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INCOMPLETE FACTORIZATIONS OF MATRICES AND CONNECTIONS WITH H-MATRICES*

R. S. VARGA†, E. B. SAFF‡ AND V. MEHRMANN§

Dedicated to Professor Garrett Birkhoff on the occasion of his seventieth birthday

Abstract. There has been much recent interest in the use of incomplete factorizations of matrices, in conjunction with applications of the generalized conjugate gradient method, for approximating solutions of large sparse systems of linear equations. Underlying many of these recent developments is the theory of H-matrices, introduced by A. M. Ostrowski. In this note, further connections of the theory of incomplete factorizations of matrices with the theory of H-matrices are derived.

1. Introduction. Suppose we wish to approximate the solution of the matrix equation

$$A\mathbf{x}=\mathbf{k}, \tag{1.1}$$

where A is typically a large sparse real symmetric and positive definite $n\times n$ matrix. Because the unique Cholesky factorization of A as $L\cdot L^T$ may, as a result of fill-in, produce a lower triangular matrix L (with positive diagonal entries) which is considerably less sparse than A, it is then convenient to consider a matrix splitting of A, i.e.,

$$A=M-N, \tag{1.2}$$

where M is also a real symmetric and positive definite $n\times n$ matrix. The matrix M should be chosen, for purposes of actual computations, so that matrix equations of the form

$$M\mathbf{g}=\mathbf{h} \tag{1.3}$$

can be readily solved for $\mathbf{g}$, given $\mathbf{h}$. (One interpretation of this could be that the Cholesky decomposition $\tilde{M}\cdot \tilde{M}^T$ of $\tilde{M}$ has an acceptable or controlled degree of sparseness.) In addition, the matrix M should be chosen in (1.2) so that $M^{-1}A$ is, in some sense, a reasonable approximation of the identity operator I, so that the preconditioned system

$$M^{-1}A\mathbf{x}=M^{-1}\mathbf{k} \tag{1.4}$$

may be better conditioned than that of (1.1). What is an important recent observation is that a good choice of the matrix M in the splitting (1.2), when coupled with the generalized (or implicit) conjugate gradient method applied to (1.4), produces a *powerful* numerical method for the solution of the matrix equation (1.1). In this regard, see Concus, Golub, and O'Leary [2], Greenbaum [3], Kershaw [4], Manteuffel [5], Meijerink and van der Vorst [6], and Reid [8].

One method which has received recent support in selecting the matrix M in the splitting (1.2), is to control the fill-in of the Cholesky decomposition of A by means of a graph, an idea which seems to have first been suggested in Varga [9] as a specific technique for generating regular splittings (cf. [10, p. 88]) of certain finite difference

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† Department of Mathematics, Kent State University, Kent, Ohio 44242. The work of this author was supported in part by the U.S. Air Force Office of Scientific Research and the U.S. Department of Energy.
‡ Department of Mathematics, University of South Florida, Tampa, Florida 33620. The work of this author was supported in part by the U.S. Air Force Office of Scientific Research.
§ Department of Mathematics, Kent State University, Kent, Ohio 44242.

operators. This method leads to *incomplete factorizations* of A, and is described below.

For n any positive integer, let $\mathbb{C}^{n,n}$ denote as usual the collection of all $n \times n$ matrices $A = [a_{i,j}]$ with complex entries $a_{i,j}$, and let G (for *graph*) denote any set (possibly empty) of ordered pairs of integers (i, j), with $1 \leqq i, j \leqq n$ and with $i \neq j$. It is convenient to call $\mathcal{G}_n$ the collection of all such sets G. Then, given any $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ and given any $G \in \mathcal{G}_n$, we attempt to produce a nonsingular matrix approximant, $A(G)$, of A, dependent on G, which is represented in factored form as

$$A(G) = L(G) \cdot \Sigma(G) \cdot U(G). \tag{1.5}$$

Here, $L(G) := [l_{i,j}(G)]$ and $U(G) := [u_{i,j}(G)]$ are to be respectively (unit) lower and upper triangular $n \times n$ matrices with all diagonal entries unity, and $\Sigma(G) := \operatorname{diag}[\sigma_1(G), \cdots, \sigma_n(G)]$ is to be an $n \times n$ nonsingular diagonal matrix, where the dependence of these matrices on G is defined recursively $(i = 1, 2, \cdots, n)$ as follows. With

$$\sigma_i(G) := a_{i,i} - \sum_{k=1}^{i-1} l_{i,k}(G)\sigma_k(G)u_{k,i}(G), \tag{1.6}$$

(where $\sum_{k=1}^{0} := 0$ when $i = 1$ above) then if $\sigma_i(G) \neq 0$, set

$$\begin{aligned} u_{i,j}(G) &:= \begin{cases} \left(a_{i,j} - \sum_{k=1}^{i-1} l_{i,k}(G)\sigma_k(G)u_{k,j}(G)\right)/\sigma_i(G), & \text{if } (i,j) \in G, \\ 0, & \text{if } (i,j) \notin G, \end{cases} \\ &\qquad\qquad \text{for each } j \text{ with } i < j \leqq n; \\ l_{j,i}(G) &:= \begin{cases} \left(a_{j,i} - \sum_{k=1}^{i-1} l_{j,k}(G)\sigma_k(G)u_{k,i}(G)\right)/\sigma_i(G), & \text{if } (j,i) \in G, \\ 0, & \text{if } (j,i) \notin G, \end{cases} \\ &\qquad\qquad \text{for each } j \text{ with } i < j \leqq n. \end{aligned} \tag{1.7}$$

We note that the set G controls the sparseness of the factors of (1.5). Next, we say that the above procedure *fails* if, at any step, $\sigma_i(G) = 0$, $i = 1, 2, \cdots, n$. If the above recursive definition does not fail at any step, we say that A *admits a regular incomplete factorization with respect to* G. Evidently, if A admits a regular incomplete factorization with respect to some $G \in \mathcal{G}_n$, then

$$\det A(G) = \prod_{i=1}^{n} \sigma_i(G) \neq 0. \tag{1.8}$$

In particular, if $\hat{G}$ denotes the maximal element in $\mathcal{G}_n$, i.e.,

$$\hat{G} := \{(i, j) : i \neq j \text{ and } 1 \leqq i, j \leqq n\}, \tag{1.9}$$

and if A admits a regular incomplete factorization with respect to $\hat{G}$, then evidently

$$A = A(\hat{G}) \quad \text{and} \quad \det A \neq 0. \tag{1.10}$$

Our interest here is in those matrices $A \in \mathbb{C}^{n,n}$ which are *regular for* $\mathcal{G}_n$, i.e., A admits a regular incomplete factorization with respect to *any* $G \in \mathcal{G}_n$, and we set

$$\mathcal{F}_n := \{A \in \mathbb{C}^{n,n} : A \text{ is regular for } \mathcal{G}_n\}. \tag{1.11}$$

In terms of applications, this means that, for any $A \in \mathcal{F}_n$, any choice of $G \in \mathcal{G}_n$ produces a regular incomplete factorization $A(G)$, which is a candidate for the matrix M in (1.4).

From Meijerink and van der Vorst [6], it is known that every $n \times n$ nonsingular M-matrix is in $\mathcal{F}_n$. More recently, this result has been extended by Manteuffel [5], who in essence established that every $n \times n$ nonsingular H-matrix is also in $\mathcal{F}_n$. Our interest here is in characterizing particular subsets of $\mathcal{F}_n$, as they relate to these known results in the area. For the remainder of this section, we introduce some additional notation.

For any $A = [a_{i,j}] \in \mathbb{C}^{n,n}$, its *comparison matrix* $\mathfrak{M}(A)$ is defined by

$$(1.12) \quad \mathfrak{M}(A) = [\alpha_{i,j}] \in \mathbb{R}^{n,n}, \quad \text{where } \alpha_{i,i} := |a_{i,i}|, \quad \alpha_{i,j} := -|a_{i,j}|, \quad i \neq j, \quad 1 \leqq i, j \leqq n.$$

With $\tau := \max_{1 \leqq i \leqq n} |a_{i,i}|$, we can write $\mathfrak{M}(A) = \tau I - C^A$, where $C^A \geqq \mathcal{O}$ (i.e., C^A is an $n \times n$ matrix with nonnegative entries). Following Ostrowski [7], A is then said to be a nonsingular (singular) *H-matrix* iff $\mathfrak{M}(A)$ is a nonsingular (singular) M-matrix iff $\tau > \rho(C^A)$ $(\tau = \rho(C^A))$, where $\rho(B) := \max\{|\lambda|: \det(\lambda I - B) = 0\}$ denotes the spectral radius of any $B \in \mathbb{C}^{n,n}$. We then set

$$(1.13) \quad \mathcal{H}_n := \{A \in \mathbb{C}^{n,n} : A \text{ is a nonsingular } H\text{-matrix}\},$$

and, for any $A = [a_{i,j}] \in \mathbb{C}^{n,n}$, we further set

$$(1.14) \quad \Omega^c(A) := \{B = [b_{i,j}] \in \mathbb{C}^{n,n} : |b_{i,j}| = |a_{i,j}| \text{ for all } i \leqq i, j \leqq n\},$$

and

$$(1.15) \quad \Omega^d(A) := \{B = [b_{i,j}] \in \mathbb{C}^{n,n} : |b_{i,i}| = |a_{i,i}| \text{ and } |b_{i,j}| \leqq |a_{i,j}| \text{ for all } 1 \leqq i, j \leqq n\},$$

(where the superscripts c and d on Ω denote "circle" and "disk", respectively).

§ 2. The set $\mathcal{F}_n^d$. As a preliminary result, we first establish

LEMMA 1. *For any $A \in \mathcal{H}_n$, then*

$$(2.1) \quad \Omega^d(A) \subseteq \mathcal{F}_n.$$

Proof. Given any $A = [a_{i,j}]$ in $\mathcal{H}_n$, then its comparison matrix $\mathfrak{M}(A) = \tau I - C^A$ is such that $\tau > \rho(C^A)$. For any $B = [b_{i,j}]$ in $\Omega^d(A)$, it follows from (1.13) and the Perron–Frobenius Theorem (cf. [10, p. 46]) that $\mathfrak{M}(B) = \tau I - C^B$, where $C^B \geqq \mathcal{O}$ and where $\rho(C^B) \leqq \rho(C^A) < \tau$. Consequently, $\mathfrak{M}(B)$ is a nonsingular M-matrix, or equivalently, $B \in \mathcal{H}_n$. Manteuffel's result ([5]) gives $B \in \mathcal{F}_n$. $\square$

If we set

$$(2.2) \quad \mathcal{F}_n^d := \{A \in \mathcal{F}_n : \Omega^d(A) \subseteq \mathcal{F}_n\},$$

then Lemma 1 asserts that

$$(2.3) \quad \mathcal{H}_n \subseteq \mathcal{F}_n^d.$$

Actually, the reverse inclusion also holds, as we now show.

THEOREM 1. *For any positive integer n,*

$$(2.4) \quad \mathcal{H}_n = \mathcal{F}_n^d.$$

Proof. From (2.3), it suffices to show that $\mathcal{F}_n^d \subseteq \mathcal{H}_n$. Consider any $A \in \mathcal{F}_n^d$. Forming $\mathfrak{M}(A) = \tau I - C^A$ where $C^A = [c_{i,j}^A] \geqq \mathcal{O}$, we wish to show that $\rho(C^A) < \tau$. Assuming the contrary, then $\tau \leqq \rho(C^A)$. From the discussion following (1.12), then

$$(2.5) \quad |a_{i,i}| = \tau - c_{i,i}^A \quad \forall 1 \leqq i \leqq n.$$

From the hypothesis that $A \in \mathcal{F}_n^d$, it follows from (1.15) and (2.2) that $\text{diag}[a_{1,1}, \cdots, a_{n,n}] \in \mathcal{F}_n$, and this readily implies that $a_{i,i} \neq 0$ for all $1 \leqq i \leqq n$.

Consequently, from (2.5),

$$\tau > \max_{1 \leqq i \leqq n} c_{i,i}^A. \tag{2.6}$$

Next, for any $t \in [0, 1]$, define $C(t) := [c_{i,j}(t)] \in \mathbb{R}^{n,n}$ by

$$c_{i,j}(t) = tc_{i,j}^A, \quad i \neq j; \qquad c_{i,i}(t) = c_{i,i}^A \quad \forall 1 \leqq i, j \leqq n, \quad \forall t \in [0, 1],$$

and set $B(t) := \tau I - C(t)$. Clearly, $C(t) \geqq \mathcal{O}$, and $\rho(C(t))$ is a nondecreasing function of t on $[0, 1]$ by the Perron–Frobenius Theorem, with

$$\rho(C(0)) = \max_{1 \leqq i \leqq n} c_{i,i}^A, \quad \text{and} \quad \rho(C(1)) = \rho(C^A).$$

Thus,

$$g(t) := \tau - \rho(C(t))$$

is a nonincreasing function on $[0, 1]$ with $g(0) = \tau - \max_{1 \leqq i \leqq n} c_{i,i}^A > 0$ from (2.6), and with $g(1) = \tau - \rho(C^A) \leqq 0$ by assumption. By continuity, there exists a $\hat{t} \in (0, 1]$ for which $g(\hat{t}) = 0$, so that $B(\hat{t})$ is a singular M-matrix. On the other hand, $B(\hat{t})$ is, by definition, an element of $\Omega^d(A)$, and hence, by hypothesis, is in $\mathcal{F}_n$. If we choose $\hat{G}$ of (1.9) in $\mathcal{G}_n$, it follows from (1.10) that $B(\hat{t})$ is necessarily nonsingular, a contradiction. □

We next establish an Ostrowski-like result (cf. [7]) for the set $\mathcal{H}_n$, which generalizes a recent result of Manteuffel ([5, Cor. 3.4]).

LEMMA 2. *For any $A \in \mathcal{H}_n$, then*

$$\min\{|\det B(G)| : B \in \Omega^d(A), G \in \mathcal{G}_n\} = \det \mathfrak{M}(A) > 0. \tag{2.7}$$

Proof. For $A \in \mathcal{H}_n$, each $B \in \Omega^d(A)$ is in $\mathcal{F}_n$ from Theorem 1. Thus, for any $G \in \mathcal{G}_n$, $B(G)$ admits a regular factorization (cf. (1.5))

$$B(G) = L^B(G)\Sigma^B(G)U^B(G),$$

where $\Sigma^B(G) := \text{diag}[\sigma_1^B(G), \cdots, \sigma_n^B(G)]$ satisfies $|\sigma_i^B(G)| > 0$ for all $1 \leqq i \leqq n$. Now, using the same induction argument as originally used by Ostrowski [7], it follows that

$$|\sigma_i^B(G)| \geqq \sigma_i(\mathfrak{M}(A)) \quad \forall B \in \Omega^d(A), \quad \forall G \in \mathcal{G}_n, \quad \forall 1 \leqq i \leqq n, \tag{2.8}$$

so that on taking products over i, we have

$$\min\{|\det B(G)| : B \in \Omega^d(A), G \in \mathcal{G}_n\} \geqq \det \mathfrak{M}(A) > 0. \tag{2.9}$$

Obviously, since $\mathfrak{M}(A)$ is itself an element of $\Omega^d(A)$, the first inequality in (2.9) reduces to equality, thereby establishing (2.7). □

3. The set $\mathcal{F}_n^c$. Thus far, all published investigations concerning matrices A which admit regular incomplete factorizations for all $G \in \mathcal{G}_n$ have involved explicitly the subset $\mathcal{F}_n^d$ of (2.2). Using the notation of (1.14), we now consider the larger subset of $\mathcal{F}_n$ defined by

$$\mathcal{F}_n^c := \{A \in \mathcal{F}_n : \Omega^c(A) \subseteq \mathcal{F}_n\}. \tag{3.1}$$

As $\Omega^c(A) \subseteq \Omega^d(A)$ from (1.14) and (1.15), then obviously $\mathcal{F}_n^d \subseteq \mathcal{F}_n^c$. More precisely, it is clear that $\mathcal{F}_1^d = \mathcal{F}_1^c = \mathcal{F}_1$, and simple examples show that

$$\mathcal{F}_n^d \subsetneqq \mathcal{F}_n^c \subsetneqq \mathcal{F}_n \quad \forall n \geqq 2.$$

It is convenient to make the following definitions. Given any $G \in \mathcal{G}_n$ and given any

$A=[a_{i,j}]\in\mathbb{C}^{n,n}$, then the matrix $A_G:=[c_{i,j}]\in\mathbb{C}^{n,n}$ is defined by

$$\begin{cases} c_{i,i}=a_{i,i} & \forall 1\leqq i\leqq n; \\ c_{i,j}=a_{i,j} & \text{if } (i,j)\in G, \quad i\neq j; \\ c_{i,j}=0 & \text{if } (i,j)\notin G, \quad i\neq j. \end{cases} \tag{3.2}$$

Next, given any $G\in\mathscr{G}_n$ and given any $A\in\mathbb{C}^{n,n}$, then G is said to be a *masking graph for* A if i) A admits a regular incomplete factorization with respect to G, and ii) $A(G)=A_G$.

We remark that if A admits a regular incomplete factorization with respect to G, then $A(G)=L(G)\cdot\Sigma(G)\cdot U(G)$ from (1.5), when multiplied out, is *not* in general equal to A_G, so that all graphs are not necessarily masking graphs for A. On the other hand, we have the following necessary and sufficient condition for a graph G to be a *masking graph for* $\mathscr{F}_n$, i.e., for every $A\in\mathscr{F}_n$.

PROPOSITION 1. *Given $G\in\mathscr{G}_n$, then G is a masking graph for $\mathscr{F}_n$ iff*

$$\begin{cases} \textit{if } (i,j)\notin G \textit{ with } i\neq j, \textit{ then, for each positive integer } k \textit{ with} \\ 1\leqq k\leqq \min(i;j), \textit{ at least one of the pairs } (i,k) \textit{ and } (k,j) \textit{ is not in } G. \end{cases} \tag{3.3}$$

Proof. Assuming $G\in\mathscr{G}_n$ satisfies (3.3), consider any $A\in\mathscr{F}_n$. Then, A admits (cf. (1.5)) the regular factorization $A(G)=L(G)\cdot\Sigma(G)\cdot U(G)=:[e_{i,j}]$, where

$$e_{i,j}=\sum_{k=1}^{\min(i;j)} l_{i,k}(G)\sigma_k(G)u_{k,j}(G). \tag{3.4}$$

Applying (3.3), we see that $e_{i,j}=0$ for every $(i,j)\notin G$ with $i\neq j$. Similarly, the corresponding (i,j)th entry of A_G is also zero from (3.2). On the other hand, if $(i,j)\in G$, then from (1.7), $e_{i,j}=a_{i,j}$, whence $A(G)=A_G$. Thus, G is a masking graph for every $A\in\mathscr{F}_n$.

Conversely, suppose that G is a masking graph for every $A\in\mathscr{F}_n$. On choosing A to be any nonsingular M-matrix, all of whose off-diagonal entries are negative, it follows (cf. (1.7)) that every off-diagonal entry of $L(G)$ and $U(G)$ is necessarily negative or zero, while $\sigma_k(G)$ is positive for all $1\leqq k\leqq n$. Consequently, each product term in the sum (3.4) is either positive or zero. Now, because G by hypothesis is a masking graph for A, then $A(G)=A_G$. Thus, if $(i,j)\notin G$ with $i\neq j$, the sum in (3.4), consisting of positive or zero terms, must also vanish from (3.2), from which it is evident that (3.3) is valid. □

As an immediate consequence of Proposition 1 and (3.1), we have

COROLLARY 1. *If $G\in\mathscr{G}_n$ is a masking graph for $\mathscr{F}_n$ and if $A\in\mathscr{F}_n^c$, then $B(G)=B_G$ for all $B\in\Omega^c(A)$.*

Combining the above with the Camion-Hoffmann Theorem [1], we next establish

THEOREM 2. *Let $A\in\mathscr{F}_n$, and let $G\in\mathscr{G}_n$ be a masking graph for each $B\in\Omega^c(A)$. Then, there exists a unique $n\times n$ permutation matrix P (dependent on G) such that $PA_G\in\mathscr{H}_n$.*

Proof. By hypothesis, $B(G)$ is nonsingular and $B(G)=B_G$ for all $B\in\Omega^c(A)$. Thus, all matrices equimodular (cf. (1.14)) to A_G are nonsingular. Using the main result of Camion and Hoffman [1], there is a unique $n\times n$ permutation matrix P such that $PA_G\in\mathscr{H}_n$. □

From Theorem 2, we easily deduce

COROLLARY 2. *Let $A\in\mathscr{F}_n^c$, and let $G\in\mathscr{G}_n$ be any masking graph for $\mathscr{F}_n$. Then, there exists a unique $n\times n$ permutation matrix P (dependent on G) such that $PA_G\in\mathscr{H}_n$. In* *particular, if S is any $r \times r$ principal submatrix of A (with $1 \leqq r \leqq n$), there exists a unique $r \times r$ permutation matrix P (dependent on S) such that $PS \in \mathcal{H}_r$.*

Proof. The first part follows immediately from Corollary 1 and Theorem 2. If S is any principal submatrix of A, we can select $G \in \mathcal{G}_n$ satisfying (3.3) such that A_G is the direct sum of S and other principal submatrices of A. The desired result then follows from the first part above. □

Turning the first part of Corollary 2 around, it is natural to ask whether, if $A \in \mathcal{F}_n$, and if every $G \in \mathcal{G}_n$ which is a masking graph for $\mathcal{F}_n$ has the property that there exists a permutation matrix P such that $PA_G \in \mathcal{H}_n$, then is $A \in \mathcal{F}_n^c$? This is true for $n = 1$ and $n = 2$, and is an open question for $n > 2$. The similar (but stronger) question can be asked for the necessary condition of the second part of Corollary 2; i.e., if $A \in \mathbb{C}^{n,n}$ and if every $r \times r$ principal submatrix S of A (with $1 \leqq r \leqq n$) has the property that there exists an $r \times r$ permutation matrix P such that $PS \in \mathcal{H}_r$, is $A \in \mathcal{F}_n^c$? This is again true for $n = 1$ and $n = 2$, but *fails* for $n \geqq 3$. To illustrate this, consider the matrix

$$A = \begin{bmatrix} 1 & 6 & -1 \\ 6 & 1 & -1 \\ 1 & 1 & 4 \end{bmatrix},$$

which has the property that, for every principal submatrix S of A, there is a permutation matrix $P = P(S)$ such that PS is a nonsingular H-matrix. Choosing $\tilde{G} := \hat{G} \backslash \{(1, 2)\}$ where $\hat{G}$ is defined in (1.9), we see $\tilde{G}$ satisfies (3.3) of Proposition 1. However, $A(\tilde{G})$ can be factored as

$$\begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix},$$

and is evidently singular. As such, A is not an element of $\mathcal{F}_3$, and hence, is not, a fortiori, in $\mathcal{F}_3^c$.

However, Theorem 2 is useful in another direction. For any $A \in \mathcal{F}_n^c$, we evidently have that

$$|\det B(G)| > 0 \quad \forall B \in \Omega^c(A), \quad \forall G \in \mathcal{G}_n, \tag{3.5}$$

so that in analogy with (2.7), we have

$$0 < \mu^c(A) := \min\{|\det B(G)| : B \in \Omega^c(A), G \in \mathcal{G}_n\}. \tag{3.6}$$

We further know from Theorem 2 that there is a unique $n \times n$ permutation matrix P such that $PA \in \mathcal{H}_n$, or equivalently, that $\mathfrak{M}(PA)$ is a nonsingular M-matrix. Thus, given an $A \in \mathcal{F}_n^c$, we have a choice of considering incomplete factorizations relative either to A or to PA. *Is there an advantage in general with the latter choice*? With this question in mind, we establish

THEOREM 3. *For any $A \in \mathcal{F}_n^c$, let P be the unique $n \times n$ permutation matrix such that $PA \in \mathcal{H}_n$. Then (cf. (3.6)),*

$$\mu^c(A) \leqq \mu^c(PA) = \det \mathfrak{M}(PA). \tag{3.7}$$

Remark. At least from the point of view of determinants, (3.7) suggests that it may be preferable to consider incomplete factorizations relative to PA, rather than to A.

Proof. First, note that the equality in (3.7) is a consequence of Lemma 2, and the fact that $\mathfrak{M}(PA)$ is an element of $\Omega^c(PA)$, which is a subset of $\Omega^d(PA)$. Suppose that the first inequality of (3.7) is not valid, i.e., $\mu^c(A) > \det \mathfrak{M}(PA)$, which implies from (3.6) that

$$|\det B(G)| > \det \mathfrak{M}(PA) \quad \forall B \in \Omega^c(A), \quad \forall G \in \mathcal{G}_n. \tag{3.8}$$

Next, define $B := P^T \mathfrak{M}(PA)$, which is clearly an element of $\Omega^c(A)$. On choosing $\hat{G}$ of (1.9) in $\mathcal{G}_n$, we obtain from (1.10) that $B = B(\hat{G})$. But, since $|\det P^T| = 1$, we see that

$$|\det B| = |\det P^T \cdot \det \mathfrak{M}(PA)| = |\det P^T| \cdot |\det \mathfrak{M}(PA)| = \det \mathfrak{M}(PA),$$

which contradicts (3.8). $\square$

It would be tempting to conjecture that if the unique permutation matrix P in Theorem 3, for which $PA \in \mathcal{H}_n$, is such that $P \not\equiv I$, then strict inequality would hold for the first inequality in (3.7). This, however, is not the case, which can be seen by examining the particular matrix of

$$A_1 := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

In this case, $A_1 \in \mathcal{F}_2^c$ and equality can be shown to be valid throughout in (3.7), with $\det \mathfrak{M}(PA) = 2$. On the other hand, the first inequality of (3.7) *can* be strict. To show this, consider

$$A_2 := \begin{bmatrix} \varepsilon & 1 \\ 1 & \varepsilon \end{bmatrix}, \qquad 0 < \varepsilon < \frac{1}{\sqrt{2}},$$

which is an element of $\mathcal{F}_2^c$. In this case, $\mu^c(A) = \varepsilon^2$, while $\det \mathfrak{M}(PA) = 1 - \varepsilon^2$, which is significantly larger for ε small.

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