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On the zeros of generalized Bessel polynomials. II

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Dedicated to Professor Emil Grosswald

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§ 9. PROOFS OF NEW RESULTS

Because it is fundamental in deriving Theorem 3.1 of this paper, we state for completeness the following result of Saff and Varga [15]. (Note that $y_n(z;\, a)$ of this present article (cf. (2.1)) corresponds to $Y_n^{(a-2)}(-z)$ of [15].)

THEOREM 9.1. *Let $\{p_k(z)\}_{k=0}^{n}$ be a sequence of polynomials of respective degrees k which satisfy the three-term recurrence relation*

$$(9.1)\qquad p_k(z) = \left(\frac{z}{b_k} + 1\right) p_{k-1}(z) - \frac{z}{c_k}\, p_{k-2}(z)\ (k = 1, 2, \ldots, n),$$

where the b_k's and c_k's are positive real numbers for all k, $1 \le k \le n$, and where $p_{-1}(z) := 0$, $p_0(z) := p_0 \neq 0$. Set

$$(9.2)\qquad \alpha := \min\,\{b_k(1 - b_{k-1}c_k^{-1}) : k = 1, 2, \ldots, n\},\ b_0 := 0.$$

Then, if $\alpha > 0$, the parabolic region

$$(9.3)\qquad \mathcal{P}_\alpha := \{z = x + iy \in \mathbb{C} : y^2 \le 4\alpha(x + \alpha), x > -\alpha\}$$

contains no zeros of $p_1(z), p_2(z), \ldots, p_n(z)$.

To give the reader more insight in the connection between the GBP and Theorem 9.1, it first can be verified from the definition of $\theta_n(z;\, a)$ in (2.4) that, for $n + a - 1 > 0$, the sequence

$$(9.4)\qquad \left\{\frac{\Gamma(n+a-1)}{\Gamma(n+a-1+k)}\, 2^k\theta_k\left(\frac{z}{2};\, n+a-k\right)\right\}_{k=1}^{n}$$

satisfies the recurrence relation (9.1) with

$$b_k = n+a-2+k,\ k=1,2,\ldots,n;\ c_1 = 1,$$

$$c_k = \frac{(n+a-2+k)(n+a-3+k)}{k=1},\ k=2,3,\ldots,n.$$

Thus, it follows in this case from (9.2) that $\alpha = n+a-1>0$, so that the polynomials in (9.4) have no zeros in $\mathscr{P}_{n+a-1}$. With the transformation $z\to 2/z$, we see from (2.4) that all zeros of the GBP $y_n(z;\,a)$ lie inside the cardioidal region (3.1), as proved in [15].

Before proceeding to the proofs of our new results, we first establish a slight generalization of a lemma in [14].

LEMMA 9.2 *Let the sequences* $\{p_k(z)\}_{k=0}^{n}$ *be as in Theorem* 9.1 *with* $p_0=1$, *and let the real numbers* $\lambda_0 := 1, \lambda_1, \lambda_2, \ldots, \lambda_n$ *satisfy*

$$(9.5)\qquad 0<\lambda_k<1 \text{ for } k=1,2,\ldots,n-1;\ 0\le\lambda_n<1.$$

Then, $p_n(z)$ *is different from zero at any point* z *which satisfies the* n *inequalities*

$$(9.6)\qquad |z| + \frac{\operatorname{Re} z}{|z|}\left\{\frac{b_k(2c_k\lambda_{k-1}-b_{k-1})}{2c_k\lambda_{k-1}(1-\lambda_k)}\right\} > \frac{b_k b_{k-1}}{2c_k\lambda_{k-1}(1-\lambda_k)},\ k=1,2,\ldots,n.$$

PROOF. This follows by imitating the argument in the proof of Lemma 3.1 of [14] which uses special values of b_k, c_k (viz., those for certain Padé approximants to e^z). In the proof, one needs the fact that $p_n(z)$ and $p_{n-1}(z)$ have no zeros in common, which follows, on assuming the contrary, from using the recurrence relation backwards to establish the contradiction that $p_0(z)=0$. □

We now apply Lemma 9.2 to the polynomials given in (9.4) to reach the intermediate results of

COROLLARY 9.3. *The polynomial* $\theta_n(z/2;\,a)$ *has, for* $n+a-1>0$, *no non-real zeros in*

$$(9.7)\qquad \mathscr{S} := \left\{z = re^{i\theta}\in\mathbb{C}:\ r>2(2n+a-2)-\frac{2(n+a-1)}{1-\cos\theta}\right\}.$$

PROOF. Putting $\lambda_k := (1+\cos\theta)/2$ for $k=1,2,\ldots,n$, then the inequalities (9.6) show that any non-real point $z=re^{i\theta}\in\mathbb{C}$ satisfying (9.7) is not a zero of $\theta_n(z/2;\,a)$. □

With the preceding result, we then come to the

PROOF OF THEOREM 4.1(i). As the sector $S(n,a)$ of (4.1) is invariant under the transformation $z \to 2/z$, it suffices to prove the assertion for the polynomial $\theta_n(z/2;\, a)$, instead of for the polynomial $y_n(z;\, a)$. First consider the following set of points

$$G_1 := \left\{ z = re^{i\theta} \in \mathbb{C}:\ r > 2n + a - 2,\ 0 < |\theta| \le \cos^{-1}\left(\frac{-a}{2n+a-2} \right) \right\}.$$

One easily verifies from (9.7) that $G_1 \subset \mathscr{S}$, which establishes that $\theta_n(z/2;\, a)$ is zero-free in G_1. From (9.4) and Theorem 9.1, it follows that $\theta_n(z/2;\, a)$ is also zero-free in $\mathscr{P}_{n+a-1}$. Rewriting $\mathscr{P}_{n+a-1}$, we obtain that

$$\mathscr{P}_{n+a-1} \supset G_2 := \left\{ z = re^{i\theta} \in \mathbb{C}:\ r \le 2n + a - 2,\ 0 \le |\theta| \le \cos^{-1}\left(\frac{-a}{2n+a-2} \right) \right\}.$$

Combining these results, we see that $\theta_n(z/2;\, a)$ is zero-free in $G_1 \cup G_2$, or, equivalently, all zeros of $\theta_n(z/2;\, a)$ lie in $\mathbb{C} \setminus (G_1 \cup G_2) = S(n,a)$.

The second assertion of Theorem 4.1(i) follows from the fact that (4.2) implies $(1-\sigma)/(1+\sigma) \ge (-a)/(2n+a-2)$, and hence, $S(n,a) \subset S_\sigma$ for those values of n and a which satisfy (4.2), as well as $n + a - 1 > 0$ and $n \ge 2$. □

The proof of the sharpness result in Theorem 4.1(ii), along with the proof of Theorem 3.3, will be given after the other theorems of Section 4 have been established. Now, we give the

PROOF OF THEOREM 4.3. As the open left half-plane is itself a sector, namely the sector S_1 from (4.3), we shall prove the result for $\theta_n(z/2;\, a)$, since, after the transformation $z \to 2/z$, this will imply the assertion for $y_n(z;\, a)$.

As in [14, p. 9], we apply Lemma 9.2 to the polynomials of (9.4). For the choice $\lambda_k = \frac{1}{2}$, $k = 1, 2, \ldots, n-1$, $\lambda_n = 0$, the inequalities (9.6) imply that, for $n \ge 3$, the polynomial $\theta_n(z/2;\, a)$ is zero-free on

$$B_1 := \{ z \in \mathbb{C}:\ \operatorname{Re} z \ge 0,\ |z| > 2(n-2) \}.$$

Again invoking the result of Theorem 9.1, we also find that $\theta_n(z/2;\, a)$ is zero-free on

$$B_2 := \{ z \in \mathbb{C}:\ \operatorname{Re} z \ge 0,\ |z| \le 2(n+a-1) \}.$$

Thus, for $a \ge -1$, we find that all the zeros of $\theta_n(z/2;\, a)$ belong to $\mathbb{C} \setminus (B_1 \cup B_2) = S_1$.

For $n = 2$, one directly verifies that $y_2(z;\, a)$ has all its zeros in the open left half-plane iff $a \ge -1$. (It should be remarked that the case $a = -1$ is degenerate, with $y_2(z;\, -1) \equiv 1$.) For $a < -1$, one similarly verifies that $y_2(z;\, a)$ has a zero in the right half-plane. □

By yet another choice for the λ_k's in Lemma 9.2 and by treating particular cases by the Wall criterion for stability, we can give the

PROOF OF THEOREM 4.4. For $n \geq 7$, $a \geq -2$, we use a method of proof, similar to that of [14, p. 10], by choosing, in Lemma 9.2,

$$\lambda_k = \tfrac{1}{2},\ k = 1, 2, \ldots, n-2;\ \lambda_{n-1} = \frac{n-1}{3n-5};\ \lambda_n = 0.$$

This time, Lemma 9.2 implies that the $\theta_n(z/2;\, a)$ are zero-free in the closed right half-plane, outside a disk with radius $2(n-3)$. Using $a \geq -2$ in combination with the zero-free parabolic region, we find that $\theta_n(z/2;\, a)$ is zero-free in the closed right half-plane for $n \geq 7$. Next, for $n = 6$, we use the following values of λ_k in Lemma 9.2:

$$\lambda_1 = \lambda_2 = \tfrac{1}{2};\ \lambda_3 = \tfrac{2}{3};\ \lambda_4 = \tfrac{5}{8},\ \lambda_5 = \tfrac{5}{12},\ \lambda_6 = 0,$$

and the proof proceeds as above, with the radius of the disk from (9.6) now being 6.

The remaining cases, $n = 3$, 4, and 5, then follow, together with the sharpness result, by applying the Wall criterion [20] and checking degenerate cases separately. □

PROOF OF THEOREM 4.5. To prove the Grosswald conjecture [5, p. 162, number 6], concerning the stability of the $y_n(z;\, a)$ for arbitrary (but fixed) a and sufficiently large n, it suffices, in view of Theorem 4.4, to restrict ourselves to the case $a < -2$.

This time, Lemma 9.2 will be applied using

$$\lambda_{n-j} = (2^j - 1)/(2^{j+1} - 1),\ j = 0, 1, \ldots, n-1. \tag{9.8}$$

With the well-known inequalities

$$2^{n+2} > n,\ 2^{n+5} \geq n^2 \text{ for all } n = 1, 2, \ldots,$$

one can easily show that the coefficient of $\operatorname{Re} z/|z|$ in (9.6) is nonnegative for $n \geq 2^{3-a}$ and $k = 1, 2, \ldots, n$. Lemma 9.2 then leads to a zero-free region in the closed right half-plane, outside the disk of radius R given by

$$R := \max\left\{\frac{k-1}{2\lambda_{k-1}(1-\lambda_k)} : k = 2, \ldots, n\right\}.$$

Inserting the values of λ_k from (9.8), we find that

$$R = \max\{(2 - 2^{-j-1})(n-j-1) : j = 0, 1, \ldots, n-2\}.$$

Now, for $n \geq 32$, it is easy to show that the maximum of the function

$$f(x) = (2 - 2^{-x-1})(n - x - 1)$$

on $[0,\ n-2]$ occurs at a point $\hat{x}$ with $n - 2 > \hat{x} > \log_2 n - 3$, and, furthermore, that $f(\hat{x}) < 2(n - \hat{x} - 1)$. Hence, $\theta_n(z/2;\, a)$ is zero-free in $\{z \in \mathbb{C} : |z| \geq 2(n - \hat{x} - 1),\ \operatorname{Re} z \geq 0\}$. Combining this result with the zero-free parabolic region, one finds that all the zeros of $\theta_n(z/2;\, a)$ lie in the open left half-plane if $\log_2 n + a - 3 \geq 0$, from which the bound $n_0(a)$, given in Theorem 4.5, follows. □

We now return to the sharpness results that have been left unproven up to now.

PROOF OF THEOREMS 3.3 AND 4.1(ii). The proof follows by imitating the argument given in Saff and Varga [16] for establishing Theorems 2.2 and 2.3 of [16]. It depends heavily upon the existence of zeros of a certain form for $y_n(z;\, a)$ which can be proved by adapting the proof of Theorem 2.1 from [16] by replacing the parameter v_j appearing there by $n+a_n-2$. Specifically, let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers satisfying

$$n+a_n-2\geq 0\ (n\in\mathbb{N}),\quad \lim_{n\to\infty}\frac{a_n}{n}=\sigma-1\ (\sigma\in(0,\infty)).$$

Then, $y_n(z;\, a_n)$ has zeros of the form

$$\frac{2}{(2n+a_n-1)}\exp\left[\pm i\cos^{-1}\left(\frac{2-a_n}{2n+a_n-1}\right)\right]+\mathcal{O}((2n+a_n-1)^{-5/3})\text{ as } n\to\infty.$$

The proof then continues as in [16]. □

For the proof of the main result of Section 5, we can again refer to a technique developed for the study of the zeros of Padé approximants to the exponential function e^z.

PROOF OF THEOREM 5.1. As previously stated in Section 5, the upper bound of (5.1) is a known result. To establish the lower bound of (5.1), it suffices to imitate the method of proof in [17, Theorems 2.1 and 2.2], using now the polynomials $\Gamma(n+a-1)2^n\theta_n(z/2;\, a)/\Gamma(2n+a-1)$. To this end, one has to replace the parameter v in [17] by $n+a-2$. □

PROOF OF THEOREM 6.1. Apart from the upper bound for ξ in (6.2), the only part of this theorem that does not follow from the preceding sections on taking $a=2$, is the lower bound for the modulus of the zeros appearing in the set A_2, defined by

$$(9.9)\qquad A_2:=\left\{z=re^{i\theta}:\ \tfrac{2}{3}\pi\geq|\theta|\geq\cos^{-1}\left(-\frac{1}{n}\right),\ \frac{1}{\sqrt{n(n+1)}}<r<\frac{1-\cos\theta}{n+1}\right\}.$$

To establish this lower bound, we recall from (2.6) of Theorem 2.2 that

$$y_n(z)=y_n(z;\, 2)=e^{1/z}W_{0,n+\frac{1}{2}}\left(\frac{2}{z}\right),$$

which leads to the fact that the function

$$w_n(z):=e^{-(z/2)}y_n\left(\frac{2}{z}\right)$$

satisfies the differential equation

$$\frac{d^2w(z)}{dz^2}=\left\{\frac{1}{4}+\frac{n(n+1)}{z^2}\right\}w(z).$$

Using a path of integration wholly within $\mathbb{C}\backslash\{0\}$, one can easily show that any solution of this differential equation satisfies

$$(9.10)\qquad \int_{z_1}^{z_2}\left\{\tfrac{1}{4}+\frac{n(n+1)}{z^2}\right\}|w(z)|^2dz=\frac{dw}{dz}\,\bar{w}\,\Big|_{z_1}^{z_2}-\int_{z_1}^{z_2}\left|\frac{dw}{dz}\right|^2\overline{dz}.$$

Now, let $\tau=\varrho e^{i\varphi}$, with $\pi/2<\varphi<\pi$, be any zero of $w_n(z)$ and consider the path of integration given by the half-line

$$\tau\cdot(1+e^{-i\varphi/2}x),\ 0\le x<\infty,$$

which is the same path of integration as employed in [17, Theorem 2.2]. Because of the restriction on φ, it is easy to prove that the integrals in (9.10) converge. Also, since $dw/dz\ \bar{w}|_\tau^\infty=0$, we find that

$$(9.11)\qquad \int_0^\infty \tau^2e^{-i\varphi}\left\{\tfrac{1}{4}+\frac{n(n+1)}{\tau^2(1+e^{-i\varphi/2}x)^2}\right\}|\tilde{w}(x)|^2dx=-\int_0^\infty\left|\frac{d\tilde{w}}{dx}\right|^2dx,$$

where $\tilde{w}(x)$ is given by

$$\tilde{w}(x):=w_n(\tau(1+e^{-i\varphi/2}x)),\ 0\le x<\infty.$$

Taking imaginary parts in (9.11), we deduce, using $\lambda:=n(n+1)$, that

$$(9.12)\qquad \int_0^\infty\left\{\frac{\varrho^2}{4\lambda}\cos\frac{\varphi}{2}-\frac{x+\cos\varphi/2}{(x^2+2x\cos(\varphi/2)+1)^2}\right\}|\tilde{w}(x)|^2\,dx=0.$$

As the limit as $x\to\infty$ of the expression in braces in (9.12) is *positive* $((\pi/2)<\varphi<\pi)$, the minimum of this function on the real axis must be *negative*. Making the restriction $(2\pi/3)\ge\varphi>\pi/2$, this leads to the fact that the integrand must be negative for $x=0$, i.e.,

$$\frac{\varrho^2}{4\lambda}\cos\frac{\varphi}{2}-\cos\frac{\varphi}{2}<0,$$

or, equivalently

$$\varrho<2\sqrt{\lambda}.$$

This shows that the zeros of $y_n(z)$ with $(2\pi/3)\ge|\theta|>\pi/2$ must satisfy

$$\frac{2}{|z|}<2\sqrt{\lambda},$$

which gives the desired lower bound for r in (9.9).

To complete the proof of Theorem 6.1, it remains to establish the upper bound for ξ in (6.2). However, this upper bound for ξ is a simple consequence of the first part of this theorem when $n\ge3$. For the remaining case $n=2$, this upper bound follows by direct computation. □

Finally, we sketch the proofs of the theorems in Section 7.

PROOF OF THEOREM 7.1. First, one can directly verify from (2.1) and (2.4) that the following integral representation is valid:

$$(9.13)\qquad \Gamma(n+a-1)2^n\theta_n(z/2;\,a)=\int_0^\infty e^{-t}(t+z)^n t^{n+a-2}dt,\ (n+a-1>0),$$

the path of integration being the nonnegative real axis. Similarly, it is known (cf. [18, eq. (4.7)]) that the Padé numerator $P_{n,\nu}(z)$ of the (n,ν)-th Padé rational approximation to e^z of (2.9), has the integral representation

$$(9.14)\qquad (n+\nu)!\,P_{n,\nu}(z)=\int_0^\infty e^{-t}(t+z)^n t^\nu dt.$$

Because of this, the asymptotic methods of [18], based on steepest descent methods applied to the integral of (9.14), can be analogously applied to the integral representation of (9.13) for any fixed real a. More precisely, with

$$(9.15)\qquad \nu_n:=n+a-2,$$

then

$$\lim_{n\to\infty}\nu_n/n=1 \text{ for any fixed real } a.$$

Thus, we deduce, as similarly in the special case $\sigma=1$ of [18, Theorem 2.2], that $\hat{z}$ is a limit point of the zeros of the normalized polynomials

$$\theta_n((2n+a-2)z/2;\,a)$$

iff

$$\hat{z}\in D_1:=\left\{z\in\mathbb{C}:\ \left|\frac{ze^{\sqrt{1+z^2}}}{1+\sqrt{1+z^2}}\right|=1,\ |z|\le 1,\ \text{and}\ \operatorname{Re} z\le 0\right\}.$$

For any fixed real a, it then follows, by means of the inversion $z\to 1/z$ that z is a limit point of zeros of the normalized GPB $y_n(2z/(2n+a-2);\,a)$ iff $z\in\Gamma$, where Γ is defined in (7.1) and (7.2). This establishes the first part of Theorem 7.1. The second part of Theorem 7.1 similarly follows as in the special case $\sigma=1$ of [18, Theorem 2.3]. □

PROOF OF THEOREM 7.2. To establish Theorem 7.2, we again apply the asymptotic results of [18], and, for convenience, we use the same notations and definitions as in [18]. We further set

$$(9.16)\qquad \lambda_\tau:=\frac{1-\tau}{1+\tau}\ \text{for any}\ 0\le\tau<\infty.$$

As shown in [18, eq. (4.2)], for any τ with $0\le\tau<\infty$,

$$(9.17)\qquad \frac{zw_\tau'(z)}{w_\tau(z)}=g_\tau(z):=\sqrt{1+z^2-2\lambda_\tau z},\ \text{for all}\ z\in\mathbb{C}\setminus\mathscr{B}_\tau,$$

from which, after some calculations, it follows that for any fixed z with $0<|z|<1$,

$$(9.18)\qquad \frac{d}{d\tau}\ln|w_\tau(z)| = \frac{2}{(1+\tau)^2}\ln|\zeta+\sqrt{1+\zeta^2}| \text{ where } \zeta := \frac{z-\lambda_\tau}{\sqrt{1-\lambda_\tau^2}}.$$

Setting $t=\zeta+\sqrt{1+\zeta^2}$, so that $\zeta=\frac{1}{2}(t-(1/t))$, it can be verified that any ζ with $\operatorname{Re}\zeta<0$ has its image in the t-plane in the open unit disk. Consequently, from (9.18)

$$(9.19)\qquad \frac{d}{d\tau}\ln|w_\tau(z)|<0 \text{ for any fixed } z \text{ with } 0<|z|<1, \text{ and } \operatorname{Re} z<\lambda_\tau.$$

This can be applied as follows. If

$$(9.20)\qquad D_\tau := \{z\in\mathbb{C}: |w_\tau(z)|=1,\ |z|\le 1, \text{ and } \operatorname{Re} z\le\lambda_\tau\},\ 0\le\tau<\infty,$$

it is known from [18] that D_τ is a Jordan arc which lies interior to the unit disk, except for the points $z_\tau^{\pm} := \exp\{\pm i\cos^{-1}\lambda_\tau\}$. Thus, (9.19) establishes that $D_{\tau'}$ lies "strictly to the left" of D_τ for any $0\le\tau<\tau'<\infty$, as indicated in fig. 6 below.

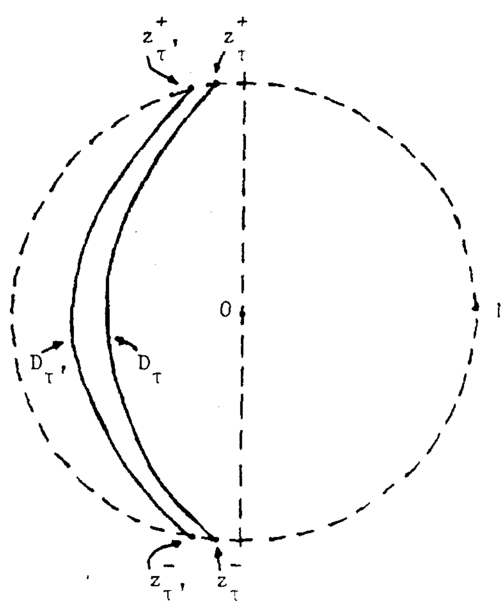

Fig. 6. $\tau'>\tau>1$.

Using the notation of [18], define

$$(9.21)\qquad \hat{N}_\tau(z) := \left(\frac{h_\tau''(t_\tau^-(z))}{h_\tau''(t_\tau^+(z))}\right)^{1/2},$$

which is analytic and single-valued on $\mathbb{C}\setminus(\mathscr{A}_\tau\cup\{0\})$ for $0\le\tau<\infty$.

It can be verified from (9.21) and the definitions of [18] that

$$\hat{N}_\tau(z) = \frac{g_\tau(z)+1-\lambda_\tau z}{z\sqrt{1-\lambda_\tau^2}}, \quad \forall z \in \mathbb{C}\backslash(\mathcal{R}_\tau \cup \{0\}), \tag{9.22}$$

and that

$$\frac{z\hat{N}'_\tau(z)}{\hat{N}_\tau(z)} = -\frac{1}{g_\tau(z)}, \quad \forall z \in \mathbb{C}\backslash(\mathcal{R}_\tau \cup \{0\}). \tag{9.23}$$

Because (cf. [18], eq. (4.1)) Re $g_\tau(z)>0$ on $\mathbb{C}\backslash\mathcal{R}_\tau$, it follows from (9.23) that $|\hat{N}_\tau(z)|$ is strictly decreasing, for any fixed θ, on the ray $\{z=re^{i\theta}: 0<r<\infty\}$ in $\mathbb{C}\backslash(\mathcal{R}_\tau \cup \{0\})$. Furthermore, as Im $g_\tau(z)<0$ along the (open) arc of the unit circle from $z=z_\tau^+$ to $z=-1$, it also follows from (9.23) that $|\hat{N}_\tau(z)|$ is strictly increasing along this arc. (Similarly, $|\hat{N}_\tau(z)|$ is strictly decreasing along the arc of the unit circle from $z=-1$ to $z=z_\tau^-$.) These observations will be useful below.

Considering now any zero $z_{k,n}$ of $\theta_n((2n+a-2)z/2;\, a)$, we must show that, for all n sufficiently large, $z_{k,n}$ lies to the left of D_1 of (9.20). Since D_1 is a Jordan arc in $|\ |\le 1$, we may assume, without loss of generality, that $|z_{k,n}|\le 1$. Next, from [18, eq. (4.30)],

$$|w_{\sigma(n)}(z_{k,n})|^{n+\nu_n} = |\hat{N}_{\sigma(n)}(z_{k,n})|\left\{1+\mathcal{O}\left(\frac{1}{n+\nu_n}\right)\right\}, \text{ as } n\to\infty, \tag{9.24}$$

where (cf. (9.15))

$$\sigma(n) := \nu_n/n = (n+a-2)/n, \text{ for all } n \text{ sufficiently large}, \tag{9.25}$$

and where the term $\mathcal{O}(1/(n+\nu_n))$ in (9.24) holds uniformly on any compact subset of $\mathbb{C}\backslash(\mathcal{R}_1 \cup \{0\})$. On the other hand, it follows from Theorem 4.1 that this zero $z_{k,n}$ satisfies $|\text{Arg } z_{k,n}| > \cos^{-1}(-a/(2n+a-2))$. Thus, it follows from previous observations from (9.23) that, with $\alpha_n := \cos^{-1}(-a/(2n+a-2))$ and $\mu_n := \exp(i\alpha_n)$,

$$\min\{|\hat{N}_{\sigma(n)}(z)| : |z|\le 1 \text{ and } |\text{Arg } z|\ge \alpha_n\} = |\hat{N}_{\sigma(n)}(\mu_n)|. \tag{9.26}$$

Furthermore, direct calculations with the right side of (9.26), using (9.22), show that there exists a positive constant c such that

$$|\hat{N}_{\sigma(n)}(\mu_n)| = \frac{|g_{\sigma(n)}(\mu_n)+1-\lambda_{\sigma(n)}\mu_n|}{\sqrt{1-\lambda_{\sigma(n)}^2}} \ge 1+\frac{c}{\sqrt{n}}, \text{ as } n\to\infty. \tag{9.27}$$

Thus, (9.26) and (9.27) together imply, with (9.24), that there exists a constant $c'>0$ such that

$$|w_{\sigma(n)}(z_{k,n})|^{n+\nu_n} \ge \left(1+\frac{c}{\sqrt{n}}\right)\left\{1+\mathcal{O}\left(\frac{1}{n}\right)\right\} \ge 1+\frac{c'}{\sqrt{n}}, \text{ as } n\to\infty. \tag{9.28}$$

so that $z_{k,n}$ must lie strictly to the left of $D_{\sigma(n)}$, for n sufficiently large. But, as (cf. (9.25))

$$\sigma(n) = \frac{n+a-2}{n} = 1 + \frac{a-2}{n},$$

then $\sigma(n)$ strictly decreases to unity as $n \to \infty$, for any fixed $a \geq 2$. Hence, from our previous discussion, each zero $z_{k,n}$ of $\theta_n((2n+a-2)z/2;\, a)$ must lie, for each n sufficiently large, to the left of D_1. Performing the inversion $z \to 1/z$, this implies that each zero z of $y_n(2z/(2n+a-2);\, a)$ for n sufficiently large, must satisfy (cf. (7.1) and (7.2))

$$|\omega(z)| > 1 \text{ and } \operatorname{Re} z < 0,$$

which establishes Theorem 7.2. □

PROOF OF THEOREM 7.3. For n a positive odd integer, let $z_n(a)$ denote the negative real zero of $\theta_n((2n+a-2)z/2;\, a)$ so that (cf. (9.24))

$$(9.29) \qquad |w_{\sigma(n)}(z_n(a))|^{n+\nu_n} = |\hat{N}_{\sigma(n)}(z_n(a))| \left\{1 + \mathcal{O}\left(\frac{1}{n+\nu_n}\right)\right\}, \text{ as } n \to \infty,$$

where ν_n and $\sigma(n)$ are given respectively by (9.15) and (9.25). Then, let $\tilde{z}_n(a)$ be defined as the unique negative real number such that

$$(9.30) \qquad |w_{\sigma(n)}(\tilde{z}_n(a))|^{n+\nu_n} = |\hat{N}_{\sigma(n)}(\tilde{z}_n(a))|,$$

for each $n \geq 1$. That $\tilde{z}_n(a)$ is uniquely defined for each $n \geq 1$ follows from the fact that $|\hat{N}_\tau(z)|$ is, from (9.23), strictly decreasing on the ray $\{z = -r,\ 0 < r < \infty\}$, while $|w_\tau(z)|$ is, from (9.17), strictly increasing (from zero to infinity) on this same ray. Next, by means of Taylor series expansions and identities involving $w_{\sigma(n)}(z)$ and $\hat{N}_{\sigma(n)}(z)$ (which we omit for reasons of brevity), it can be verified that

$$(9.31) \qquad z_n(a) = \tilde{z}_n(a) + \mathcal{O}\left(\frac{1}{(n+\nu_n)^2}\right), \text{ as } n \to \infty.$$

Thus, if we can express $\tilde{z}_n(a)$ as

$$(9.32) \qquad \tilde{z}_n(a) = \hat{r} + \frac{\gamma_1(a)}{(n+\nu_n)} + \mathcal{O}\left(\frac{1}{(n+\nu_n)^2}\right), \text{ as } n \to \infty,$$

where $\hat{r}$ is defined in (7.5) and $\gamma_1(a)$ is independent of n, then a consequence of (9.31) is that

$$(9.33) \qquad z_n(a) = \hat{r} + \frac{\gamma_1(a)}{(n+\nu_n)} + \mathcal{O}\left(\frac{1}{(n+\nu_n)^2}\right), \text{ as } n \to \infty.$$

To establish (9.32), we have from (9.30) that

$$(9.34) \qquad |w_{\sigma(n)}(\tilde{z}_n(a))| = |\hat{N}_{\sigma(n)}(\tilde{z}_n(a))|^{1/(n+\nu_n)}.$$

Assuming the form (9.32), it follows from (9.17) that

$$(9.35)\qquad w_{\sigma(n)}(\tilde{z}_n(a)) = w_{\sigma(n)}(\hat{r})\left\{1 + \frac{\gamma_1(a)g_{\sigma(n)}(\hat{r})}{(n+\nu_n)\hat{r}} + \mathcal{O}\left(\frac{1}{(n+\nu_n)^2}\right)\right\}, \text{ as } n\to\infty.$$

Now, from (9.18), it can be verified that

$$|w_{\sigma(n)}(\hat{r})| = |w_1(\hat{r})| + \frac{(a-2)|w_1(\hat{r})|\cdot \ln(\hat{r}+\sqrt{1+\hat{r}^2})}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right), \text{ as } n\to\infty.$$

But as $|w_1(\hat{r})| = 1$ from [18, eq. (4.40)], the above reduces to

$$(9.36)\qquad |w_{\sigma(n)}(\hat{r})| = 1 + \frac{(a-2)\ln(\hat{r}+\sqrt{1+\hat{r}^2})}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right), \ n\to\infty.$$

Similarly, using the definition of $g_r(z)$ in (9.17), it can be shown that

$$(9.37)\qquad 1 + \frac{\gamma_1(a)g_{\sigma(n)}(\hat{r})}{(n+\nu_n)\hat{r}} + \mathcal{O}\left(\frac{1}{(n+\nu_n)^2}\right) = 1 + \frac{\gamma_1(a)\sqrt{1+\hat{r}^2}}{2n\hat{r}} + \mathcal{O}\left(\frac{1}{n^2}\right), \ n\to\infty.$$

Thus, combining (9.35)–(9.37) yields

$$(9.38)\qquad \begin{cases} |w_{\sigma(n)}(\tilde{z}_n(a))| = \Bigg|1 + \dfrac{1}{2n}\Bigg\{(a-2)\ln(\hat{r}+\sqrt{1+\hat{r}^2}) + \\ \quad + \dfrac{\gamma_1(a)\sqrt{1+\hat{r}^2}}{\hat{r}}\Bigg\} + \mathcal{O}\left(\dfrac{1}{n^2}\right)\Bigg|. \end{cases}$$

Next, using (9.23), it can be verified that

$$(9.39)\qquad |\hat{N}_{\sigma(n)}(\tilde{z}_n(a))|^{1/(n+\nu_n)} = 1 + \frac{\sqrt{1+\hat{r}^2}}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right), \text{ as } n\to\infty.$$

Thus, on combining (9.38) and (9.39) in (9.34), we deduce that

$$(9.40)\qquad \gamma_1(a) = \hat{r}\left\{\frac{\sqrt{1+\hat{r}^2} + (2-a)\ln(\hat{r}+\sqrt{1+\hat{r}^2})}{\sqrt{1+\hat{r}^2}}\right\}.$$

Now, as $n+\nu_n = 2n+a-2$, it follows that the negative real zero of $\theta_n((2n+a-2)z/2;\, a)$, for n odd, is given by

$$(9.41)\qquad z_n(a) = \hat{r} + \frac{\gamma_1(a)}{(2n+a-2)} + \mathcal{O}\left(\frac{1}{(2n+a-2)^2}\right), \text{ as } n\to\infty.$$

Recalling (2.4), then (9.41) implies that the negative real zero $\alpha_n(a)$ of the non-normalized GBP $y_n(z;\, a)$, for n odd, satisfies the desired result (7.4) of Theorem 7.3. □

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