ \alpha_j \}_{j=1}^{\nu}$ are the ν poles of F in $E_\rho \setminus E$ (listed according to multiplicity), then the rational interpolants $r_{n,\nu}(z; F)$ of* (2.5) *and $\tilde{r}_{n,\nu}(z; F)$ of* (2.7) *satisfy*

$$\lim_{n\to\infty} [\tilde{r}_{n,\nu}(z; F) - r_{n,\nu}(z; F)] = 0, \quad \forall z \in E\rho^{2-\lambda} \setminus \bigcup_{j=1}^{\nu} \{ \alpha_j \}, \tag{2.12}$$

the convergence being uniform and geometric on any closed subset of

$$E\rho^{2-\lambda} \setminus \bigcup_{j=1}^{\nu} \{ \alpha_j \}.$$

More precisely, on any closed subset $\mathscr{H}$ of any $\overline{E}_\tau \setminus \bigcup_{j=1}^{\nu} \{ \alpha_j \}$ with $\rho \leqslant \tau < \infty$, there holds

$$\limsup_{n\to\infty} \{ \max_{z\in\mathscr{H}} | \tilde{r}_{n,\nu}(z; F) - r_{n,\nu}(z; F) | \}^{1/n} \leqslant \tau/\rho^{2-\lambda}. \tag{2.13}$$

We remark that while the rows of tableau (2.2) are defined for every $n = 0, 1, 2, \ldots$, the tableau of (2.3) need only be defined for some infinite increasing subsequence of nonnegative integers n, and the conclusions (2.12) and (2.13) of Theorem 2.1 remain valid for that subsequence. As we shall see in § 3, this observation will be useful in studying Hermite interpolation.

Essential to the proof of Theorem 2.1 is the following extension, due to Saff [5], of the Montessus de Ballore Theorem [3].

THEOREM B : *Suppose that* $F \in M(E_\rho; \nu)$ *for some* $1 < \rho < \infty$, *and* $\nu \geqslant 0$, *and let* $\{ \alpha_j \}_{j=1}^{\nu}$ *denote the* ν *poles of* F *in* $E_\rho \diagdown E$. *Suppose further that the points of the triangular scheme*

$$\begin{array}{l} b_1^{(0)} \\ b_1^{(1)}, b_2^{(1)} \\ \cdots\cdots \\ b_1^{(n)}, b_2^{(n)}, \ldots, b_{n+1}^{(n)} \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array} \tag{2.14}$$

(*which need not be distinct in any row*) *have no limit points exterior to* E, *and that*

$$\lim_{n \to \infty} \left| \prod_{i=1}^{n+1} (z - b_i^{(n)}) \right|^{1/n} = \Delta \exp G(z), \tag{2.15}$$

uniformly on each closed and bounded subset of K. *Then, for all* n *sufficiently large, there exists a unique rational function* $s_{n,\nu}(z)$ *of the form*

$$s_{n,\nu}(z) = \frac{g_{n,\nu}(z)}{h_{n,\nu}(z)}, \quad g_{n,\nu} \in \pi_n, \ h_{n,\nu} \in \pi_\nu, \ h_{n,\nu} \text{ monic}, \tag{2.16}$$

which interpolates $F(z)$ *in the points* $b_1^{(n+\nu)}, b_2^{(n+\nu)}, \ldots, b_{n+\nu+1}^{(n+\nu)}$. *Each* $s_{n,\nu}(z)$ *has precisely* ν *finite poles, and as* $n \to \infty$, *these poles approach, respectively, the* ν *poles of* $F(z)$ *in* $E_\rho \diagdown E$. *The sequence* $\{ s_{n,\nu}(z) \}_{n=n_0}^{\infty}$ *converges to* $F(z)$ *on*

$$E_\rho \setminus \bigcup_{j=1}^{\nu} \{ \alpha_j \},$$

uniformly and geometrically on any closed subset of $E_\rho \setminus \bigcup_{j=1}^{\nu} \{ \alpha_j \}$. *More precisely, on any closed subset* $\mathscr{H}$ *of any* $\overline{E}_\tau \setminus \bigcup_{j=1}^{\nu} \{ \alpha_j \}$ *with* $1 < \tau < \rho$, *there holds*

$$\limsup_{n \to \infty} \{ \max_{z \in \mathscr{H}} | F(z) - s_{n,\nu}(z) | \}^{1/n} \leqslant \tau/\rho . \tag{2.17}$$

Theorem B in particular implies that the monic denominator polynomials of the $s_{n,\nu}(z)$ satisfy

$$\lim_{n\to\infty} h_{n,\nu}(z) = \prod_{i=1}^{\nu} (z - \alpha_i) =: B(z)\,, \tag{2.18}$$

uniformly on each compact set of the plane. In the proof of Theorem 2.1, we also need the following quantitative property.

LEMMA 2.2 : *With the hypotheses of Theorem B, suppose that* $F(z)$ *has a pole of order* $m(\leqslant \nu)$ *at* α_j, *where* $\alpha_j \in \Gamma_{\sigma_j}(\sigma_j < \rho)$. *Then* (*cf.* (2.16)),

$$\limsup_{n\to\infty} \left| \frac{d^k}{dz^k} h_{n,\nu}(\alpha_j) \right|^{1/n} \leqslant \sigma_j/\rho, \text{ for each } k = 0, 1, \ldots, m-1\,. \tag{2.19}$$

Proof : With $B(z)$ as defined in (2.18), the function $f(z) := B(z)\,F(z)$ is analytic throughout E_ρ, and is nonzero at each point α_i, $i = 1, \ldots, \nu$. On multiplying $F(z) - s_{n,\nu}(z)$ by $B(z)\,h_{n,\nu}(z)$, it follows from (2.17) and (2.18) that, for each τ with $1 < \tau < \rho$, there holds

$$\limsup_{n\to\infty} \{ \max_{z\in\Gamma_\tau} | f(z)\,h_{n,\nu}(z) - B(z)\,g_{n,\nu}(z) | \}^{1/n} \leqslant \tau/\rho\,. \tag{2.20}$$

More generally, on setting

$$D_n(z) := f(z)\,h_{n,\nu}(z) - B(z)\,g_{n,\nu}(z)\,,$$

so that $D_n(z)$ is analytic throughout E_ρ, it follows from (2.20) and Cauchy's formula that, for each nonnegative integer k,

$$\limsup_{n\to\infty} \left\{ \max_{z\in\Gamma_\tau} \left| \frac{d^k}{dz^k} D_n(z) \right| \right\}^{1/n} \leqslant \tau/\rho\,, \quad \text{for} \quad 1 < \tau < \rho\,. \tag{2.21}$$

Since $B(\alpha_j) = 0$, then taking $z = \alpha_j$ and $\tau = \sigma_j$ in (2.20) yields

$$\limsup_{n\to\infty} | f(\alpha_j)\,h_{n,\nu}(\alpha_j) |^{1/n} \leqslant \sigma_j/\rho\,,$$

and since $f(\alpha_j) \neq 0$, inequality (2.19) follows for the case $k = 0$. For $k = 1, \ldots, m-1$, inequality (2.19) is easily proved by induction, using the more general estimates of (2.21), the Leibniz formula for differentiating products, and the fact that $B^{(k)}(\alpha_j) = 0$ for $k = 0, 1, \ldots, m-1$. ∎

As a consequence of (2.19), on expanding each $h_{n,\nu}(z)$ in terms of a fixed Lagrange basis of polynomials, there holds on each compact set $\mathscr{H} \subset \mathbb{C}$,

$$\limsup_{n\to\infty} \{ \max_{z\in\mathscr{H}} | h_{n,\nu}(z) - B(z) | \}^{1/n} \leqslant (\max_{i=1,\ldots,\nu} \sigma_i)/\rho\,, \tag{2.22}$$

where $\alpha_i \in \Gamma_{\sigma_i}$ for each $i = 1, \ldots, \nu$.

It is clear from the hypothesis (2.9) of Theorem 2.1 that the results of Theorem B and Lemma 2.2 apply to the triangular scheme of (2.2). The next lemma establishes that the same is true for the triangular scheme of (2.3).

LEMMA 2.3 : *With the hypotheses of Theorem* 2.1, *the polynomials* $\tilde{w}_n(z)$ *of* (2.8) *satisfy*

$$\lim_{n\to\infty} |\tilde{w}_n(z)|^{1/n} = \Delta \exp G(z), \tag{2.23}$$

uniformly in z on each closed bounded subset of K.

Proof : By assumption, the zeros of the polynomials $\tilde{w}_n(z)$ have no limit point exterior to E. Hence, on each compact set in K, the harmonic functions $\frac{1}{n}\log|\tilde{w}_n(z)|$ are, for n sufficiently large, uniformly bounded, and so they form a normal family in K. Now, let R be any fixed number with $\max\{1, \rho^\lambda\} < R < \rho$. Since from (2.9),

$$\lim_{j\to\infty} [\max_{z\in\Gamma_R} |w_j(z)|]^{1/j} = \Delta R < \Delta\rho,$$

it follows from the assumption of (2.11) that

$$\limsup_{n\to\infty}\left[\max_{z\in\Gamma_R}\sum_{j=0}^{n}|\gamma_j(n)|\,|w_{j-1}(z)|\right]^{1/n} \leqslant \Delta\rho^\lambda < \Delta R,$$

and hence (*cf.* (2.10)), we have

$$\lim_{n\to\infty}[\max_{z\in\Gamma_R}|\tilde{w}_n(z)|]^{1/n} = \lim_{n\to\infty}[\max_{z\in\Gamma_R}|w_n(z)|]^{1/n} = \Delta R. \tag{2.24}$$

Noting that ΔR is the transfinite diameter of $\bar{E}_R$, the result of (2.24) implies, by a theorem of Walsh [8, Theorem 4, p. 163], that

$$\lim_{n\to\infty}\frac{1}{n}\log|\tilde{w}_n(z)| = \log\Delta + G(z), \tag{2.25}$$

uniformly on each compact set exterior to Γ_R. But, as the functions $\frac{1}{n}\log|\tilde{w}_n(z)|$ form a normal family in K, then (2.25) necessarily holds uniformly on each compact set in K, which gives (2.23). ■

We can now give the

Proof of Theorem 2.1 : By Lemma 2.3 and the assumption of (2.9), it follows from Theorem B that, for each n sufficiently large, the rational interpolants

$r_{n,\nu}(z)$ of (2.5) and $\tilde{r}_{n,\nu}(z)$ of (2.7) exist and are unique. Furthermore, the monic denominator polynomials $q_{n,\nu}(z)$ and $\tilde{q}_{n,\nu}(z)$ satisfy

$$\lim_{n\to\infty} q_{n,\nu}(z) = \lim_{n\to\infty} \tilde{q}_{n,\nu}(z) = \prod_{i=1}^{\nu} (z - \alpha_i) =: B(z), \tag{2.26}$$

uniformly on every compact set of the plane.

Next, for convenience, set

$$J_n(z) := q_{n,\nu}(z)\, \tilde{q}_{n,\nu}(z)\, F(z). \tag{2.27}$$

Since $r_{n,\nu}(z)$ interpolates $F(z)$ in the points $\{\beta_i^{(n+\nu)}\}_{i=1}^{n+\nu+1}$ from (2.5), it follows, on multiplication by $q_{n,\nu}(z)\,\tilde{q}_{n,\nu}(z)$, that $\tilde{q}_{n,\nu}(z)\,p_{n,\nu}(z)$ is the unique polynomial in $\pi_{n+\nu}$ which interpolates $J_n(z)$ in these $n + \nu + 1$ points. Similarly, $q_{n,\nu}(z)\,\tilde{p}_{n,\nu}(z)$ is from (2.7) the unique polynomial in $\pi_{n+\nu}$ which interpolates $J_n(z)$ in the points $\{\tilde{\beta}_i^{(n+\nu)}\}_{i=1}^{n+\nu+1}$. Since $F(z)$ is, by hypothesis, analytic on E, there exists a constant $\eta > 1$ such that $F(z)$ is analytic on and interior to the level curve Γ_η. Then, for each n sufficiently large, Hermite's formula gives

$$\tilde{q}_{n,\nu}(z)\, p_{n,\nu}(z) = \frac{1}{2\pi i} \int_{\Gamma_\eta} \frac{[w_{n+\nu}(t) - w_{n+\nu}(z)]\, J_n(t)\, dt}{w_{n+\nu}(t)\,(t - z)}, \quad \forall z \in \mathbb{C}, \tag{2.28}$$

$$q_{n,\nu}(z)\, \tilde{p}_{n,\nu}(z) = \frac{1}{2\pi i} \int_{\Gamma_\eta} \frac{[\tilde{w}_{n+\nu}(t) - \tilde{w}_{n+\nu}(z)]\, J_n(t)\, dt}{\tilde{w}_{n+\nu}(t)\,(t - z)}, \quad \forall z \in \mathbb{C}. \tag{2.29}$$

On subtracting, we have

$$\tilde{q}_{n,\nu}(z)\, p_{n,\nu}(z) - q_{n,\nu}(z)\, \tilde{p}_{n,\nu}(z) = \frac{1}{2\pi i} \int_{\Gamma_\eta} \frac{K_n(t, z)\, J_n(t)\, dt}{w_{n+\nu}(t)\, \tilde{w}_{n+\nu}(t)\,(t - z)}, \tag{2.30}$$

where

$$K_n(t, z) := w_{n+\nu}(t)\, \tilde{w}_{n+\nu}(z) - w_{n+\nu}(z)\, \tilde{w}_{n+\nu}(t). \tag{2.31}$$

Next, let $\{\alpha_j^*\}_{j=1}^{s}$, $s \leqslant \nu$, denote the *distinct* poles of $F(z)$ in $E_\rho \setminus E$, so that $\bigcup_{j=1}^{s} \{\alpha_j^*\} = \bigcup_{j=1}^{\nu} \{\alpha_j\}$. Let R be any constant such that $\max\{1, \rho^\lambda\} < R < \rho$ and such that all the poles of $F(z)$ lie interior to Γ_R. Further, select s small circles $C_j := \{t \in \mathbb{C} : |t - \alpha_j^*| = \delta_j\}$ which are mutually exterior and satisfy $C_j \subset E_R \setminus E$ for each $j = 1, 2, \ldots, s$. Setting $C_{s+1} := \Gamma_R$, then Cauchy's theorem applied to the integral of (2.30) gives, for all n sufficiently large, that

$$\tilde{q}_{n,\nu}(z)\, p_{n,\nu}(z) - q_{n,\nu}(z)\tilde{p}_{n,\nu}(z) = \sum_{j=1}^{s+1} I_j^{(n)}(z), \tag{2.32}$$

where

$$I_j^{(n)}(z) := \frac{1}{2\pi i}\int_{C_j}\frac{K_n(t,z)\,J_n(t)\,dt}{w_{n+\nu}(t)\,\tilde{w}_{n+\nu}(t)\,(t-z)}, \quad j = 1, 2, \dots, s+1. \quad (2.33)$$

In (2.33), the contour C_{s+1} is taken to be positively oriented, while the remaining contours C_j, $1 \leqslant j \leqslant s$, are all negatively oriented.

To estimate the integrals in (2.33), we first note that using (2.10) we can express $K_n(t, z)$ as

$$K_n(t,z) = \sum_{i=0}^{n+\nu}\gamma_i(n+\nu)\left[w_{n+\nu}(t)\,w_{i-1}(z) - w_{n+\nu}(z)\,w_{i-1}(t)\right]. \quad (2.34)$$

From the hypotheses (2.9) and (2.11), it then follows that, for each $\tau \geqslant \rho$,

$$\limsup_{n\to\infty}\{\max|K_n(t,z)| : t\in\Gamma_R,\ z\in\Gamma_\tau\}^{1/n} \leqslant \Delta^2\,\tau\rho^\lambda,$$

and from (2.9) and (2.23) we have

$$\lim_{n\to\infty}\left[\min|w_{n+\nu}(t)\,\tilde{w}_{n+\nu}(t)| : t\in\Gamma_R\right]^{1/n} = (\Delta R)^2.$$

Further, we note from (2.26) and (2.27) that the functions $J_n(t)$ are uniformly bounded (independent of n) on the contour $C_{s+1} = \Gamma_R$. Putting the above facts together yields from (2.33) that

$$\limsup_{n\to\infty}\left[\max|I_{s+1}^{(n)}(z)| : z\in\Gamma_\tau\right]^{1/n} \leqslant \tau\rho^\lambda/R^2, \quad \tau\geqslant\rho. \quad (2.35)$$

Next, to estimate the integrals around the poles α_j^*, we note that for each $j = 1, 2, \dots, s$, $I_j^{(n)}(z)$ is just the negative of the residue at $t = \alpha_j^*$ of the function

$$\frac{K_n(t,z)\,J_n(t)}{w_{n+\nu}(t)\,\tilde{w}_{n+\nu}(t)\,(t-z)}. \quad (2.36)$$

If $\alpha_j^*\in\Gamma_{\sigma_j^*}$, then it follows from (2.9), (2.11), (2.23), and (2.34) that for each $k = 0, 1, \dots$,

$$\limsup_{n\to\infty}\left[\max\left|\frac{\partial^k}{\partial t^k}K_n(\alpha_j^*;z)\right| : z\in\Gamma_\tau\right]^{1/n} \leqslant \Delta^2\,\tau\rho^\lambda,\ \tau\geqslant\rho, \quad (2.37)$$

and

$$\limsup_{n\to\infty}\left|\frac{d^k}{dt^k}\,\frac{1}{w_{n+\nu}(t)\,\tilde{w}_{n+\nu}(t)}\right|^{1/n} \leqslant 1/(\Delta\sigma_j^*)^2 \quad \text{at} \quad t = \alpha_j^*. \quad (2.38)$$

Furthermore, if α_j^* is a pole of $F(t)$ of order m, then from Lemma 2.2 we have, for each $k = 0, 1, \ldots, m-1$,

$$\limsup_{n\to\infty} | q_{n,\nu}^{(k)}(\alpha_j^*) |^{1/n} \leqslant \sigma_j^*/\rho , \quad \limsup_{n\to\infty} | \tilde{q}_{n,\nu}^{(k)}(\alpha_j^*) |^{1/n} \leqslant \sigma_j^*/\rho ,$$

and, consequently, for such k

$$\limsup_{n\to\infty} \left| \frac{d^k}{dt^k} [J_n(t)(t-\alpha_j^*)^m] \right|^{1/n} \leqslant (\sigma_j^*/\rho)^2 \quad \text{at} \quad t = \alpha_j^* . \tag{2.39}$$

On combining (2.37), (2.38), and (2.39) to estimate the residue at $t = \alpha_j^*$ of the function in (2.36), we find that, for each $j = 1, 2, \ldots, s$,

$$\limsup_{n\to\infty} [\max | I_j^{(n)}(z) | : z \in \Gamma_\tau]^{1/n} \leqslant \frac{\Delta^2 \tau \rho^\lambda}{(\Delta\sigma_j^*)^2} \left(\frac{\sigma_j^*}{\rho}\right)^2 = \frac{\tau}{\rho^{2-\lambda}}, \tau \geqslant \rho . \tag{2.40}$$

Thus, from (2.32) and the estimate of (2.35), it follows that

$$\limsup_{n\to\infty} [\max_{z\in\Gamma_\tau} | \tilde{q}_{n,\nu}(z) \, p_{n,\nu}(z) - q_{n,\nu}(z) \, \tilde{p}_{n,\nu}(z) |]^{1/n} \leqslant \tau\rho^\lambda/R^2, \tau \geqslant \rho ,$$

and so, on letting R approach ρ and applying the Maximum Principle, we have

$$\limsup_{n\to\infty} [\max_{z\in E_\tau} | \tilde{q}_{n,\nu}(z) \, p_{n,\nu}(z) - q_{n,\nu}(z) \, \tilde{p}_{n,\nu}(z) |]^{1/n} \leqslant \tau/\rho^{2-\lambda}, \quad \tau \geqslant \rho . \tag{2.41}$$

Finally, appealing to the equations (2.26), the desired conclusions (2.12) and (2.13) of Theorem 2.1 then follow. ∎

COROLLARY 2.4 : *With the hypotheses of Theorem* 2.1, *there holds on every compact set* $\mathscr{H} \subset \mathbb{C}$,

$$\limsup_{n\to\infty} [\max_{z\in\mathscr{H}} | \tilde{q}_{n,\nu}(z) - q_{n,\nu}(z) |]^{1/n} \leqslant 1/\rho^{1-\lambda} . \tag{2.42}$$

Proof : Since $\tilde{q}_{n,\nu}(z)$ and $q_{n,\nu}(z)$ are, for n large, each monic polynomials of degree ν, the difference $d_n(z) := \tilde{q}_{n,\nu}(z) - q_{n,\nu}(z)$ is a polynomial of degree at most $\nu - 1$. Moreover, $d_n(z)$ is the unique polynomial in $\pi_{\nu-1}$ which interpolates the function

$$G_n(z) := (\tilde{q}_{n,\nu}(z) \, p_{n,\nu}(z) - q_{n,\nu}(z) \, \tilde{p}_{n,\nu}(z))/p_{n,\nu}(z) \tag{2.43}$$

in the ν zeros of $q_{n,\nu}(z)$. From Theorem B (*cf.* (2.26)), these zeros approach, respectively, the ν poles of $F(z)$ in $E_\rho \setminus E$. Also, as

$$\lim_{n\to\infty} p_{n,\nu}(z) = B(z) \, F(z) =: f(z) , \tag{2.44}$$

uniformly on compact subsets of E_ρ, and as $f(z)$ is analytic and different from zero in each pole of $F(z)$, then there exist s small circles C_j :

$$C_j : | z - \alpha_j^* | = \delta_j , \qquad j = 1, ..., s$$

(as in the proof of Theorem 2.1) such that for n sufficiently large, $p_{n,\nu}(z)$ is different from zero on the closed interior of each C_j. Consequently, for n large, the function $G_n(z)$ is analytic inside and on each C_j, $j = 1, ..., s$. Since the zeros of $q_{n,\nu}(z)$ will eventually all be contained in the union of the interiors of the circles C_j, Hermite's formula again gives

$$d_n(z) = \frac{1}{2\pi i} \sum_{j=1}^{s} \int_{C_j} \frac{(q_{n,\nu}(t) - q_{n,\nu}(z))\, G_n(t)\, dt}{q_{n,\nu}(t)\,(t - z)}, \quad \forall z \in \mathbb{C}, \tag{2.45}$$

where now the integration is taken in the positive sense around each C_j. But, from (2.44) and from (2.41) with $\tau = \rho$, we have for $1 \leqslant j \leqslant s$,

$$\limsup_{n\to\infty} [\max_{t\in C_j} | G_n(t) |]^{1/n} \leqslant \limsup_{n\to\infty} [\max_{t\in C_j} | \tilde{q}_{n,\nu}(t)\, p_{n,\nu}(t) - q_{n,\nu}(t)\, \tilde{p}_{n,\nu}(t) |]^{1/n}$$
$$\leqslant \rho/\rho^{2-\lambda} = 1/\rho^{1-\lambda} .$$

Using this estimate together with the limiting behavior (2.26) of the polynomials $q_{n,\nu}(z)$, it follows from (2.45) that

$$\limsup_{n\to\infty} [\max_{z\in\mathcal{H}} | d_n(z) |]^{1/n} \leqslant 1/\rho^{1-\lambda} ,$$

where $\mathcal{H}$ is any compact set in the plane, which establishes (2.42). ■

If only the triangular interpolation schemes are specified, but not the point set E, then D. D. Warner has shown [9] that, under rather mild regularity conditions, the schemes determine a geometric setting in which Saff's Theorem B remains valid. Such assumptions lead to further generalizations of Theorem 2.1.

3. SOME EXAMPLES

In this section, we discuss some special cases of Theorem 2.1 and Corollary 2.4. We begin with the results quoted in the introduction concerning rational interpolation in the origin and in the roots of unity.

Example 1 : Let E be the closed unit disk $| z | \leqslant 1$, so that E has capacity $\Delta = 1$. The associated Green's function is then simply $G(z) = \log | z |$, and the level curves Γ_σ are the circles $| z | = \sigma$. Next, select the n-th rows of the tableaux (2.2) and (2.3) to consist, respectively, of all zeros and of the $(n+1)$-st roots of unity; that is, with the notation of (2.8),

$$w_n(z) = z^{n+1}, \qquad \tilde{w}_n(z) = z^{n+1} - 1. \tag{3.1}$$

Trivially, $w_n(z)$ satisfies (2.9) and, furthermore, the inequality of (2.11) is valid, for every $\rho > 1$, with $\lambda = 0$. Thus, Theorem 2.1 gives the conclusions (1.10) and (1.11) of Theorem 1.1, provided that $F(z) \in M_\rho(\nu)$ has all of its ν poles *exterior* to $E : |z| \leqslant 1$. However, slight modifications in the proof of Theorem 2.1 show that, for these special interpolation schemes, we can indeed allow some or all of the ν poles of $F(z)$ to lie in the punctured disk $0 < |z| < 1$, and this will not effect the validity of Theorem 2.1.

Next, we establish the sharpness assertion (1.12) of Theorem 1.1. For any given ρ with $1 < \rho < \infty$, and any fixed complex α with $0 < |\alpha| < \rho$, $|\alpha| \neq 1$, the particular meromorphic function

$$\hat{F}(z) := \frac{1}{z-\alpha} + \frac{1}{z-\rho} \tag{3.2}$$

is evidently an element of $M_\rho(1)$. Because $\nu = 1$ in this example, the associated interpolants (*cf.* (1.6) and (1.8)) of $\hat{F}(z)$ are

$$S_{n,1}(z;\hat{F}) = \frac{U_{n,1}(z)}{V_{n,1}(z)}, \quad \text{and} \quad R_{n,1}(z;\hat{F}) = \frac{P_{n,1}(z)}{Q_{n,1}(z)},$$

where we write

$$V_{n,1}(z) = z + \lambda_n, \quad \text{and} \quad Q_{n,1}(z) = z + \gamma_n.$$

It can be verified that

$$\lambda_n = \frac{\alpha\rho^{n+2} + \alpha^{n+2}\rho - \rho - \alpha}{2 - \rho^{n+2} - \alpha^{n+2}}, \quad \gamma_n = -\rho\alpha\left(\frac{\rho^{n+1} + \alpha^{n+1}}{\rho^{n+2} + \alpha^{n+2}}\right), \tag{3.3}$$

and that

$$\left.\begin{aligned} U_{n,1}(z) &= 2 - \frac{\rho V_{n,1}(\rho)(z^{n+1} - \rho^{n+1})}{(\rho^{n+2} - 1)(z-\rho)} - \frac{\alpha V_{n,1}(\alpha)(z^{n+1} - \alpha^{n+1})}{(\alpha^{n+2} - 1)(z-\alpha)}, \\ P_{n,1}(z) &= 2 - \frac{Q_{n,1}(\rho)(z^{n+1} - \rho^{n+1})}{\rho^{n+1}(z-\rho)} - \frac{Q_{n,1}(\alpha)(z^{n+1} - \alpha^{n+1})}{\alpha^{n+1}(z-\alpha)}. \end{aligned}\right\} \tag{3.4}$$

Note that since $\rho > |\alpha|$, both λ_n and γ_n tend, from (3.3), to $-\alpha$ as $n \to \infty$. This, of course, implies that the poles of $S_{n,1}(z;\hat{F})$ and $R_{n,1}(z;\hat{F})$ both tend to α as $n \to \infty$, which is in agreement with Theorem B. Using (3.3) and (3.4), straight-forward (but lengthy) calculations give

$$S_{n,1}(z\,;\hat{F}) - R_{n,1}(z\,;\hat{F}) = \frac{z^{n+2}(\rho-\alpha)^2(\rho+\alpha-2z)}{\rho^{2n+4}(z-\alpha)^3(z-\rho)} + \mathcal{O}\left(\frac{1}{\rho^n}\right), \quad \text{as} \quad n\to\infty \tag{3.5}$$

the last term holding uniformly on any bounded set in $\mathbb{C}\setminus(\{\alpha\}\cup\{\rho\})$. From this, it follows that

$$\lim_{n\to\infty}\left\{\min_{|z|=\rho^2}|S_{n,1}(z\,;\hat{F}) - R_{n,1}(z\,;\hat{F})|\right\} =$$

$$= \min_{0\leqslant\theta\leqslant 2\pi}\frac{|\rho-\alpha|^2\,|\rho+\alpha-2\rho^2e^{i\theta}|}{|\rho^2e^{i\theta}-\alpha|^3\,|\rho^2e^{i\theta}-\rho|} \geqslant \frac{|\rho-\alpha|^2(2\rho^2-\rho-|\alpha|)}{(\rho^2+|\alpha|)^3(\rho^2+\rho)} > 0\,. \tag{3.6}$$

Thus, for the particular function $\hat{F}(z)$ of (3.2), we see that (3.6) implies (1.12) of Theorem 1.1, for the case $\nu = 1$. It thus remains to establish (1.12) for any integer $\nu \geqslant 2$ and any $1 < \rho < \infty$. This is done by using the previous construction as follows.

Let us regard the function $\hat{F}(z)$ of (3.2) as a function of z, α, and ρ, i.e.,

$$\hat{F}(z) = \hat{F}(z\,;\alpha,\rho)\,.$$

For any ρ with $1 < \rho < \infty$, and for any positive integer ν, we set

$$\hat{F}_\nu(z) := \hat{F}(z^\nu\,;\alpha^\nu,\rho^\nu) = \frac{1}{z^\nu-\alpha^\nu} + \frac{1}{z^\nu-\rho^\nu} \in M_\rho(\nu)\,, \tag{3.7}$$

where, as in (3.2), $0 < |\alpha| < \rho$ and $|\alpha| \neq 1$. Then, the rational interpolants $S_{n,\nu}(z\,;\hat{F}_\nu)$ and $R_{n,\nu}(z\,;\hat{F}_\nu)$ of $\hat{F}_\nu$ are easily seen to be related to the rational interpolants $S_{n,1}(z\,;\hat{F})$ and $R_{n,1}(z\,;\hat{F})$ of $\hat{F}$ as follows :

$$\begin{aligned} S_{(m+1)\nu-1,\nu}(z\,;\hat{F}_\nu) &\equiv S_{m,1}(z^\nu\,;\hat{F}(\cdot\,;\alpha^\nu,\rho^\nu))\,,\\ R_{(m+1)\nu-1,\nu}(z\,;\hat{F}_\nu) &\equiv R_{m,1}(z^\nu\,;\hat{F}(\cdot\,;\alpha^\nu,\rho^\nu)),\ m = 1, 2, \ldots\,. \end{aligned} \tag{3.8}$$

Because of the relationships of (3.8), it follows from (3.6) that

$$\lim_{m\to\infty}\left\{\min_{|z|=\rho^2}|S_{(m+1)\nu-1,\nu}(z\,;\hat{F}_\nu) - R_{(m+1)\nu-1,\nu}(z\,;\hat{F}_\nu)|\right\}$$

$$\geqslant \frac{|\rho^\nu-\alpha^\nu|^2(2\rho^{2\nu}-\rho^\nu-|\alpha|^\nu)}{(\rho^{2\nu}+|\alpha|^\nu)^3(\rho^{2\nu}+\rho^\nu)} > 0\,,$$

and hence

$$\limsup_{n\to\infty} \{ \min_{|z|=\rho^2} | S_{n,\nu}(z; \hat{F}_\nu) - R_{n,\nu}(z; \hat{F}_\nu) | \} > 0 ,$$

for each positive integer ν, and each ρ with $1 < \rho < \infty$. This completes the proof of the sharpness assertion of Theorem 1.1.

Finally, we remark that Corollary 1.2 is an immediate consequence of Corollary 2.4 with $\lambda = 0$.

Example 2 : If we wish to compare (Padé) rational interpolation in the origin with *Hermite* rational interpolation of order $k(\geqslant 2)$ in the roots of unity, we again take E to be the closed unit disk and we set

$$w_{n-1}(z) := z^n, n = 1, 2, \ldots; \quad \tilde{w}_{km-1}(z) := (z^m - 1)^k , \quad m = 1, 2, \ldots .$$

Then, it can be verified that the inequality of (2.11) (with $n = km$) holds for every $\rho > 1$ with $\lambda = 1 - 1/k$. Thus, Theorem 2.1 (modified to allow poles in the punctured disk $0 < | z | < 1$, as discussed in example 1) gives for any $F \in M_\rho(\nu)$,

$$\lim_{m\to\infty} \{ \hat{S}_{km-1-\nu,\nu}(z; F) - R_{km-1-\nu,\nu}(z; F) \} = 0, \forall z \in D_{\rho^{1+1/k}} / \bigcup_{j=1}^{\nu} \{ \alpha_j \} , \tag{3.9}$$

where $\hat{S}_{km-1-\nu,\nu}(z; F)$ is the rational function of type $(km - 1 - \nu, \nu)$ which interpolates $F(z)$ in the m-th roots of unity, each considered of multiplicity k, and where $R_{km-1-\nu,\nu}(z; F)$ is the corresponding Padé approximant to $F(z)$. We note that the result (3.9) for the case $\nu = 0$ appears as the case $l = 1$ in Cavaretta, Sharma, and Varga [2, Theorem 3].

Example 3 : Here we take E to be the real interval $[-1, 1]$, which has capacity $\Delta = 1/2$. The level curve Γ_σ $(\sigma > 1)$ for E is the ellipse in the z-plane with foci ± 1, and semi-major axis $(\sigma + 1/\sigma)/2$. With $T_n(x) = \cos(n \arccos x)$ denoting the familiar Chebyshev polynomial (of the first kind) of degree n, we shall compare Lagrange interpolation in the Chebyshev zeros with Hermite interpolation of order k $(\geqslant 2)$ in these zeros. For this purpose, we define (*cf.* (2.8)) the monic polynomials

$$w_{n-1}(z) := 2^{1-n} T_n(z), n = 1, 2, \ldots, \tilde{w}_{km-1}(z) := (2^{1-m} T_m(z))^k, m = 1, 2, \ldots .$$

It is well-known (*cf.* [8, p. 163]) that the $w_n(z)$ satisfy (2.9), and moreover, it can be verified that the inequality of (2.11) (with $n = km - 1$) holds with $\lambda = (k - 2)/k$ for every $\rho > 1$. Hence, if $F(z)$ is analytic on $[-1, 1]$ and meromorphic with precisely ν poles $\{ \alpha_j \}_{j=1}^{\nu}$ inside the ellipse Γ_ρ (i.e., $F \in M(E_\rho ; \nu)$), then Theorem 2.1 gives

$$\lim_{m \to \infty} \{ \tilde{r}_{km-1-\nu,\nu}(z ; F) - r_{km-1-\nu,\nu}(z ; F) \} = 0, \forall z \in E_{\rho^{(k+2)/k}} \setminus \bigcup_{j=1}^{\nu} \{ \alpha_j \} . \tag{3.10}$$

As a special case, we see that the choice $k = 2$ gives convergence to zero in $E_{\rho^2} \setminus \bigcup_{j=1}^{\nu} \{ \alpha_j \}$, which is reminiscent of the result of Theorem 1.1.

Example 4 : Let E be a closed bounded point set (containing more than one point) whose complement K is simply connected, and let $\mathscr{F}_n(z)$, for $n = 0, 1, \ldots$, denote the Faber polynomials [6, Chap. 2] for E. For simplicity, we assume that E has capacity $\Delta = 1$. If $w = \varphi(z)$ maps, one-to-one and conformally, the complement K onto the domain $| w | > 1$ so that $\varphi(\infty) = \infty$, then $\mathscr{F}_n(z)$ is the principal part of the expansion of $[\varphi(z)]^n$ as a Laurent series in a neighborhood of $z = \infty$. Specifically, if

$$\varphi(z) = z + c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \cdots \tag{3.11}$$

in a neighborhood of $z = \infty$, then

$$[\varphi(z)]^n = z^n + c_{n-1}^{(n)} z^{n-1} + c_{n-2}^{(n)} z^{n-2} + \cdots + c_0^{(n)} + \frac{c_{-1}^{(n)}}{z} + \frac{c_{-2}^{(n)}}{z^2} + \cdots, \tag{3.12}$$

and, by definition,

$$\mathscr{F}_n(z) := z^n + c_{n-1}^{(n)} z^{n-1} + \cdots + c_0^{(n)}, n = 0, 1, \ldots . \tag{3.13}$$

It is known that the zeros of $\mathscr{F}_n(z)$ have no limit points in K and, moreover (*cf.* [6, p. 135])

$$\lim_{n \to \infty} | \mathscr{F}_n(z) |^{1/n} = | \varphi(z) | , \tag{3.14}$$

uniformly on each compact set in K. Choosing the interpolation scheme of (2.2) to consist of the zeros of the Faber polynomials, i.e., setting

$$w_{n-1}(z) := \mathscr{F}_n(z) ,$$

then the condition of (2.9), with $G(z) = \log | \varphi(z) |$, is clearly satisfied. For a comparison scheme, we consider Hermite interpolation of order 2 in these Faber polynomial zeros, i.e., we set

$$\tilde{w}_{2m-1}(z) := [\mathscr{F}_m(z)]^2, \quad m = 1, 2, \ldots \tag{3.15}$$

Now, if $z = \psi(w)$ denotes the inverse of the function $\varphi(z)$, we have (*cf.* [6, p. 138])

$$[\mathscr{F}_m(z)]^2 = \sum_{j=0}^{2m} a_j^{(m)} \mathscr{F}_j(z), \tag{3.16}$$

where, for any $r > 1$,

$$a_j^{(m)} = \frac{1}{2\pi i} \int_{|w|=r} \frac{[\mathscr{F}_m(\psi(w))]^2}{w^{j+1}} dw, \quad j = 0, 1, \ldots, 2m. \tag{3.17}$$

Now, it is known [6, p. 132] that

$$\mathscr{F}_m(\psi(w)) = w^m + \frac{1}{w} M_m\left(\frac{1}{w}\right), \quad \forall\, |w| > 1, \tag{3.18}$$

where $M_m(1/w)$ is analytic at $w = \infty$ and has a Laurent series converging for all $|w| > 1$. Substituting (3.18) in (3.17) gives

$$a_j^{(m)} = \frac{1}{2\pi i} \int_{|w|=r} \frac{(w^{2m} + 2w^{m-1} M_m(1/w))}{w^{j+1}} dw, \quad j = 0, 1, \ldots, 2m. \tag{3.19}$$

From this, we immediately see that

$$a_{2m}^{(m)} = 1; \quad a_j^{(m)} = 0 \quad \text{for} \quad m \leqslant j < 2m. \tag{3.20}$$

Next, we estimate the remaining coefficients $a_j^{(m)}$, $0 \leqslant j < m$. For $0 \leqslant j < m$, we have from (3.19) that

$$a_j^{(m)} = \frac{1}{2\pi i} \int_{|w|=r} \frac{2w^{m-1} M_m(1/w)\, dw}{w^{j+1}}. \tag{3.21}$$

Writing $M_m\left(\frac{1}{w}\right) = \sum_{k=0}^{\infty} \gamma_k^{(m)} w^{-k}$ for all $|w| > 1$, then it is evident that

$$a_j^{(m)} = 2\gamma_{m-j-1}^{(m)}, \quad 0 \leqslant j < m. \tag{3.22}$$

Let $1 < \sigma < \rho$. Then, we can obviously write

$$\gamma_{m-j-1}^{(m)} = \frac{1}{2\pi i} \int_{|w|=\sigma} w^{m-j-1} \left\{ \frac{1}{w} M_m\left(\frac{1}{w}\right) \right\} dw. \tag{3.23}$$

From [6, p. 134, inequality (2)], $\left|\frac{1}{w} M_m\left(\frac{1}{w}\right)\right| \leqslant \mu(\sigma)\,\sigma^m$ for $|w| = \sigma$, where $\mu(\sigma)$ is a positive constant, independent of m. Thus, from (3.23),

$$|\gamma_{m-j-1}^{(m)}| \leqslant \mu(\sigma)\,\sigma^{2m-j}, \quad 0 \leqslant j < m.$$

Hence from (3.20) and (3.22), we have

$$\limsup_{m\to\infty}\left\{\sum_{j=0}^{2m-1} |a_j^{(m)}|\,\rho^j\right\}^{1/2m} = \limsup_{m\to\infty}\left\{\sum_{j=0}^{m-1} |a_j^{(m)}|\,\rho^j\right\}^{1/2m} \tag{3.24}$$
$$\leqslant \limsup_{m\to\infty}\left\{2\,\mu(\sigma)\sum_{j=0}^{m-1}\sigma^{2m-j}\,\rho^j\right\}^{1/2m} = \sqrt{\sigma\rho}\,.$$

Letting σ tend to unity, we see that for $n = 2m - 1$, inequality (2.11) holds with $\lambda = \frac{1}{2}$ (since $\Delta = 1$). In a similar (but more tedious fashion), it can be shown that if we consider Hermite interpolation of order k ($\geqslant 2$) in the zeros of the Faber polynomials, i.e. (*cf.* (3.15)) if

$$\tilde{w}_{km-1}(z) := [\mathscr{F}_m(z)]^k, \quad m = 1, 2, \ldots, \tag{3.25}$$

and (*cf.* (3.16)) if

$$[\mathscr{F}_m(z)]^k := \sum_{j=0}^{km} a_j^{(m)}(k)\,\mathscr{F}_j(z), \tag{3.26}$$

then (3.24) can be generalized to

$$\limsup_{m\to\infty}\left\{\sum_{j=0}^{km-1} |a_j^{(m)}(k)|\,\rho^j\right\}^{1/km} \leqslant (\rho^{k-1}\sigma)^{1/k}, \tag{3.27}$$

so that on letting σ again tend to unity, we see inequality (2.11) now holds with $\lambda = 1 - 1/k$. In other words, Theorem 2.1 gives for any $F \in M(E_\rho; \nu)$,

$$\lim_{m\to\infty}\{\check{S}_{km-1-\nu,\nu}(z;F) - \check{R}_{km-1-\nu,\nu}(z;F)\} = 0, \quad \forall z \in E_{\rho^{1+1/k}} \setminus \bigcup_{j=1}^{\nu}\{\alpha_j\}, \tag{3.28}$$

where $\check{S}_{km-1-\nu,\nu}(z;F)$ is the rational function of type $(km - 1 - \nu, \nu)$ which interpolates $F(z)$ in the zeros of the Faber polynomial $\mathscr{F}_{km}(z)$, while

$$\check{R}_{km-1-\nu,\nu}(z;F)$$

is the rational function of type $(km - 1 - \nu, \nu)$ which interpolates $F(z)$, with multiplicity k, in each of the zeros of the Faber polynomial $\mathcal{F}_m(z)$.

Finally, although the set $E = [-1, +1]$ of example 3 is a special case of example 4, we note however that the comparison of Lagrange interpolation in the zeros of the Faber polynomial $\mathcal{F}_{mk}(z)$, with that of Hermite interpolation of order k in the zeros of the Faber polynomial $\mathcal{F}_m(z)$, gives the associated exponent (*cf.* (2.11)) of example 3 as $\lambda = \frac{k-2}{k}$, which is *smaller* than the associated exponent $\lambda' = \frac{k-1}{k}$ of example 4.

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