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On the zeros of generalized Bessel polynomials. I

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Dedicated to Professor Emil Grosswald

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ABSTRACT

In this paper, the location of the zeros of generalized Bessel polynomials is studied, leading to many improvements of previous results.

§1. INTRODUCTION

In a paper published in 1949, Krall and Frink [9] studied properties of the so-called *Bessel polynomials* (BP), and they defined a generalization which has become known under the name of *generalized Bessel polynomials* (GBP).

In his recent monograph, Grosswald [5] has given a systematic treatment of the GPB, including a chapter on the location of their zeros. Because of the still growing interest in GBP for their many applications, it is important to study the location of zeros of GBP even more closely than has been done by Grosswald [5] or [6].

The aim of this paper is to improve upon previous results concerning the location of the zeros of GBP, using techniques developed in a series of papers by Saff and Varga [14–18] for i) studying the zeros of sequences of polynomials satisfying a three-term recurrence relation, and for ii) studying the zeros and poles of Padé approximants to the exponential function e^z. In particular, we will prove (cf. Theorem 4.5) a conjecture of Grosswald on the stability of GBP,

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and will disprove (cf. Theorem 7.3) a conjecture of Luke on the asymptotic behavior of the real zero of odd degree GBP.

The outline of this paper is as follows. In Section 2, the definitions and notations for the GBP (using notations and normalizations appearing in Grosswald [5]) are given, along with some of their known properties (Theorem 2.2). In Sections 3, 4, and 5, we respectively treat cardioidal regions, infinite sectors, and annuli and rectangles containing all zeros of GBP. In Section 6, the results of the previous sections are combined with differential equations techniques, leading to even better bounds for the zeros of ordinary Bessel polynomials. In Section 7, new asymptotic results for the zeros of normalized GBP are determined, as $n\to\infty$, which generalize results of Olver [12] for the special case $a=2$.

As a consequence of our results on the location of the zeros of GBP, another proof is given, in Section 8, for the result of de Bruin [2], concerning convergence of certain sequences of Padé approximants taken from the Padé table for the confluent hypergeometric function ${}_1F_1(1;\, c;\, z)$ with $c\neq 0, -1, -2, \ldots$. In Section 9, we then present the proofs of our new results.

Finally, because Professor Grosswald's excellent monograph [5] has both enlightened us and inspired us, we respectfully dedicate this paper to him.

§ 2. NOTATIONS AND DEFINITIONS

The generalized Bessel polynomials (GBP) will now be defined by an explicit formula. As usual, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{R}$ the set of all real numbers, and $\mathbb{C}$ the set of all complex numbers.

DEFINITION 2.1. *The GBP $y_n(z;\, a)$ with $n\in\mathbb{N}\cup\{0\}$, $a\in\mathbb{R}$, is given by*

$$(2.1)\qquad y_n(z;\, a):=\sum_{k=0}^{n}\binom{n}{k}(n+a-1)_k\left(\frac{z}{2}\right)^k.$$

In this definition, the Pochhammer notation for ascending factorials is used:

$$(x)_0:=1,\quad (x)_k:=x(x+1)\ldots(x+k-1)\ \text{for}\ k\in\mathbb{N},\ \text{where}\ x\in\mathbb{C},$$

and it is immediately clear that $y_n(z;\, a)$ is of exact degree n if and only if $a\notin\{-2n+2, -2n+3, \ldots, -n, -n+1\}$. The cases where the degree is less than n will be called *degenerate*.

On taking $a=2$ in (2.1), we are led to the *ordinary Bessel polynomial*, which will be denoted by

$$(2.2)\qquad y_n(z):=y_n(z;\, 2).$$

Throughout this paper, we assume (in view of the applications), unless otherwise stated, that n and a satisfy the restrictions

$$(2.3)\qquad a\in\mathbb{R},\quad n+a-1>0.$$

Along with the GBP, we will use the *reversed polynomials*, defined by the formula

(2.4) $\theta_n(z; a) := z^n y_n(z^{-1}; a); \quad \theta_n(z) := z^n y_n(z^{-1})$.

Finally, we list for future reference some of the known properties of the GBP and their reversed polynomials (cf. Grosswald [5]):

THEOREM 2.2. *The GBP $y_n(z; a)$ of degree n satisfies*

(2.5) $z^2 y_n'' + (az+2)y_n' - n(n+a-1)y_n = 0,$

and

(2.6) $y_n(z; a) = e^{1/z}\left(\frac{z}{2}\right)^{1-a/2} W_{1-(a/2), ((a-1)/2)+n}\left(\frac{2}{z}\right),$

where $W_{k,m}(z)$ denotes the Whittaker function, satisfying the differential equation

(2.7) $w'' + \left\{-\frac{1}{4} + \frac{k}{z} + \frac{1-4m^2}{4z^2}\right\} w = 0.$

For $a \in \mathbb{N} \cup \{0\}$ and $n+a-2 \geq 0$,

(2.8) $y_n(z; a) = \frac{(2n+a-2)!}{(n+a-2)!}\left(\frac{z}{2}\right)^n P_{n,n+a-2}\left(\frac{2}{z}\right),$

where the polynomial $P_{n,\nu}(z)$ is the Padé numerator of the (n, ν)-th Padé rational approximant for e^z, given by

(2.9) $P_{n,\nu}(z) = \sum_{j=0}^{n} (n+\nu-j)!\,n!\,z^j/\{(n+\nu)!\,j!\,(n-j)!\}.$

The reversed polynomial $\theta_n(z; a)$ of degree n satisfies

(2.10) $z\theta_n'' - (2z+2n-2+a)\theta_n' + 2n\theta_n = 0,$

and

(2.11) $\theta_n(z; a) = 2^{a/2-1} e^z z^{n+a/2-1} W_{1-(a/2), ((a-1)/2)+n}(2z).$

§ 3. CARDIOIDAL REGIONS CONTAINING ALL ZEROS OF GBP

The main result of this section is the known result of

THEOREM 3.1. (Saff and Varga [15, Thm. 5.2]). *All zeros of the GBP $y_n(z; a)$ lie in the cardioidal region*

(3.1) $C(n, a) := \left\{z = re^{i\theta} \in \mathbb{C}: 0 < r < \frac{1-\cos\theta}{n+a-1}\right\} \cup \left\{\frac{-2}{n+a-1}\right\}.$

Earlier, Dočev [3] had shown that all the zeros of the GBP $y_n(z; a)$ lie in the disk $D(n, a) := \{z \in \mathbb{C}: |z| \leq 2/(n+a-1)\}$. Since $C(n, a)$ is properly contained in

$D(n, a)$ except for the point $-2/(n+a-1)$, then Theorem 3.1 gives an improvement over Dočev's result. Related to Theorem 3.1 is the following result first proved by Underhill:

THEOREM 3.2 (Underhill [19], Saff and Varga [17, Thm. 2.2]). *For any integers a and $n \geq 1$ with $n+a-2 \geq 0$, the zeros of the GBP $y_n(z; a)$ satisfy*:

$$|z| < 2/[\mu(2n+a-2)], \tag{3.2}$$

where $\mu \doteqdot 0.278465$ is the unique positive root of $\mu e^{1+\mu} = 1$.

It can be seen that, for certain rather restricted values of n, a, and z (i.e., $n \geq 4$, a an integer, $-n+2 \leq a \leq -(1-2\mu)(n-1)/(1-\mu)$, $z = re^{i\theta}$ with θ sufficiently close to π), (3.2) can give partial improvements over (3.1). However, if one considers the collection of all GBP for all n and all a, the cardioidal region of (3.1) is *sharp*, in the following sense.

Let $\mathscr{S}$ denote the set of all zeros of the normalized polynomials

$$\left\{y_n\left(\frac{2z}{n+a-1}; a\right): n \in \mathbb{N},\ a \in \mathbb{R},\ n+a-1>0\right\}.$$

Then, by Theorem 3.1, $\mathscr{S}$ is contained in the normalized cardioidal region

$$C := \left\{z = re^{i\theta} \in \mathbb{C}: 0 < r < \frac{1-\cos\theta}{2}\right\} \cup \{-1\}. \tag{3.3}$$

Our new result (whose proof is given in Section 9) is

THEOREM 3.3. *Each boundary point of C of* (3.3) *is an accumulation point of the set $\mathscr{S}$ of all zeros of the normalized polynomials*

$$\left\{y_n\left(\frac{2z}{n+a-1}; a\right): n \in \mathbb{N},\ a \in \mathbb{R},\ n+a-1>0\right\}.$$

§ 4. INFINITE SECTORS CONTAINING ALL ZEROS

In this section, we give sectors with vertex at the origin which contain all zeros of the GBP. In Theorem 4.5, a conjecture of Grosswald [5] concerning the stability of the GBP will be settled.

The basic result of this section is the new result of

THEOREM 4.1 (i) *For $n \geq 2$, all zeros of $y_n(z; a)$ belong to the sector*

$$S(n, a) := \left\{z = re^{i\theta} \in \mathbb{C}: |\theta| > \cos^{-1}\left(\frac{-a}{2n+a-2}\right),\ \text{where } -\pi < \theta \leq \pi\right\}. \tag{4.1}$$

In particular, if n and a satisfy

$$\frac{n+a-1}{n-1} \geq \sigma\ (n \geq 2) \tag{4.2}$$

for a fixed real number σ, $0<\sigma<\infty$, then all zeros of the polynomial $y_n(z;\,a)$ belong to the sector

$$(4.3)\qquad S_\sigma := \left\{ z = re^{i\theta} \in \mathbb{C}\colon\ |\theta| > \cos^{-1}\left(\frac{1-\sigma}{1+\sigma}\right),\ \text{where } -\pi<\theta\le\pi \right\}.$$

(ii) *The sector S_σ from the first part of the theorem is sharp in the following sense. Consider any sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers satisfying* (4.2) *and*

$$(4.4)\qquad n+a_n-2\ge 0,\quad \lim_{n\to\infty}\frac{a_n}{n}=\sigma-1.$$

Then, all zeros of the sequence of polynomials $\{y_n(z;\,a_n)\}_{n=1}^{\infty}$ belong to S_σ, and any smaller infinite sector with vertex at the origin (i.e., an $S_{\sigma+\varepsilon}$ with $0<\varepsilon$) fails to contain all zeros of the polynomials mentioned. In fact, for each $\varepsilon>0$ sufficiently small, there are infinitely many zeros of this sequence in $S_\sigma\setminus S_{\sigma+\varepsilon}$.

It is obvious that Theorem 4.1(i) also holds for $n=1$ if one replaces the inequality sign in the definition of $S(n,a)$ in (4.1) by "$\ge$". Furthermore, Theorem 4.1 implies immediately a result on stability for the GBP:

COROLLARY 4.2. *For $n\ge 2$, $a\ge 0$, all zeros of $y_n(z;\,a)$ are in the open left half-plane.*

This corollary improves upon the result by Bottema [1], Wimp [21] and re-establishes the result of Martinez [11], in that they proved stability of the GBP with $n\ge 2$ for $a\ge 2$, for $a\ge 1$ and for $a\ge 0$, respectively.

Further improvements on Corollary 4.2 are given in the following two new results.

THEOREM 4.3. *For $n\ge 2$, $a\ge -1$, all zeros of $y_n(z;\,a)$ are in the open left half-plane. This result is sharp in the following sense: for $a<-1$, the polynomial $y_2(z;\,a)$ has at least one zero in the right half-plane.*

THEOREM 4.4. *For $n\ge 4$, $a\ge -2$, all zeros of $y_n(z;\,a)$ are in the open left half-plane. This result is sharp in the following sense: $y_3(z;\,a)$ does not have all its zeros in the open left half-plane for all values of a with $a\ge -2$. (In fact, for $a>-1.75$ and in the degenerate case $a=-2$, the zeros are in the open left half-plane; for $-2<a<-1.75$, they are not).*

The preceding theorems are in fact special cases of the following, which proves a conjecture of Grosswald [5; page 162, number 6]:

THEOREM 4.5. *For each $a\in\mathbb{R}$, there exists an integer $n_0=n_0(a)$ such that all zeros of $y_n(z;\,a)$ are in the open left half-plane for $n\ge n_0$. For $a<-2$, one can take $n_0(a)=1+[\![2^{3-a}]\!]$, where $[\![\cdot]\!]$ denotes the greatest integer function.*

Knowing the existence of an integer $n_0(a)$ from Theorem 4.5, we can then *define* $N_s(a)$ to be the smallest positive integer n_0 such that $\{y_m(z; a)\}_{m=n_0}^{\infty}$ has all its zeros in the open left-half plane. Numerical evidence, as given in table 1 below, supports the conjecture that $N_s(a)$ is a nonincreasing function of a.

Table 1

a	-4.5	-3.5	-2.5	-1.5	-0.5	>0
$N_s(a)$	11	8	4	3	2	1

§ 5. ANNULI AND RECTANGLES CONTAINING ALL ZEROS

Using differential equations techniques, we arrive at the following new result:

THEOREM 5.1. *For $n \geq 1$, $n+a-1>0$, all zeros of $y_n(z; a)$ lie in the annulus*

$$(5.1) \qquad A(n,a) := \left\{ z \in \mathbb{C}: \frac{2}{2n+a-\frac{2}{3}} < |z| \leq \frac{2}{n+a-1} \right\}.$$

We remark that the outer radius for the annulus $A(n,a)$ of Theorem 5.1 comes from the result of Dočev [3], as well as from the cardioidal region (3.1) of Theorem 3.1. Our essential contribution in Theorem 5.1 is the lower bound $2/(2n+a-\frac{2}{3})$ of (5.1) for the moduli of the zeros of $y_n(z; a)$ (which is of order n^{-1} for n large). This is a considerable improvement over the known (cf. Grosswald [5, p. 82]) lower bound $2/[n(n+a-1)]$ (which is of order n^{-2} for n large), this latter bound having been derived by applying the well-known Eneström-Kakeya Theorem [4, 7] to $y_n(z; a)$. In this regard, this improvement in the lower bound of (5.1) can be viewed as giving a partial answer to Problem 5 in Grosswald [5, p. 162].

A certain refinement in the proof of the lower bound of (5.1) shows that, for each $\varepsilon>0$, there exists an integer $m_0(\varepsilon)$ such that the term $-\frac{2}{3}$ may be replaced by $-1+\varepsilon$ for $n \geq m_0(\varepsilon)$. Numerical evidence, however, seems to indicate that $-\frac{2}{3}$ in (5.1), may be replaced by -1 without any further restrictions on n and a, i.e., $n \geq 2$ and $n+a-1>0$.

Next, on combining the cardioidal region of (3.1), the sector of (4.1), the result of Theorem 4.3, and the annulus given in (5.1), we easily deduce the following:

COROLLARY 5.2. *Let $z_0 = \xi + i\eta$ be a zero of $y_n(z; a)$ where $n \geq 2$, $n+a-1>0$. Then,*

$$(5.2) \qquad |\eta| < \frac{3\sqrt{3}}{4(n+a-1)}; \quad \frac{-2}{n+a-1} \leq \xi < \xi(n,a),$$

where

$$(5.3)\quad \xi(n,a):=\begin{cases} -2a/\{(2n+a-2)(2n+a-\tfrac{2}{3})\} & \text{for } a\geq 0,\\ 0 & \text{for } -1\leq a<0,\\ -2a/(2n+a-2)^2 & \text{for } -\tfrac{2}{3}n+\tfrac{2}{3}\leq a<-1,\\ 1/\{4(n+a-1)\} & \text{for } -n+1<a<-\tfrac{2}{3}n+\tfrac{2}{3}.\end{cases}$$

§ 6. THE ZEROS OF THE ORDINARY BESSEL POLYNOMIALS

As was pointed out in Section 2, the choice $a=2$ in the GBP leads to the ordinary Bessel polynomials

$$(6.1)\qquad y_n(z):=\sum_{k=0}^{n}\binom{n}{k}(n+1)_k\left(\frac{z}{2}\right)^k.$$

Putting $a=2$ in the results of the previous sections and refining the argument using differential equations techniques, we arrive at the new result of

THEOREM 6.1. *For $n\geq 1$, the zeros of $y_n(z)$ belong to $\mathscr{R}:=A_1\cup A_2\cup A_3$, where*

$$A_1:=\left\{z=re^{i\theta}:|\theta|\geq\frac{2\pi}{3},\ \frac{1}{n+\frac{2}{3}}<r<\frac{1-\cos\theta}{n+1}\right\},$$

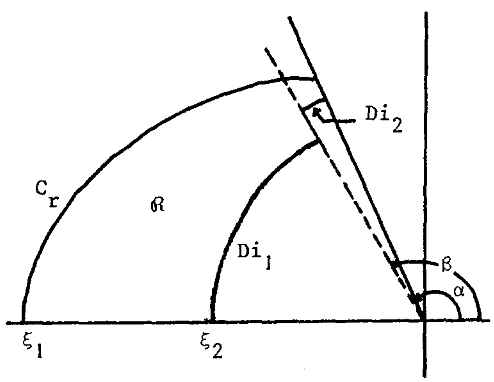
$$A_2:=\left\{z=re^{i\theta}:\frac{2\pi}{3}\geq|\theta|\geq\cos^{-1}\left(-\frac{1}{n}\right),\ \frac{1}{\sqrt{n(n+1)}}<r<\frac{1-\cos\theta}{n+1}\right\},$$

$$A_3:=\left\{-\frac{2}{n+1}\right\}.$$

For $n\geq 2$, any zero $z_0=\xi+i\eta$ of $y_n(z)$ satisfies

$$(6.2)\qquad |\eta|\leq\frac{3\sqrt{3}}{4(n+1)};\quad -\frac{2}{n+1}\leq\xi<\frac{-1}{\sqrt{n^3(n+1)}}.$$

In fig. 1 below, the symmetric region $\mathscr{R}$ is sketched only in the upper-half plane.



C_r: $r=\dfrac{1-\cos\theta}{n+1}$

Di_1: $r=\dfrac{1}{n+\frac{2}{3}}$

Di_2: $r=\dfrac{1}{\sqrt{n(n+1)}}$

$\cos\alpha:=-\dfrac{1}{n}$

$\beta:=\dfrac{2\pi}{3}$

$\xi_1:=-\dfrac{2}{n+1}$

$\xi_2:=-\dfrac{1}{n+\frac{2}{3}}$

Fig. 1.

To indicate how the results of Theorem 6.1 compare with known results about zeros of ordinary Bessel polynomials, we first remark that Wragg and Underhill [22] have given the following upper bound for ξ:

$$\xi \le -2/\{(2n-3)(2n-1)\}. \tag{6.3}$$

A short calculation shows that the related upper bound in (6.2) is an improvement over the above inequality (6.3) for all $n \ge 5$.

Next, Wragg and Underhill [22] have also given the following upper bound:

$$|\eta| \le 8/15 \doteqdot 0.533\,333 \text{ (for all } n \ge 3), \tag{6.4}$$

a bound, independent of n, which was derived using determinantal representations for Bessel polynomials. The corresponding upper bound in (6.2) is an improvement over (6.4) in two ways: first, it has a dependence on n which makes this upper bound strictly decreasing with n; second, this upper bound in (6.2) gives that

$$|\eta| \le 3\sqrt{3}/16 \doteqdot 0.324\,760 \text{ (for all } n \ge 3),$$

which is better than that in (6.4).

§ 7. ASYMPTOTIC BEHAVIOR FOR n LARGE

In Olver [12], it was shown that the zeros of the normalized ordinary Bessel polynomials $y_n(z/n) = y_n(z/n; 2)$ tend, as $n \to \infty$, to a curve Γ in the closed left half plane, defined by

$$\Gamma := \{z \in \mathbb{C} : |\omega(z)| = 1 \text{ and } \operatorname{Re} z \le 0\}, \tag{7.1}$$

where

$$\omega(z) := \frac{e^{\sqrt{1+z^{-2}}}}{z\{1+\sqrt{1+z^{-2}}\}} \tag{7.2}$$

so that $\{\pm i\}$ are the endpoints of Γ. The representation (7.2) is derived in Saff and Varga [18]. Using results of [18], Olver's result can be both substantially sharpened for the case $a = 2$, as well as generalized to any real a.

THEOREM 7.1. *For any fixed $a \in \mathbb{R}$, $\hat{z}$ is a limit point of zeros of the normalized GBP $y_n(2z/(2n+a-2); a)$ as $n \to \infty$ iff $\hat{z} \in \Gamma$. Moreover, if γ is a closed arc of $\Gamma \setminus \{\pm i\}$ with endpoints μ_1 and μ_2 (with $\pi/2 < \arg \mu_1 \le \arg \mu_2 < 3\pi/2$), where $\omega(\mu_j) = e^{i\phi_j}$, $j = 1, 2$, $(\pi/2 < \phi_2 \le \phi_1 < 3\pi/2)$, let $\tau_n(\gamma)$ denote the number of zeros z of $y_n(z; a)$ which satisfy* $\arg \mu_1 \le \arg z \le \arg \mu_2$. *Then,*

$$\lim_{n \to \infty} \frac{\tau_n(\gamma)}{n} = \frac{\phi_1 - \phi_2}{\pi}. \tag{7.3}$$

Further, again using results of [18], it can be shown that the zeros of $y_n(2z/(2n+a-2); a)$ must, for fixed $a \ge 2$ and n large, approach the curve Γ from the *inside*, i.e., through points with $|\omega(z)| > 1$. More precisely, we have

THEOREM 7.2. *For any fixed $a \geq 2$, there exists an integer $j(a)$ such that for each $n > j(a)$, any zero z of the normalized GBP $y_n(2z/(2n+a-2); a)$ satisfies* $|\omega(z)| > 1$ *and* $\operatorname{Re} z < 0$.

To indicate the results of Theorems 7.1 and 7.2 for several different values of a, we have included in figs. 2-5 the zeros of the normalized GBP $y_n(2z/(2n+a-2); a)$ for $n = 2, 3, \ldots, 15$, in relation to the curve Γ of (7.1).

Next, for a any fixed real number and for n any odd positive integer, let $\alpha_n(a)$ denote the unique (negative) real zero of the unnormalized GBP $y_n(z; a)$. Then, we establish the new result of

THEOREM 7.3. *For any fixed $a \in \mathbb{R}$,*

$$(7.4) \qquad \frac{2}{\alpha_n(a)} = (2n+a-2)\hat{r} + K(\hat{r}; a) + \mathcal{O}\left(\frac{1}{2n+a-2}\right), \text{ as } n \to \infty,$$

where $\hat{r}$ is the unique negative root of

$$(7.5) \qquad -\hat{r}e^{\sqrt{1+\hat{r}^2}} = 1 + \sqrt{1+\hat{r}^2} \qquad (\hat{r} \doteq -0.662\,743\,419),$$

and where

$$(7.6) \qquad K(\hat{r}; a) := \frac{\hat{r}[\sqrt{1+\hat{r}^2} + (2-a)\ln(\hat{r} + \sqrt{1+\hat{r}^2})]}{\sqrt{1+\hat{r}^2}}.$$

Note that $1/\hat{r}$ is from (7.5) just the real point of the curve Γ of (7.1) and (7.2), and hence, the result of Theorem 7.3 can be regarded (after the appropriate normalization) as a sharpened special case of Theorem 7.2.

With the approximate value of $\hat{r}$ from (7.5), we can express the result of (7.4) as

$$(7.7) \qquad \frac{2}{\alpha_n(a)} \doteq 2n\hat{r} - 1.006\,289\,950a + 1.349\,836\,480 + \mathcal{O}\left(\frac{1}{2n+a-2}\right).$$

In this regard, it is interesting to remark that, based on the examination of numerical results, it had been conjectured by Luke [10, p. 194], as well as Grosswald [5, p. 93], that

$$(7.8) \qquad \frac{2}{\alpha_n(a)} \sim 2n\hat{r} - a + (\pi+1)/\pi, \; a > 0, \text{ as } n \to \infty.$$

Now, the dominant term, i.e., $2n\hat{r}$, in both (7.7) and (7.8) comes (cf. Theorem 7.1) directly from the known real point $1/\hat{r}$ of the curve Γ. Thus, the essence of the conjecture (7.8) concerns the accuracy of the next two constant terms of (7.8), as they compare with the corresponding two terms of (7.7). Of course, as $(\pi+1)/\pi \doteq 1.318\,309\,886$, neither of these terms of (7.8) is correct, but these conjectured constants nonetheless had a maximum relative error of only 2%.

Next, on deleting the term $\mathcal{O}(1/(2n+a-2))$ in (7.7) and denoting the resulting approximate value of $\alpha_n(a)$ by $\tilde{\alpha}_n(a)$, i.e.,

$$(7.9) \qquad \tilde{\alpha}_n(a) \doteq \frac{-2}{\{1.325\,486\,838n + 1.006\,289\,950a - 1.349\,836\,480\}},$$

then $\tilde{\alpha}_n(a)$ is a surprisingly accurate approximation of $\alpha_n(a)$, even for very small values of n, as table 2 below, for the special case $a=2$, shows.

Table 2

n	$\alpha_n(2)$	$\tilde{\alpha}_n(2)$
1	−1.000 000 000	−1.005 919 708
3	−0.430 628 846	−0.431 108 446
5	−0.274 217 626	−0.274 341 739
7	−0.201 134 930	−0.201 183 942
9	−0.158 805 297	−0.158 829 428
11	−0.131 193 311	−0.131 206 918
13	−0.111 760 443	−0.111 768 857
15	−0.097 341 509	−0.097 347 068

From (7.4), we obtain the representation

$$(7.10) \qquad \frac{(2n+a-2)\alpha_n(a)}{2} = \frac{1}{\hat{r}} - \frac{K(\hat{r};\,a)}{\hat{r}^2(2n+a-2)} + \mathcal{O}\left(\frac{1}{(2n+a-2)^2}\right), \text{ as } n\to\infty,$$

the left-hand side being the real zero of the normalized GBP $y_n(2z/(2n+a-2);\,a)$. Noting that $K(\hat{r};\,a)$ is linear in a, let a^* be the unique value of a such that $K(\hat{r};\,a^*)=0$. Then, we see from (7.10) that this choice $a=a^*$ produces a second-order correct approximation to $1/\hat{r} \doteq -1.508\,879\,562$. From (7.6), we find that

$$(7.11) \qquad a^* \doteq 0.070\,877\,276,$$

and in table 3, we give values of

$$(7.12) \qquad \beta_n := \frac{(2n+a^*-2)\alpha_n(a^*)}{2}.$$

Note that the last column of table 3 indicates the $\mathcal{O}(1/(2n+a^*-2)^2)$ convergence of β_n to $1/\hat{r}$, as $n\to\infty$. Note moreover from fig. 4 that this choice of $a=a^*$

Table 3

n	β_n	$(2n+a^*-2)^2(\beta_n-1/\hat{r})$
3	−1.508 922 977	$-0.719\cdot10^{-3}$
5	−1.508 911 191	$-2.060\cdot10^{-3}$
7	−1.508 896 790	$-2.510\cdot10^{-3}$
9	−1.508 890 144	$-2.733\cdot10^{-3}$
11	−1.508 886 676	$-2.866\cdot10^{-3}$
13	−1.508 884 660	$-2.954\cdot10^{-3}$
15	−1.508 883 390	$-3.017\cdot10^{-3}$

is such that not only the real zeros, but *all* the zeros of the normalized GBP $y_n(2z/(2n+a^*-2);\, a^*)$ are amazingly close to the curve Γ.

Finally, on the basis of further computations, it appears that a considerable improvement over Theorem 7.2 is possible. With the notation

$$\Lambda := \{z = iy : -1 \le y \le 1\},$$

so that $\Gamma \cup \Lambda$ is a closed curve in $\mathbb{C}$, we make the following two conjectures.

CONJECTURE 1. *There exists a real number* $\hat{a} \doteqdot \frac{1}{4}$ *such that for all* $n \ge 1$ *and all* $a \ge \hat{a}$, *every zero of the normalized GBP* $y_n(2z/(2n+a-2);\, a)$ *lies inside* $\Gamma \cup \Lambda$.

CONJECTURE 2. *For all* $n \ge 2$ *and for all* $a \le a^*$ (cf. (7.11)) *so that* $n+a-1>1$, *every zero of the normalized GBP* $y_n(2z/(2n+a-2);\, a)$ *lies outside* $\Gamma \cup \Lambda$.

In this regard, figs. 2–5 give some indication of the validity of these conjectures.

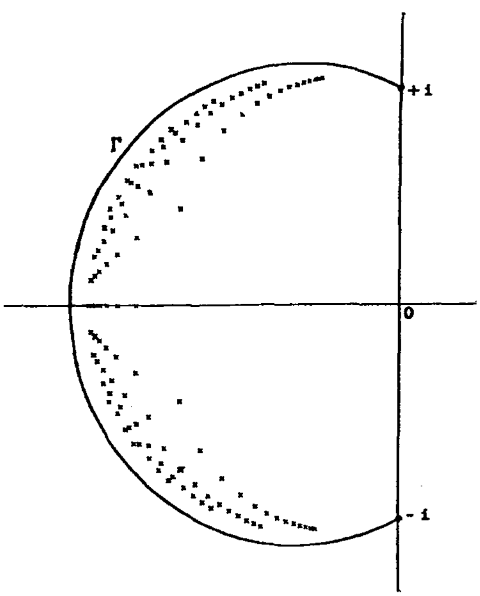


Fig. 2. Zeros of Normalized GBP; $a=4$.

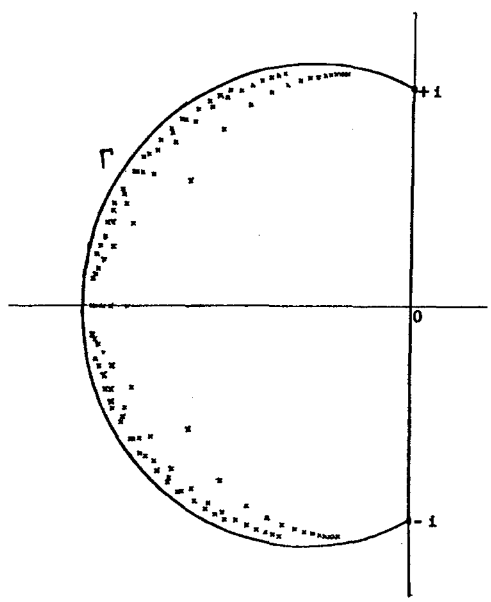


Fig. 3. Zeros of Normalized GBP; $a=2$.

Zeros of Normalized GBP $y_n(z;\, a)$ for $n=2, 3, \ldots, 15$.

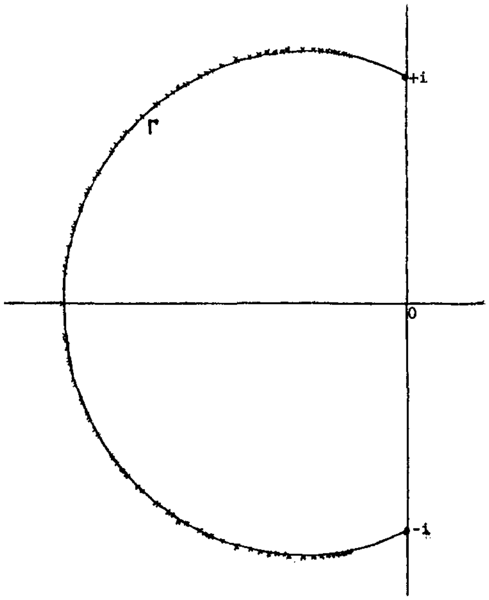


Fig. 4. Zeros of Normalized GBP; $a = 0.070\ 877\ 276$.

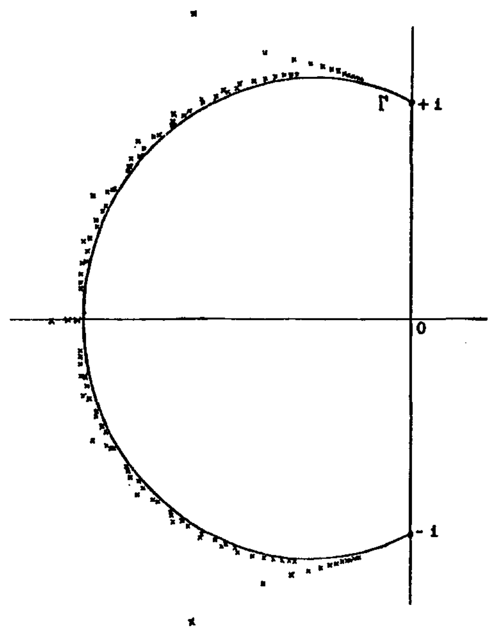


Fig. 5. Zeros of Normalized GBP; $a = -0.5$.

Zeros of Normalized GBP $y_n(z;\ a)$ for $n = 2, 3, \ldots, 15$.

§ 8. AN APPLICATION TO THE PADÉ TABLE FOR ${}_1F_1(1;\ c;\ z)$

We will show that the preceding results can be used to give an alternate proof for the convergence properties of the Padé table for the confluent hypergeometric function

$$(8.1)\qquad {}_1F_1(1;\ c;\ z) := \sum_{k=0}^{\infty} z^k/(c)_k,\ c \neq 0, -1, -2, \ldots,$$

as given in de Bruin [2].

For any ordered pair (ν, n) of nonnegative integers, the (ν, n)-th Padé approximant $r_{\nu,n}(z) = p_{\nu,n}(z)/q_{\nu,n}(z)$ of ${}_1F_1(1;\ c;\ z)$ is defined by the conditions

$$\begin{cases} \text{degree } p_{\nu,n}(z) \leq \nu;\ \text{degree } q_{\nu,n}(z) \leq n; \\ q_{\nu,n}(z)\,{}_1F_1(1;\ c;\ z) - p_{\nu,n}(z) = \mathcal{O}(|z|^{n+\nu+1}),\ \text{as } z \to 0. \end{cases}$$

The doubly infinite array $\{r_{\nu,n}(z)\}_{\nu=0,\ n=0}^{\infty\quad\ \infty}$ is called the *Padé table* (cf. Perron [13]) for ${}_1F_1(1;\ c;\ z)$.

For $\nu \geq n-1$ and the normalization $q_{\nu,n}(0) = 1$, one easily derives (cf. de Bruin [2]) that

$$(8.2)\quad q_{\nu,n}(z) = {}_1F_1(-n;\ -c-n-\nu+1;\ -z) = \frac{(-z)^n}{(\nu+c)_n}\, y_n\!\left(-\frac{2}{z};\ \nu-n+1+c\right),$$

which, from Theorem 5.1, immediately implies

THEOREM 8.1. *For any ν and n with $\nu \geq n-1$ and $\nu + c > 0$, the zeros of the Padé denominator $q_{\nu,n}(z)$ for ${}_1F_1(1; c; z)$ satisfy*

(8.3) $$\nu + c \leq |z| < \nu + c + n + \tfrac{1}{3}.$$

The fact that ${}_1F_1(1; c; z)$, with $c \neq 0, -1, -2, \ldots$, is an entire function, and the fact that the poles of its Padé approximants $r_{\nu,n}(z)$ tend to infinity as $\nu \to \infty$, together imply, by a simple application of the Hermite formula (cf. Karlsson and Saff [8]), the result of

COROLLARY 8.2. *Let $\{r_{\nu_j,n_j}(z)\}_{j=1}^{\infty}$ be any sequence of Padé approximants for ${}_1F_1(1; c; z)$ $(c \neq 0, -1, \ldots)$, satisfying*

(8.4) $$\nu_j \geq n_j - 1 \text{ for all } j \text{ large, and } \lim_{j\to\infty} \nu_j = +\infty.$$

Then,

(8.5) $$\lim_{j\to\infty} r_{\nu_j,n_j}(z) = {}_1F_1(1; c; z),$$

uniformly on compact subsets of $\mathbb{C}$.

(To be continued)

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On the zeros of generalized Bessel polynomials. II

by M.G. de Bruin, E.B. Saff and R.S. Varga

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Dedicated to Professor Emil Grosswald

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§ 9. PROOFS OF NEW RESULTS

Because it is fundamental in deriving Theorem 3.1 of this paper, we state for completeness the following result of Saff and Varga [15]. (Note that $y_n(z;\, a)$ of this present article (cf. (2.1)) corresponds to $Y_n^{(a-2)}(-z)$ of [15].)

THEOREM 9.1. *Let $\{p_k(z)\}_{k=0}^n$ be a sequence of polynomials of respective degrees k which satisfy the three-term recurrence relation*

$$(9.1)\qquad p_k(z)=\left(\frac{z}{b_k}+1\right)p_{k-1}(z)-\frac{z}{c_k}\,p_{k-2}(z)\quad (k=1,2,\ldots,n),$$

where the b_k's and c_k's are positive real numbers for all k, $1\le k\le n$, and where $p_{-1}(z):=0$, $p_0(z):=p_0\neq 0$. Set

$$(9.2)\qquad \alpha:=\min\,\{b_k(1-b_{k-1}c_k^{-1}):\ k=1,2,\ldots,n\},\ b_0:=0.$$

Then, if $\alpha>0$, the parabolic region

$$(9.3)\qquad \mathscr{P}_\alpha:=\{z=x+iy\in\mathbb{C}:\ y^2\le 4\alpha(x+\alpha),\, x>-\alpha\}$$

contains no zeros of $p_1(z), p_2(z),\ldots,p_n(z)$.

To give the reader more insight in the connection between the GBP and Theorem 9.1, it first can be verified from the definition of $\theta_n(z;\, a)$ in (2.4) that, for $n+a-1>0$, the sequence

(9.4) $$\left\{\frac{\Gamma(n+a-1)}{\Gamma(n+a-1+k)}\, 2^k \theta_k\left(\frac{z}{2};\, n+a-k\right)\right\}_{k=1}^{n}$$

satisfies the recurrence relation (9.1) with

$$b_k = n+a-2+k,\ k=1,2,\ldots,n;\ c_1=1,$$

$$c_k = \frac{(n+a-2+k)(n+a-3+k)}{k-1}, \quad k=2,3,\ldots,n.$$

Thus, it follows in this case from (9.2) that $\alpha = n+a-1>0$, so that the polynomials in (9.4) have no zeros in $\mathscr{P}_{n+a-1}$. With the transformation $z \to 2/z$, we see from (2.4) that all zeros of the GBP $y_n(z;\, a)$ lie inside the cardioidal region (3.1), as proved in [15].

Before proceeding to the proofs of our new results, we first establish a slight generalization of a lemma in [14].

LEMMA 9.2 *Let the sequence* $\{p_k(z)\}_{k=0}^{n}$ *be as in Theorem* 9.1 *with* $p_0 = 1$, *and let the real numbers* $\lambda_0 := 1, \lambda_1, \lambda_2, \ldots, \lambda_n$ *satisfy*

(9.5) $$0<\lambda_k<1 \text{ for } k=1,2,\ldots,n-1;\ 0\le \lambda_n<1.$$

Then, $p_n(z)$ *is different from zero at any point* z *which satisfies the* n *inequalities*

(9.6) $$|z| + \frac{\operatorname{Re} z}{|z|}\left\{\frac{b_k(2c_k\lambda_{k-1}-b_{k-1})}{2c_k\lambda_{k-1}(1-\lambda_k)}\right\} > \frac{b_k b_{k-1}}{2c_k\lambda_{k-1}(1-\lambda_k)},\ k=1,2,\ldots,n.$$

PROOF. This follows by imitating the argument in the proof of Lemma 3.1 of [14] which uses special values of b_k, c_k (viz., those for certain Padé approximants to e^z). In the proof, one needs the fact that $p_n(z)$ and $p_{n-1}(z)$ have no zeros in common, which follows, on assuming the contrary, from using the recurrence relation backwards to establish the contradiction that $p_0(z)=0$. □

We now apply Lemma 9.2 to the polynomials given in (9.4) to reach the intermediate results of

COROLLARY 9.3. *The polynomial* $\theta_n(z/2;\, a)$ *has, for* $n+a-1>0$, *no non-real zeros in*

(9.7) $$\mathscr{D} := \left\{z = re^{i\theta} \in \mathbb{C}:\ r > 2(2n+a-2) - \frac{2(n+a-1)}{1-\cos\theta}\right\}.$$

PROOF. Putting $\lambda_k := (1+\cos\theta)/2$ for $k=1,2,\ldots,n$, then the inequalities (9.6) show that any non-real point $z = re^{i\theta} \in \mathbb{C}$ satisfying (9.7) is not a zero of $\theta_n(z/2;\, a)$. □

With the preceding result, we then come to the

PROOF OF THEOREM 4.1(i). As the sector $S(n,a)$ of (4.1) is invariant under the transformation $z \to 2/z$, it suffices to prove the assertion for the polynomial $\theta_n(z/2;\, a)$, instead of for the polynomial $y_n(z;\, a)$. First, consider the following set of points

$$G_1 := \left\{ z = re^{i\theta} \in \mathbb{C}:\ r > 2n+a-2,\ 0 < |\theta| \leq \cos^{-1}\left(\frac{-a}{2n+a-2}\right)\right\}.$$

One easily verifies from (9.7) that $G_1 \subset \mathscr{D}$, which establishes that $\theta_n(z/2;\, a)$ is zero-free in G_1. From (9.4) and Theorem 9.1, it follows that $\theta_n(z/2;\, a)$ is also zero-free in $\mathscr{P}_{n+a-1}$. Rewriting $\mathscr{P}_{n+a-1}$, we obtain that

$$\mathscr{P}_{n+a-1} \supset G_2 := \left\{ z = re^{i\theta} \in \mathbb{C}:\ r \leq 2n+a-2,\ 0 \leq |\theta| \leq \cos^{-1}\left(\frac{-a}{2n+a-2}\right)\right\}.$$

Combining these results, we see that $\theta_n(z/2;\, a)$ is zero-free in $G_1 \cup G_2$, or, equivalently, all zeros of $\theta_n(z/2;\, a)$ lie in $\mathbb{C} \setminus (G_1 \cup G_2) = S(n,a)$.

The second assertion of Theorem 4.1(i) follows from the fact that (4.2) implies $(1-\sigma)/(1+\sigma) \geq (-a)/(2n+a-2)$, and hence, $S(n,a) \subset S_\sigma$ for those values of n and a which satisfy (4.2), as well as $n+a-1>0$ and $n \geq 2$. □

The proof of the sharpness result in Theorem 4.1(ii), along with the proof of Theorem 3.3, will be given after the other theorems of Section 4 have been established. Now, we give the

PROOF OF THEOREM 4.3. As the open left half-plane is itself a sector, namely the sector S_1 from (4.3), we shall prove the result for $\theta_n(z/2;\, a)$, since, after the transformation $z \to 2/z$, this will imply the assertion for $y_n(z;\, a)$.

As in [14, p. 9], we apply Lemma 9.2 to the polynomials of (9.4). For the choice $\lambda_k = \frac{1}{2}$, $k = 1, 2, \ldots, n-1$, $\lambda_n = 0$, the inequalities (9.6) imply that, for $n \geq 3$, the polynomial $\theta_n(z/2;\, a)$ is zero-free on

$$B_1 := \{ z \in \mathbb{C}:\ \operatorname{Re} z \geq 0,\ |z| > 2(n-2) \}.$$

Again invoking the result of Theorem 9.1, we also find that $\theta_n(z/2;\, a)$ is zero-free on

$$B_2 := \{ z \in \mathbb{C}:\ \operatorname{Re} z \geq 0,\ |z| \leq 2(n+a-1) \}.$$

Thus, for $a \geq -1$, we find that all the zeros of $\theta_n(z/2;\, a)$ belong to $\mathbb{C} \setminus (B_1 \cup B_2) = S_1$.

For $n=2$, one directly verifies that $y_2(z;\, a)$ has all its zeros in the open left half-plane iff $a \geq -1$. (It should be remarked that the case $a = -1$ is degenerate, with $y_2(z;\, -1) \equiv 1$.) For $a < -1$, one similarly verifies that $y_2(z;\, a)$ has a zero in the right half-plane. □

By yet another choice for the λ_k's in Lemma 9.2 and by treating particular cases by the Wall criterion for stability, we can give the

PROOF OF THEOREM 4.4. For $n \geq 7$, $a \geq -2$, we use a method of proof, similar to that of [14, p. 10], by choosing, in Lemma 9.2,

$$\lambda_k = \tfrac{1}{2},\ k = 1, 2, \ldots, n-2; \quad \lambda_{n-1} = \frac{n-1}{3n-5}; \quad \lambda_n = 0.$$

This time, Lemma 9.2 implies that the $\theta_n(z/2;\ a)$ are zero-free in the closed right half-plane, outside a disk with radius $2(n-3)$. Using $a \geq -2$ in combination with the zero-free parabolic region, we find that $\theta_n(z/2;\ a)$ is zero-free in the closed right half-plane for $n \geq 7$. Next, for $n = 6$, we use the following values of λ_k in Lemma 9.2:

$$\lambda_1 = \lambda_2 = \tfrac{1}{2};\ \lambda_3 = \tfrac{2}{3};\ \lambda_4 = \tfrac{5}{8},\ \lambda_5 = \tfrac{5}{12},\ \lambda_6 = 0,$$

and the proof proceeds as above, with the radius of the disk from (9.6) now being 6.

The remaining cases, $n = 3$, 4, and 5, then follow, together with the sharpness result, by applying the Wall criterion [20] and checking degenerate cases separately. □

PROOF OF THEOREM 4.5. To prove the Grosswald conjecture [5, p. 162, number 6], concerning the stability of the $y_n(z;\ a)$ for arbitrary (but fixed) a and sufficiently large n, it suffices, in view of Theorem 4.4, to restrict ourselves to the case $a < -2$.

This time, Lemma 9.2 will be applied using

$$\lambda_{n-j} = (2^j - 1)/(2^{j+1} - 1),\ j = 0, 1, \ldots, n-1. \tag{9.8}$$

With the well-known inequalities

$$2^{n+2} > n, \quad 2^{n+5} \geq n^2 \text{ for all } n = 1, 2, \ldots,$$

one can easily show that the coefficient of $\operatorname{Re} z/|z|$ in (9.6) is nonnegative for $n \geq 2^{3-a}$ and $k = 1, 2, \ldots, n$. Lemma 9.2 then leads to a zero-free region in the closed right half-plane, outside the disk of radius R given by

$$R := \max\left\{\frac{k-1}{2\lambda_{k-1}(1-\lambda_k)}:\ k = 2, \ldots, n\right\}.$$

Inserting the values of λ_k from (9.8), we find that

$$R = \max\{(2 - 2^{-j-1})(n - j - 1):\ j = 0, 1, \ldots, n-2\}.$$

Now, for $n \geq 32$, it is easy to show that the maximum of the function

$$f(x) = (2 - 2^{-x-1})(n - x - 1)$$

on $[0,\ n-2]$ occurs at a point $\hat{x}$ with $n - 2 > \hat{x} > \log_2 n - 3$, and, furthermore, that $f(\hat{x}) < 2(n - \hat{x} - 1)$. Hence, $\theta_n(z/2;\ a)$ is zero-free in $\{z \in \mathbb{C}:\ |z| \geq 2(n - \hat{x} - 1),\ \operatorname{Re} z \geq 0\}$. Combining this result with the zero-free parabolic region, one finds that all the zeros of $\theta_n(z/2;\ a)$ lie in the open left half-plane if $\log_2 n + a - 3 \geq 0$, from which the bound $n_0(a)$, given in Theorem 4.5, follows. □

We now return to the sharpness results that have been left unproven up to now.

PROOF OF THEOREMS 3.3 AND 4.1(ii). The proof follows by imitating the argument given in Saff and Varga [16] for establishing Theorems 2.2 and 2.3 of [16]. It depends heavily upon the existence of zeros of a certain form for $y_n(z; a)$ which can be proved by adapting the proof of Theorem 2.1 from [16] by replacing the parameter v_j appearing there by $n+a_n-2$. Specifically, let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers satisfying

$$n+a_n-2\geq 0 \ (n\in \mathbb{N}),\ \lim_{n\to\infty}\frac{a_n}{n}=\sigma-1\ (\sigma\in(0,\infty)).$$

Then, $y_n(z; a_n)$ has zeros of the form

$$\frac{2}{(2n+a_n-1)}\exp\left[\pm i\cos^{-1}\left(\frac{2-a_n}{2n+a_n-1}\right)\right]+\mathcal{O}((2n+a_n-1)^{-5/3})\text{ as } n\to\infty.$$

The proof then continues as in [16]. □

For the proof of the main result of Section 5, we can again refer to a technique developed for the study of the zeros of Padé approximants to the exponential function e^z.

PROOF OF THEOREM 5.1. As previously stated in Section 5, the upper bound of (5.1) is a known result. To establish the lower bound of (5.1), it suffices to imitate the method of proof in [17, Theorems 2.1 and 2.2], using now the polynomials $\Gamma(n+a-1)2^n\theta_n(z/2; a)/\Gamma(2n+a-1)$. To this end, one has to replace the parameter v in [17] by $n+a-2$. □

PROOF OF THEOREM 6.1. Apart from the upper bound for ξ in (6.2), the only part of this theorem that does not follow from the preceding sections on taking $a=2$, is the lower bound for the modulus of the zeros appearing in the set A_2, defined by

$$A_2:=\left\{z=re^{i\theta}: \frac{2}{3}\pi\geq|\theta|\geq\cos^{-1}\left(-\frac{1}{n}\right), \frac{1}{\sqrt{n(n+1)}}<r<\frac{1-\cos\theta}{n+1}\right\}. \tag{9.9}$$

To establish this lower bound, we recall from (2.6) of Theorem 2.2 that

$$y_n(z)=y_n(z;\ 2)=e^{1/z}W_{0,n+\frac{1}{2}}\left(\frac{2}{z}\right),$$

which leads to the fact that the function

$$w_n(z):=e^{-(z/2)}y_n\left(\frac{2}{z}\right)$$

satisfies the differential equation

$$\frac{d^2w(z)}{dz^2}=\left\{\frac{1}{4}+\frac{n(n+1)}{z^2}\right\}w(z).$$

Using a path of integration wholly within $\mathbb{C}\backslash\{0\}$, one can easily show that any solution of this differential equation satisfies

$$\text{(9.10)} \quad \int_{z_1}^{z_2} \left\{\frac{1}{4} + \frac{n(n+1)}{z^2}\right\} |w(z)|^2 dz = \frac{dw}{dz} \bar{w} \Big|_{z_1}^{z_2} - \int_{z_1}^{z_2} \left|\frac{dw}{dz}\right|^2 \overline{dz}.$$

Now, let $\tau = \varrho e^{i\varphi}$, with $\pi/2 < \varphi < \pi$, be any zero of $w_n(z)$ and consider the path of integration given by the half-line

$$\tau \cdot (1 + e^{-i\varphi/2} x), \; 0 \leq x < \infty,$$

which is the same path of integration as employed in [17, Theorem 2.2]. Because of the restriction on φ, it is easy to prove that the integrals in (9.10) converge. Also, since $(dw/dz)\bar{w}\,|_\tau^\infty = 0$, we find that

$$\text{(9.11)} \quad \int_0^\infty \tau^2 e^{-i\varphi} \left\{\frac{1}{4} + \frac{n(n+1)}{\tau^2(1+e^{-i\varphi/2}x)^2}\right\} |\tilde{w}(x)|^2 dx = -\int_0^\infty \left|\frac{d\tilde{w}}{dx}\right|^2 dx,$$

where $\tilde{w}(x)$ is given by

$$\tilde{w}(x) := w_n(\tau(1 + e^{-i\varphi/2} x)), \; 0 \leq x < \infty.$$

Taking imaginary parts in (9.11), we deduce, using $\lambda := n(n+1)$, that

$$\text{(9.12)} \quad \int_0^\infty \left\{\frac{\varrho^2}{4\lambda} \cos\frac{\varphi}{2} - \frac{x + \cos(\varphi/2)}{(x^2 + 2x\cos(\varphi/2) + 1)^2}\right\} |\tilde{w}(x)|^2 \, dx = 0.$$

As the limit as $x \to \infty$ of the expression in braces in (9.12) is *positive* $(\pi/2 < \varphi < \pi)$, the minimum of this function on the real axis must be *negative*. Making the restriction $2\pi/3 \geq \varphi > \pi/2$, this leads to the fact that the integrand must be negative for $x = 0$, i.e.,

$$\frac{\varrho^2}{4\lambda} \cos\frac{\varphi}{2} - \cos\frac{\varphi}{2} < 0,$$

or, equivalently

$$\varrho < 2\sqrt{\lambda}.$$

This shows that the zeros of $y_n(z)$ with $2\pi/3 \geq |\theta| > \pi/2$ must satisfy

$$\frac{2}{|z|} < 2\sqrt{\lambda},$$

which gives the desired lower bound for r in (9.9).

To complete the proof of Theorem 6.1, it remains to establish the upper bound for ξ in (6.2). However, this upper bound for ξ is a simple consequence of the first part of this theorem when $n \geq 3$. For the remaining case $n = 2$, this upper bound follows by direct computation. □

Finally, we sketch the proofs of the theorems in Section 7.

PROOF OF THEOREM 7.1. First, one can directly verify from (2.1) and (2.4) that the following integral representation is valid:

$$(9.13)\qquad \Gamma(n+a-1)2^n\theta_n(z/2;\, a)=\int_0^\infty e^{-t}(t+z)^n t^{n+a-2}dt,\ (n+a-1>0),$$

the path of integration being the nonnegative real axis. Similarly, it is known (cf. [18, eq. (4.7)]) that the Padé numerator $P_{n,\nu}(z)$ of the (n, ν)-th Padé rational approximation to e^z of (2.9), has the integral representation

$$(9.14)\qquad (n+\nu)!\, P_{n,\nu}(z)=\int_0^\infty e^{-t}(t+z)^n t^\nu dt.$$

Because of this, the asymptotic methods of [18], based on steepest descent methods applied to the integral of (9.14), can be analogously applied to the integral representation of (9.13) for any fixed real a. More precisely, with

$$(9.15)\qquad \nu_n := n+a-2,$$

then

$$\lim_{n\to\infty} \nu_n/n=1 \text{ for any fixed real } a.$$

Thus, we deduce, as similarly in the special case $\sigma=1$ of [18, Theorem 2.2], that $\hat{z}$ is a limit point of the zeros of the normalized polynomials

$$\theta_n((2n+a-2)z/2;\, a)$$

iff

$$\hat{z}\in D_1 := \left\{z\in\mathbb{C}: \left|\frac{ze^{\sqrt{1+z^2}}}{1+\sqrt{1+z^2}}\right|=1,\ |z|\le 1, \text{ and } \operatorname{Re} z\le 0\right\}.$$

For any fixed real a, it then follows, by means of the inversion $z\to 1/z$ that z is a limit point of zeros of the normalized GPB $y_n(2z/(2n+a-2);\, a)$ iff $z\in\Gamma$, where Γ is defined in (7.1) and (7.2). This establishes the first part of Theorem 7.1. The second part of Theorem 7.1 similarly follows as in the special case $\sigma=1$ of [18, Theorem 2.3]. □

PROOF OF THEOREM 7.2. To establish Theorem 7.2, we again apply the asymptotic results of [18], and, for convenience, we use the same notations and definitions as in [18]. We further set

$$(9.16)\qquad \lambda_\tau := \frac{1-\tau}{1+\tau} \text{ for any } 0\le\tau<\infty.$$

As shown in [18, eq. (4.2)], for any τ with $0\le\tau<\infty$,

$$(9.17)\qquad \frac{zw_\tau'(z)}{w_\tau(z)}=g_\tau(z) := \sqrt{1+z^2-2\lambda_\tau z}, \text{ for all } z\in\mathbb{C}\setminus \mathscr{R}_\tau,$$

from which, after some calculations, it follows that for any fixed z with $0<|z|<1$,

$$(9.18)\qquad \frac{d}{d\tau}\ln|w_\tau(z)| = \frac{2}{(1+\tau)^2}\ln|\zeta+\sqrt{1+\zeta^2}| \text{ where } \zeta := \frac{z-\lambda_\tau}{\sqrt{1-\lambda_\tau^2}}.$$

Setting $t=\zeta+\sqrt{1+\zeta^2}$, so that $\zeta=\frac{1}{2}(t-(1/t))$, it can be verified that any ζ with $\operatorname{Re}\zeta<0$ has its image in the t-plane in the open unit disk. Consequently, from (9.18),

$$(9.19)\qquad \frac{d}{d\tau}\ln|w_\tau(z)|<0 \text{ for any fixed } z \text{ with } 0<|z|<1, \text{ and } \operatorname{Re} z<\lambda_\tau.$$

This can be applied as follows. If

$$(9.20)\qquad D_\tau := \{z\in\mathbb{C}: |w_\tau(z)|=1,\ |z|\le 1, \text{ and } \operatorname{Re} z\le\lambda_\tau\},\ 0\le\tau<\infty,$$

it is known from [18] that D_τ is a Jordan arc which lies interior to the unit disk, except for the points $z_\tau^\pm := \exp\{\pm i\cos^{-1}\lambda_\tau\}$. Thus, (9.19) establishes that $D_{\tau'}$ lies "strictly to the left" of D_τ for any $0\le\tau<\tau'<\infty$, as indicated in fig. 6 below.

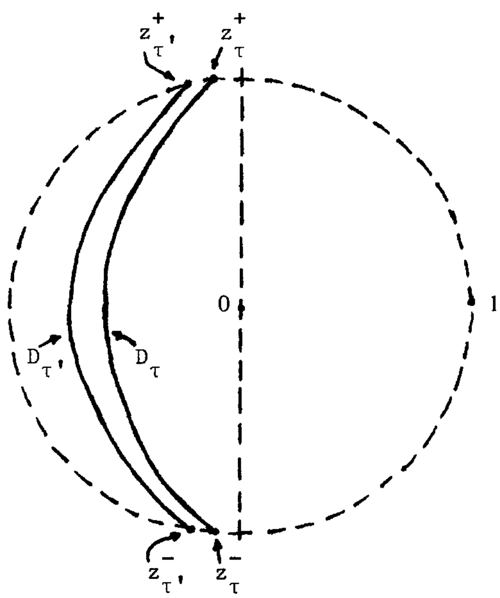

Fig. 6. $\tau'>\tau>1$.

Using the notation of [18], define

$$(9.21)\qquad \hat{N}_\tau(z) := \left(\frac{h_\tau''(t_\tau^-(z))}{h_\tau''(t_\tau^+(z))}\right)^{1/2},$$

which is analytic and single-valued on $\mathbb{C}\backslash(\mathscr{R}_\tau\cup\{0\})$ for $0\le\tau<\infty$.

It can be verified from (9.21) and the definitions of [18] that

$$(9.22)\qquad \hat{N}_\tau(z)=\frac{g_\tau(z)+1-\lambda_\tau z}{z\sqrt{1-\lambda_\tau^2}},\quad \forall z\in\mathbb{C}\setminus(\mathscr{R}_\tau\cup\{0\}),$$

and that

$$(9.23)\qquad \frac{z\hat{N}'_\tau(z)}{\hat{N}_\tau(z)}=-\frac{1}{g_\tau(z)},\quad \forall z\in\mathbb{C}\setminus(\mathscr{R}_\tau\cup\{0\}).$$

Because (cf. [18], eq. (4.1)) Re $g_\tau(z)>0$ on $\mathbb{C}\setminus\mathscr{R}_\tau$, it follows from (9.23) that $|\hat{N}_\tau(z)|$ is strictly decreasing, for any fixed θ, on the ray $\{z=re^{i\theta}: 0<r<\infty\}$ in $\mathbb{C}\setminus(\mathscr{R}_\tau\cup\{0\})$. Furthermore, as Im $g_\tau(z)<0$ along the (open) arc of the unit circle from $z=z_\tau^+$ to $z=-1$, it also follows from (9.23) that $|\hat{N}_\tau(z)|$ is strictly increasing along this arc. (Similarly, $|\hat{N}_\tau(z)|$ is strictly decreasing along the arc of the unit circle from $z=-1$ to $z=z_\tau^-$.) These observations will be useful below.

Considering now any zero $z_{k,n}$ of $\theta_n((2n+a-2)z/2;\, a)$, we must show that, for all n sufficiently large, $z_{k,n}$ lies to the left of D_1 of (9.20). Since D_1 is a Jordan arc in $|z|\le 1$, we may assume, without loss of generality, that $|z_{k,n}|\le 1$. Next, from [18, eq. (4.30)],

$$(9.24)\qquad |w_{\sigma(n)}(z_{k,n})|^{n+\nu_n}=|\hat{N}_{\sigma(n)}(z_{k,n})|\left\{1+\mathscr{O}\left(\frac{1}{n+\nu_n}\right)\right\},\ \text{as } n\to\infty,$$

where (cf. (9.15))

$$(9.25)\qquad \sigma(n):=\nu_n/n=(n+a-2)/n,\ \text{for all } n \text{ sufficiently large},$$

and where the term $\mathscr{O}(1/(n+\nu_n))$ in (9.24) holds uniformly on any compact subset of $\mathbb{C}\setminus(\mathscr{R}_1\cup\{0\})$. On the other hand, it follows from Theorem 4.1 that this zero $z_{k,n}$ satisfies $|\text{Arg } z_{k,n}|>\cos^{-1}(-a/(2n+a-2))$. Thus, it follows from previous observations from (9.23) that, with $\alpha_n:=\cos^{-1}(-a/(2n+a-2))$ and $\mu_n:=\exp(i\alpha_n)$,

$$(9.26)\qquad \min\{|\hat{N}_{\sigma(n)}(z)|: |z|\le 1 \text{ and } |\text{Arg } z|\ge\alpha_n\}=|\hat{N}_{\sigma(n)}(\mu_n)|.$$

Furthermore, direct calculations with the right side of (9.26), using (9.22), show that there exists a positive constant c such that

$$(9.27)\qquad |\hat{N}_{\sigma(n)}(\mu_n)|=\frac{|g_{\sigma(n)}(\mu_n)+1-\lambda_{\sigma(n)}\mu_n|}{\sqrt{1-\lambda_{\sigma(n)}^2}}\ge 1+\frac{c}{\sqrt{n}},\ \text{as } n\to\infty.$$

Thus, (9.26) and (9.27) together imply, with (9.24), that there exists a constant $c'>0$ such that

$$(9.28)\qquad |w_{\sigma(n)}(z_{k,n})|^{n+\nu_n}\ge\left(1+\frac{c}{\sqrt{n}}\right)\left\{1+\mathscr{O}\left(\frac{1}{n}\right)\right\}\ge 1+\frac{c'}{\sqrt{n}},\ \text{as } n\to\infty,$$

so that $z_{k,n}$ must lie strictly to the left of $D_{\sigma(n)}$, for n sufficiently large. But, as (cf. (9.25))

$$\sigma(n) = \frac{n+a-2}{n} = 1 + \frac{a-2}{n},$$

then $\sigma(n)$ strictly decreases to unity as $n \to \infty$, for any fixed $a \geq 2$. Hence, from our previous discussion, each zero $z_{k,n}$ of $\theta_n((2n+a-2)z/2; a)$ must lie, for each n sufficiently large, to the left of D_1. Performing the inversion $z \to 1/z$, this implies that each zero z of $y_n(2z/(2n+a-2); a)$ for n sufficiently large, must satisfy (cf. (7.1) and (7.2))

$$|\omega(z)| > 1 \text{ and } \operatorname{Re} z < 0,$$

which establishes Theorem 7.2. □

PROOF OF THEOREM 7.3. For n a positive odd integer, let $z_n(a)$ denote the negative real zero of $\theta_n((2n+a-2)z/2; a)$ so that (cf. (9.24))

$$(9.29) \qquad |w_{\sigma(n)}(z_n(a))|^{n+v_n} = |\hat{N}_{\sigma(n)}(z_n(a))| \left\{1 + \mathcal{O}\left(\frac{1}{n+v_n}\right)\right\}, \text{ as } n \to \infty,$$

where v_n and $\sigma(n)$ are given respectively by (9.15) and (9.25). Then, let $\tilde{z}_n(a)$ be defined as the unique negative real number such that

$$(9.30) \qquad |w_{\sigma(n)}(\tilde{z}_n(a))|^{n+v_n} = |\hat{N}_{\sigma(n)}(\tilde{z}_n(a))|,$$

for each $n \geq 1$. That $\tilde{z}_n(a)$ is uniquely defined for each $n \geq 1$ follows from the fact that $|\hat{N}_\tau(z)|$ is, from (9.23), strictly decreasing on the ray $\{z = -r, 0 < r < \infty\}$, while $|w_\tau(z)|$ is, from (9.17), strictly increasing (from zero to infinity) on this same ray. Next, by means of Taylor series expansions and identities involving $w_{\sigma(n)}(z)$ and $\hat{N}_{\sigma(n)}(z)$ (which we omit for reasons of brevity), it can be verified that

$$(9.31) \qquad z_n(a) = \tilde{z}_n(a) + \mathcal{O}\left(\frac{1}{(n+v_n)^2}\right), \text{ as } n \to \infty.$$

Thus, if we can express $\tilde{z}_n(a)$ as

$$(9.32) \qquad \tilde{z}_n(a) = \hat{r} + \frac{\gamma_1(a)}{(n+v_n)} + \mathcal{O}\left(\frac{1}{(n+v_n)^2}\right), \text{ as } n \to \infty,$$

where $\hat{r}$ is defined in (7.5) and $\gamma_1(a)$ is independent of n, then a consequence of (9.31) is that

$$(9.33) \qquad z_n(a) = \hat{r} + \frac{\gamma_1(a)}{(n+v_n)} + \mathcal{O}\left(\frac{1}{(n+v_n)^2}\right), \text{ as } n \to \infty.$$

To establish (9.32), we have from (9.30) that

$$(9.34) \qquad |w_{\sigma(n)}(\tilde{z}_n(a))| = |\hat{N}_{\sigma(n)}(\tilde{z}_n(a))|^{1/(n+v_n)}.$$

Assuming the form (9.32), it follows from (9.17) that

$$(9.35)\qquad w_{\sigma(n)}(\tilde{z}_n(a)) = w_{\sigma(n)}(\hat{r})\left\{1 + \frac{\gamma_1(a)g_{\sigma(n)}(\hat{r})}{(n+\nu_n)\hat{r}} + \mathscr{O}\left(\frac{1}{(n+\nu_n)^2}\right)\right\}, \text{ as } n\to\infty.$$

Now, from (9.18), it can be verified that

$$|w_{\sigma(n)}(\hat{r})| = |w_1(\hat{r})| + \frac{(a-2)|w_1(\hat{r})|\cdot\ln(\hat{r}+\sqrt{1+\hat{r}^2})}{2n} + \mathscr{O}\left(\frac{1}{n^2}\right), \text{ as } n\to\infty.$$

But as $|w_1(\hat{r})| = 1$ from [18, eq. (4.40)], the above reduces to

$$(9.36)\qquad |w_{\sigma(n)}(\hat{r})| = 1 + \frac{(a-2)\ln(\hat{r}+\sqrt{1+\hat{r}^2})}{2n} + \mathscr{O}\left(\frac{1}{n^2}\right), \ n\to\infty.$$

Similarly, using the definition of $g_\tau(z)$ in (9.17), it can be shown that

$$(9.37)\qquad 1 + \frac{\gamma_1(a)g_{\sigma(n)}(\hat{r})}{(n+\nu_n)\hat{r}} + \mathscr{O}\left(\frac{1}{(n+\nu_n)^2}\right) = 1 + \frac{\gamma_1(a)\sqrt{1+\hat{r}^2}}{2n\hat{r}} + \mathscr{O}\left(\frac{1}{n^2}\right), \ n\to\infty.$$

Thus, combining (9.35)–(9.37) yields

$$(9.38)\qquad \left\{\begin{aligned} |w_{\sigma(n)}(\tilde{z}_n(a))| = \Bigg|1 + \frac{1}{2n}\Bigg\{&(a-2)\ln(\hat{r}+\sqrt{1+\hat{r}^2}) \\ &+ \frac{\gamma_1(a)\sqrt{1+\hat{r}^2}}{\hat{r}}\Bigg\} + \mathscr{O}\left(\frac{1}{n^2}\right)\Bigg|. \end{aligned}\right.$$

Next, using (9.23), it can be verified that

$$(9.39)\qquad |\hat{N}_{\sigma(n)}(\tilde{z}_n(a))|^{1/(n+\nu_n)} = 1 + \frac{\sqrt{1+\hat{r}^2}}{2n} + \mathscr{O}\left(\frac{1}{n^2}\right), \text{ as } n\to\infty.$$

Thus, on combining (9.38) and (9.39) in (9.34), we deduce that

$$(9.40)\qquad \gamma_1(a) = \hat{r}\left\{\frac{\sqrt{1+\hat{r}^2} + (2-a)\ln(\hat{r}+\sqrt{1+\hat{r}^2})}{\sqrt{1+\hat{r}^2}}\right\}.$$

Now, as $n+\nu_n = 2n+a-2$, it follows that the negative real zero of $\theta_n((2n+a-2)z/2;\, a)$, for n odd, is given by

$$(9.41)\qquad z_n(a) = \hat{r} + \frac{\gamma_1(a)}{(2n+a-2)} + \mathscr{O}\left(\frac{1}{(2n+a-2)^2}\right), \text{ as } n\to\infty.$$

Recalling (2.4), then (9.41) implies that the negative real zero $\alpha_n(a)$ of the non-normalized GBP $y_n(z;\, a)$, for n odd, satisfies the desired result (7.4) of Theorem 7.3. □

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