CONSTRUCTIVE APPROACHES TO MATHEMATICAL MODELS
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Bounds for Incomplete Polynomials Vanishing at Both Endpoints of an Interval

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1. INTRODUCTION

At the *December 1976 Tampa Conference on Rational Approximation with Emphasis on Applications of Padé Approximants*, Lorentz [3] presented results and open problems concerning *incomplete polynomials of type θ*, i.e., for a fixed θ with $0 < \theta \leqq 1$, the set of all real or complex polynomials of the form

$$\sum_{k=s}^{n} \alpha_k x^k, \qquad \text{where} \quad s \geqq \theta \cdot n, \quad n \text{ an arbitrary nonnegative integer.} \quad (1.1)$$

These incomplete polynomials of type θ have been further studied by Saff and Varga [7–9], by Kemperman and Lorentz [1], and by Lorentz [3]. Note that any incomplete polynomial of type θ has, from (1.1), a zero at $x = 0$ of order at least $[n \cdot \theta]$.

In this paper, we consider the more general problem of polynomials constrained to have zeros at *both* endpoints of a finite interval. Without loss of generality, we take this interval to be $[-1, +1]$. By analogy with (1.1), for any $\theta_1 > 0$ and $\theta_2 > 0$ fixed with $\theta_1 + \theta_2 \leqq 1$, we consider here the set of all real or complex polynomials of the form

$$(x-1)^{s_1}(x+1)^{s_2} \sum_{k=0}^{m} \beta_k x^k \qquad \text{where} \quad s_1 \geqq \theta_1 \cdot (s_1 + s_2 + m),$$
$$s_2 \geqq \theta_2 \cdot (s_1 + s_2 + m). \quad (1.2)$$

In Section 2, upper bounds for the growth of polynomials of the form (1.2) are determined. In Section 3, an analog of the classical Chebyshev polynomials for constrained polynomials of the form (1.2) is given, and it is shown that the upper bounds of Section 2 are, in a certain limiting sense, *best possible.*

2. GROWTH ESTIMATES FOR CONSTRAINED POLYNOMIALS

In the spirit of two lemmas of Walsh [12, p. 250], we prove the following result of Lemma 2.1 on bounding above the moduli of constrained polynomials. For a related result, see Kemperman and Lorentz [1].

Lemma 2.1. *Let $\mathscr{E}$ be a closed bounded point set, not a single point, whose complement K with respect to the extended complex plane is simply connected. Let $w = \varphi(z)$ denote a function which maps K onto $|w| > 1$, so that the points at infinity correspond to each other. Let the (not necessarily distinct) points α_k, $k = 1, \ldots, m$, lie exterior to $\mathscr{E}$, and let $P(z)$ be a polynomial of degree n ($n \geqq m$) which vanishes at each of the points α_k (with each α_k listed according to its multiplicity). If, on the boundary of $\mathscr{E}$,*

$$|P(z)| \leqq L, \tag{2.1}$$

then, for all z in K,

$$|P(z)| \leqq L|\varphi(z)|^n \prod_{k=1}^{m} \left| \frac{\varphi(z) - \varphi(\alpha_k)}{\overline{\varphi(\alpha_k)}\varphi(z) - 1} \right|. \tag{2.2}$$

Proof. Define $Q(z)$ for z in K by means of

$$Q(z) := \frac{P(z)}{[\varphi(z)]^n} \prod_{k=1}^{m} \left(\frac{\overline{\varphi(\alpha_k)}\varphi(z) - 1}{\varphi(z) - \varphi(\alpha_k)} \right),$$

so that $Q(z)$ is analytic in K, even at infinity. Its modulus is continuous in the exterior of $\mathscr{E}$, and, on the boundary of $\mathscr{E}$, is bounded above by L. Hence, by the maximum principle,

$$|Q(z)| \leqq L, \qquad \text{for all } z \text{ in } K,$$

which gives the desired inequality (2.2). ∎

Here, we are interested in polynomials constrained to have certain order zeros at $x = -1$ and at $x = 1$, and we thus introduce the following notation. As usual, for each nonnegative integer n, π_n denotes the set of complex polynomials of degree at most n. For every ordered triple of non-negative integers (s_1, s_2, m), the set $\pi(s_1, s_2, m)$ is defined by

$$\pi(s_1, s_2, m) := \{(x - 1)^{s_1}(x + 1)^{s_2}q(x) : q \in \pi_m\}. \tag{2.3}$$

Finally, for each continuous g defined on a compact set B, we set

$$\|g\|_B := \max\{|g(z)| : z \in B\}. \tag{2.4}$$

To obtain growth estimates for constrained polynomials from Lemma 2.1, take the set $\mathscr{E}$ now to be some real interval $[a, b]$, with $-1 < a < b < +1$. Then,

$$z = \psi(w) := \frac{b+a}{2} + \frac{b-a}{2}\left(\frac{w + w^{-1}}{2}\right), \qquad |w| > 1, \tag{2.5}$$

maps the exterior of the unit circle in the w-plane onto $\mathbb{C}^*\backslash[a, b]$ in the z-plane (where $\mathbb{C}^*$ denotes the extended complex plane). The function $\varphi(z)$ required by Lemma 2.1 is then the inverse map of $\psi(w)$, i.e.,

$$w = \varphi(z) = \frac{\sqrt{z-a} + \sqrt{z-b}}{\sqrt{z-a} - \sqrt{z-b}}, \qquad z \in \mathbb{C}^*\backslash[a, b], \tag{2.6}$$

for some suitable branch of the square root function. For this choice of $\mathscr{E}$ and for the choice $\alpha_k = 1$, $1 \leqq k \leqq s_1$, and $\alpha_k = -1$, $s_1 + 1 \leqq k \leqq s_1 + s_2$, we have, as an immediate consequence of Lemma 2.1,

Corollary 2.2. *Let $p \in \pi(s_1, s_2, m)$, and set $n := s_1 + s_2 + m$. Then, for all $z \in \mathbb{C}^*\backslash[a, b]$,*

$$|p(z)| \leqq \|p\|_{[a,b]} |\varphi(z)|^n \left|\frac{\varphi(z) - \varphi(1)}{\varphi(1)\varphi(z) - 1}\right|^{s_1} \left|\frac{\varphi(z) - \varphi(-1)}{\varphi(-1)\varphi(z) - 1}\right|^{s_2}, \tag{2.7}$$

where $\varphi(z)$ is given by (2.6).

In a typical application, it may be known that a polynomial from $\pi(s_1, s_2, m)$ is bounded above in modulus on $[-1, 1]$ by some constant L. We then wish to choose an interval $[a, b]$, strictly contained in $[-1, 1]$, so as to optimize the upper bound of (2.7). In what follows, we examine the behavior of polynomials of large degree from $\pi(s_1, s_2, m)$ where $s_1/n \geqq \varphi_1$ and $s_2/n \geqq \theta_2$, for θ_1 and θ_2 fixed. We then make use of certain limiting results to determine the best choice for $[a, b]$, depending only on the relative orders of contact at $x = -1$ and at $x = 1$. As Jacobi polynomials play a significant role in the determination of such an optimal interval $[a, b]$ and in the results of Section 3, we briefly summarize some of their basic properties.

For the real parameters $\alpha > -1$ and $\beta > -1$, $P_n^{(\alpha,\beta)}(x)$ denotes the classical Jacobi polynomial of degree n. It is well known (cf. Szegö [11, p. 68]) that if

$$h_n^{(\alpha,\beta)} := \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta (P_n^{(\alpha,\beta)}(x))^2 \, dx, \tag{2.8}$$

then the sequence $\{P_n^{(\alpha,\beta)}(x)/\sqrt{h_n^{(\alpha,\beta)}}\}_{n=0}^{\infty}$ is orthonormal with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ in the interval $[-1, 1]$, i.e.,

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta\left(\frac{P_n^{(\alpha,\beta)}(x)}{\sqrt{h_n^{(\alpha,\beta)}}}\right)\left(\frac{P_m^{(\alpha,\beta)}(x)}{\sqrt{h_m^{(\alpha,\beta)}}}\right)dx=\delta_{n,m}.$$

The following representation for $h_n^{(\alpha,\beta)}$ will be useful (cf. Szegö [11, p. 68]):

$$h_n^{(\alpha,\beta)}=\frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}. \tag{2.9}$$

As a well-known consequence of the theory of orthogonal polynomials, (cf. Szegö [11, p. 63]), the unique monic polynomial $q(x)$ in π_n which minimizes the integral

$$\int_{-1}^{+1}(1-x)^\alpha(1+x)^\beta(q(x))^2\,dx$$

is given explicitly by the monic Jacobi polynomial

$$2^n\binom{2n+\alpha+\beta}{n}^{-1}P_n^{(\alpha,\beta)}(x). \tag{2.10}$$

It is further well known that the zeros of $P_n^{(\alpha,\beta)}(x)$, for any $n \geqq 1$ and for any choices of $\alpha > -1$ and $\beta > -1$, all lie in $(-1, +1)$. Concerning the asymptotic location of zeros of particular sequences of Jacobi polynomials, we state

Lemma 2.3. (Moak *et al.* [6]) *Let a_n and b_n denote, respectively, the smallest and largest zeros of $P_n^{(\alpha_n,\beta_n)}(x)$, where $\alpha_n > -1$ and $\beta_n > -1$. Assume that*

$$\lim_{n\to\infty}\frac{\alpha_n}{2n+\alpha_n+\beta_n}=\theta_1 \quad \textit{and} \quad \lim_{n\to\infty}\frac{\beta_n}{2n+\alpha_n+\beta_n}=\theta_2. \tag{2.11}$$

For $\mu := \theta_2+\theta_1$ and $\nu := \theta_2-\theta_1$, set

$$\begin{aligned} a &= a(\theta_1,\theta_2) := \mu\nu-\sqrt{(1-\mu^2)(1-\nu^2)},\\ b &= b(\theta_1,\theta_2) := \mu\nu+\sqrt{(1-\mu^2)(1-\nu^2)}. \end{aligned} \tag{2.12}$$

Then,

$$\lim_{n\to\infty}a_n=a \quad \textit{and} \quad \lim_{n\to\infty}b_n=b. \tag{2.13}$$

Moreover, the zeros of the sequence $\{P_n^{(\alpha_n,\beta_n)}(x)\}_{n=0}^{\infty}$ are dense in the interval $[a, b]$.

We make use of this last result in selecting the optimal interval $[a, b]$ in Corollary 2.2. To every ordered pair (θ_1, θ_2) from the set

$$\Omega := \{(\theta_1, \theta_2) : \theta_1 > 0, \theta_2 > 0, \theta_1 + \theta_2 < 1\}, \tag{2.14}$$

there corresponds a unique interval $[a, b]$ with $-1 < a < b < 1$ from (2.12). Consequently, for each (θ_1, θ_2) in Ω, there exists a unique mapping function

$$\phi(z) = \phi(z; \theta_1, \theta_2) := \frac{\sqrt{z-a} + \sqrt{z-b}}{\sqrt{z-a} - \sqrt{z-b}}, \tag{2.15}$$

mapping $\mathbb{C}^* \setminus [a, b]$ onto the exterior of the unit circle. For $(\theta_1, \theta_2) \in \Omega$ and $z \in \mathbb{C}^* \setminus [a, b]$, we now define the function $G(z; \theta_1, \theta_2)$ by

$$G(z) = G(z; \theta_1, \theta_2) := |\varphi(z)| \left| \frac{\varphi(z) - \varphi(1)}{\varphi(1)\varphi(z) - 1} \right|^{\theta_1} \cdot \left| \frac{\varphi(z) - \varphi(-1)}{\varphi(-1)\varphi(z) - 1} \right|^{\theta_2}. \tag{2.16}$$

We extend $G(z; \theta_1, \theta_2)$ continuously to the interval $[a, b]$ by defining $G(z; \theta_1, \theta_2) = 1$ for $z \in [a, b]$. We remark that $G(z; \theta_1, \theta_2)$ is continuous in the variables θ_1 and θ_2 and we extend its definition continuously to the closure $\bar{\Omega}$ of Ω. For example, when $\theta_1 + \theta_2 = 1$, we have

$$G(z) = G(z; \theta_1, \theta_2) = (2\theta_1)^{-\theta_1}(2\theta_2)^{-\theta_2} |1 - z|^{\theta_1} |1 + z|^{\theta_2}. \tag{2.17}$$

In the special case when $\theta_1 = 0$, the G function agrees with the corresponding G function defined by Saff and Varga in [7, 8]. To facilitate the statement of the main result of this section, we first mention some simple properties of the function $G(z; \theta_1, \theta_2)$.

Note that, as the points at infinity correspond to each other under the mapping $\varphi(z)$, for (θ_1, θ_2) fixed in Ω, we have

$$G(z; \theta_1, \theta_2) \to \infty \quad \text{as} \quad |z| \to \infty. \tag{2.18}$$

Moreover, it is evident from (2.16) that, for $(\theta_1, \theta_2) \in \Omega$,

$$G(-1; \theta_1, \theta_2) = 0 = G(1; \theta_1, \theta_2). \tag{2.19}$$

Next, we also have the following result of Lemma 2.4. Its proof, being similar to that of [7, Lemma 4.2], is omitted.

Lemma 2.4. *For (θ_1, θ_2) fixed in Ω, $G(x; \theta_1, \theta_2)$, considered as a function of the real variable x, is strictly decreasing on $(-\infty, -1)$ and on $(b, 1)$ and is strictly increasing on $(-1, a)$ and on $(1, +\infty)$.*

As a consequence of Lemma 2.4 and the properties of (2.18) and (2.19), for each $(\theta_1, \theta_2) \in \Omega$ there exist two unique real points

$$\rho = \rho(\theta_1, \theta_2) < -1 \quad \text{and} \quad \sigma = \sigma(\theta_1, \theta_2) > 1 \tag{2.20}$$

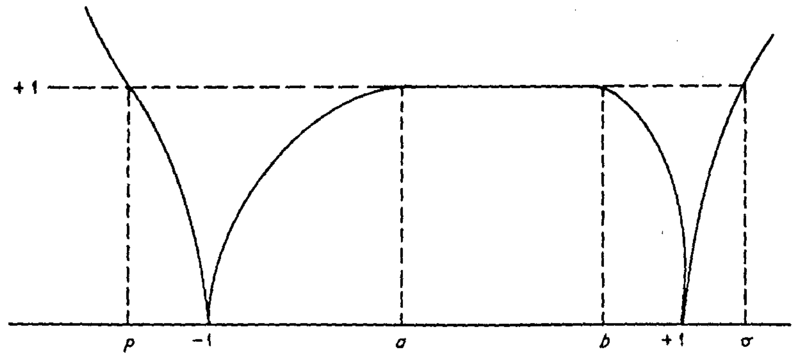

Fig. 2.1. $G(x; 3/9, 5/9)$, x real.

for which we have

$$G(\rho(\theta_1, \theta_2); \theta_1, \theta_2) = 1 = G(\sigma(\theta_1, \theta_2); \theta_1, \theta_2). \tag{2.21}$$

By way of illustration, Fig. 2.1 gives the graph of $G(x; 3/9, 5/9)$ for real values of x.

We comment that, although we do not have an explicit representation for ρ or σ in (2.20), numerical estimates for ρ and σ are easily determined.

Since, for each $(\theta_1, \theta_2) \in \Omega$, the continuous function $G(z; \theta_1, \theta_2)$ vanishes at $z = -1$ and $z = 1$, we have the existence of neighborhoods about $z = -1$ and $z = +1$ for which $G(z; \theta_1, \theta_2) < 1$. Actually, the level curve $G(z; \theta_1, \theta_2) = 1$, enclosing these neighborhoods, traces out a "barbell"-type curve. We illustrate this "barbell" configuration in Fig. 2.2 by graphing the level curve $G(z; 3/9, 5/9) = 1$.

Next, for each pair (θ_1, θ_2) in $\overline{\Omega}$, we define the open set

$$\Lambda(\theta_1, \theta_2) := \{z : G(z; \theta_1, \theta_2) < 1\}. \tag{2.22}$$

We now state the main result of this section.

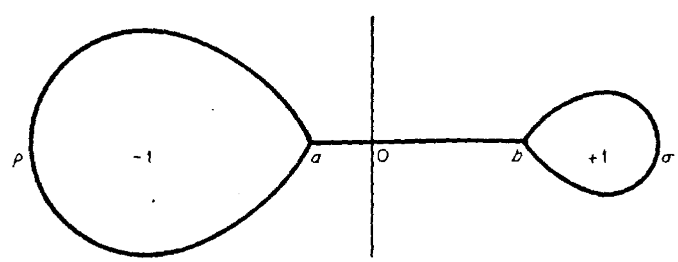

Fig. 2.2. $G(z; 3/9, 5/9) = 1$.

Theorem 2.5. *Let $p \in \pi(s_1, s_2, m)$ with p not identically constant, and set $n := s_1 + s_2 + m$. Then, for all z,*

$$|p(z)| \leqq \|p\|_{[-1,1]}(G(z;\ s_1/n,\ s_2/n))^n. \tag{2.23}$$

Consequently, for $z \in \Lambda(s_1/n, s_2/n)$,

$$|p(z)| \leqq \|p\|_{[-1,1]}(G(z; s_1/n, s_2/n))^n < \|p\|_{[-1,1]}. \tag{2.24}$$

In particular, if $\xi \in [-1, 1]$ is such that $|p(\xi)| = \|p\|_{[-1,1]}$, then

$$a(s_1/n, s_2/n) \leqq \xi \leqq b(s_1/n, s_2/n). \tag{2.25}$$

The statement (2.23) is actually a restatement of Corollary 2.2 in terms of the function $G(z; \theta_1, \theta_2)$. Furthermore, we will show in Section 3 that the inequality (2.23) is *sharp* in a certain limiting sense.

In the next two corollaries, we state convergence and interpolation results for sequences of constrained polynomials.

Corollary 2.6. *Let (θ_1, θ_2) be fixed in $\bar{\Omega}$ and let*

$$\left\{p_i(z) := (z-1)^{s_{1,i}}(z+1)^{s_{2,i}} \sum_{k=0}^{m_i} a_{k,i} z^k\right\}_{i=1}^{\infty}$$

by any infinite sequence of complex polynomials satisfying

$$\lim_{i \to \infty} n_i = \infty \qquad (n_i := s_{1,i} + s_{2,i} + m_i,\ i \geqq 1) \tag{2.26}$$

and

$$\lim_{i \to \infty} s_{1,i}/n_i = \theta_1 \qquad \textit{and} \qquad \lim_{i \to \infty} s_{2,i}/n_i = \theta_2. \tag{2.27}$$

If

$$\lim_{i \to \infty} \sup(\|p_i\|_{[-1,1]})^{1/n_i} \leqq 1, \tag{2.28}$$

then

$$\lim_{i \to \infty} p_i(z) = 0 \qquad \textit{for all} \quad z \in \Lambda(\theta_1, \theta_2). \tag{2.29}$$

Moreover, for any closed subset B of $\Lambda(\theta_1, \theta_2)$,

$$\lim_{i \to \infty} \sup(\|p_i\|_B)^{1/n_i} \leqq \|G(\cdot\,; \theta_1, \theta_2)\|_B < 1. \tag{2.30}$$

The proof of Corollary 2.6 follows from applying inequality (2.23) to each polynomial $p_i(z)$ and from the continuity of $G(z; \theta_1, \theta_2)$ in the variables θ_1 and θ_2. We will show in Section 3 that Corollary 2.6 gives the *largest possible* open set of convergence to zero for the class of such polynomials $p_i(z)$.

Our final result of this section concerns polynomials interpolating a function $f(z)$ at the points $z = -1$ and $z = 1$.

Corollary 2.7. *Let $f(z)$ be analytic at $z = -1$ and at $z = 1$. Suppose there exists a sequence of polynomials $\{p_i(z)\}_{i=1}^{\infty}$ with $p_i \in \pi_{n_i}$ $\forall i$, $n_1 < n_2 < \cdots$, where $n_{i+1}/n_i \to 1$ as $i \to \infty$, and such that*

$$\begin{aligned} p_i^{(k)}(1) &= f^{(k)}(1), \qquad k = 0, 1, \ldots, s_{1,n_i}, \\ p_i^{(k)}(-1) &= f^{(k)}(-1), \qquad k = 0, 1, \ldots, s_{2,n_i}. \end{aligned} \tag{2.31}$$

Further, suppose that

$$\lim_{i\to\infty} \frac{s_{1,n_i}}{n_i} = \theta_1 \qquad \text{and} \qquad \lim_{i\to\infty} \frac{s_{2,n_i}}{n_i} = \theta_2 \tag{2.32}$$

and that

$$\limsup_{i\to\infty} (\|p_i\|_{[-1,1]})^{1/n_i} \leqq 1. \tag{2.33}$$

Then, $f(z)$ is analytic at each point of $\Lambda(\theta_1, \theta_2)$ and $p_i(z) \to f(z)$ for $z \in \Lambda(\theta_1, \theta_2)$, the convergence being uniform on any closed subset of $\Lambda(\theta_1, \theta_2)$. On a closed subset B of $\Lambda(\theta_1, \theta_2)$ we have the following convergence rate:

$$\limsup_{i\to\infty} \|f - p_i\|_B^{1/n_i} \leqq \|G(\cdot;\ \theta_1,\ \theta_2)\|_B < 1. \tag{2.34}$$

To prove this last result, we consider consecutive differences from the sequence $\{f(z) - p_i(z)\}_{i=1}^{\infty}$ which in turn form a sequence of constrained polynomials $\{p_{i+1}(z) - p_i(z)\}_{i=1}^{\infty}$. This sequence satisfies the hypotheses of Corollary 2.6 and hence tends to zero, geometrically in $\Lambda(\theta_1, \theta_2)$.

As an application of Corollary 2.7, we mention an interpolation problem suggested by Meinardus [5], related to the design of filters. Let g be the real step function defined for all real t by

$$g(t) = \begin{cases} 1 & \text{for} \quad t < \frac{1}{2} \\ \frac{1}{2} & \text{for} \quad t = \frac{1}{2} \\ 0 & \text{for} \quad t > \frac{1}{2} \end{cases} \tag{2.35}$$

and consider the sequence of polynomials $\{q_n(x)\}_{n=0}^{\infty}$ with $q_n \in \pi_{2n+1}$ for all $n \geqq 0$, satisfying the conditions

$$\begin{aligned} q_n^{(k)}(0) &= g^{(k)}(0), \qquad k = 0, 1, \ldots, n, \\ q_n^{(k)}(1) &= g^{(k)}(1), \qquad k = 0, 1, \ldots, n. \end{aligned} \tag{2.36}$$

The polynomials q_n are uniquely determined, and, in fact, are explicitly given by

$$q_n(x) = B_{2n+1}(x; g), \qquad n \geqq 0,$$

where

$$B_n(x; f) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

is the *Bernstein polynomial* of degree n for a given real-valued function on $[0, 1]$ (cf. Lorentz [2, p. 4]). It is easily seen that $\{q_n(x)\}_{n=0}^{\infty}$ so defined are uniformly bounded by unity on $[0, 1]$. Setting $p_n(x) := q_n((x+1)/2)$, we find that the sequence $\{p_n(x)\}_{n=1}^{\infty}$ satisfies the hypotheses of Corollary 2.7 with $\theta_1 = \theta_2 = 1/2$. In this case, one readily verifies [cf. (2.20)] that $\rho(1/2, 1/2) = -\sqrt{2}$ and $\sigma(1/2, 1/2) = \sqrt{2}$, so that, by Corollary 2.7, we have for real x that the sequence $\{p_n(x)\}_{n=1}^{\infty}$ converges to 1 for $x \in (-\sqrt{2}, 0)$ and converges to 0 for $x \in (0, \sqrt{2})$, the convergence being *geometric* on any closed subinterval of $(-\sqrt{2}, 0) \cup (0, \sqrt{2})$. Consequently, as $n \to \infty$, $q_n(t) \to g(t)$, geometrically on any closed subinterval of

$$\left(\frac{1-\sqrt{2}}{2}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{1+\sqrt{2}}{2}\right).$$

More generally, for complex t, Corollary 2.7 implies that the sequence $q_n(t)$ converges geometrically on each closed subset of

$$H = \{t \in \mathbb{C} : (2t-1) \in \Lambda(1/2, 1/2)\}. \tag{2.37}$$

Meinardus [5] has been able to show moreover that the sequence $q_n(t)$ *diverges* for $t < (1-\sqrt{2})/2$, and for $t > (1+\sqrt{2})/2$. We remark that the "overconvergence" properties of Bernstein polynomials have been extensively studied (see Lorentz [2]). In the cases when $\theta_1 + \theta_2 = 1$, the convergence region given, via Corollary 2.7, by

$$\{t \in \mathbb{C} : (2t-1) \in \Lambda(\theta_1, \theta_2)\}, \tag{2.38}$$

and by that of Bernstein polynomials, *coincide*.

3. CONSTRAINED CHEBYSHEV POLYNOMIALS

In [8], the analog of the classical Chebyshev polynomials for polynomials constrained at one endpoint of an interval by a certain order zero were studied. In this section, we extend the definition of these constrained polynomials to the two endpoint case. As a consequence, we shall prove that Theorem 2.5 is best possible in a certain limiting sense.

Proposition 3.1. *For each ordered triple of nonnegative integers* (s_1, s_2, m), *there exists a unique monic polynomial* $Q_{s_1,s_2,m}(x)$ *in* $\pi(s_1, s_2, m)$ [cf. (2.3)] *of precise degree* $n := s_1 + s_2 + m$ *satisfying*

$$\begin{aligned}\|Q_{s_1,s_2,m}\|_{[-1,1]} &= \inf\{\|(x-1)^{s_1}(x+1)^{s_2}x^m \\ &\qquad - g(x)\|_{[-1,1]} : g \in \pi(s_1, s_2, m-1)\} \\ &=: E_{s_1,s_2,m}\end{aligned} \tag{3.1}$$

(where $\pi(s_1, s_2, m-1) := \{0\}$ if $m = 0$). *Furthermore, for* $n > 0$, $Q_{s_1,s_2,m}(x)$ *has an alternation set of precisely* $m+1$ *distinct points* $\xi_j^{(s_1,s_2,m)}$, $j = 0, 1, \ldots, m$, *with* [cf. (2.12)]

$$a(s_1/n, s_2/n) \leqq \xi_0^{(s_1,s_2,m)} < \xi_1^{(s_1,s_2,m)} < \cdots < \xi_m^{(s_1,s_2,m)} \leqq b(s_1/n, s_2/n), \quad (3.2)$$

for which

$$Q_{s_1,s_2,m}(\xi_j^{(s_1,s_2,m)}) = (-1)^{s_1+m-j} E_{s_1,s_2,m}, \qquad j = 0, 1, \ldots, m. \quad (3.3)$$

Proof. As the special cases $s_1 + s_2 = 0$ and $m = 0$ of Proposition 3.1 are clearly true and will in fact be explicitly covered in Proposition 3.2, assume that $s_1 + s_2 > 0$ and that $m > 0$. From general linear approximation theory, it follows that there exists a monic polynomial, say p, in $\pi(s_1, s_2, m)$ with

$$\|p\|_{[-1,1]} = E_{s_1,s_2,m}.$$

As a consequence of Theorem 2.5, the polynomial p also satisfies the extremal problem

$$\|p\|_{[a,b]} = \inf\{\|(x-1)^{s_1}(x+1)^{s_2}x^m - g(x)\|_{[a,b]} : g \in \pi(s_1, s_2, m-1)\},$$

where $a = a(s_1/n, s_2/n)$ and $b = b(s_1/n, s_2/n)$ are defined in (2.12). On this subinterval, the linear space $\pi(s_1, s_2, m-1)$, which has dimension m, satisfies the Haar condition, which guarantees (cf. Meinardus [4, p. 20]) the uniqueness of p. From the same result, we have the existence of an alternation set of at least $m+1$ distinct points in $[a, b]$. If there were $m+k$ alternation points with $k > 1$, the derivative of p, a polynomial of degree $n-1$, would have at least n zeros on $[-1, +1]$, and would, consequently, be identically zero, contradicting the fact that p is monic. Thus, there are precisely $m+1$ distinct alternation points satisfying (3.3). ∎

With the existence of $Q_{s_1,s_2,m}(x)$ for each triple of nonnegative integers, we define

$$T_{s_1,s_2,m}(x) := \frac{Q_{s_1,s_2,m}(x)}{\|Q_{s_1,s_2,m}\|_{[-1,1]}} = \frac{Q_{s_1,s_2,m}(x)}{E_{s_1,s_2,m}} \quad (3.4)$$

to be the (normalized) *constrained Chebyshev polynomial of degree* $s_1 + s_2 + m$ *associated with the set* $\pi(s_1, s_2, m)$ *on the interval* $[-1, 1]$. Let $T_n(x)$ denote as usual the classical Chebyshev polynomial (of the first kind) of degree n, given by

$$T_n(x) = \cos n\theta, \qquad x = \cos\theta.$$

We now summarize some special cases and properties for certain triples (s_1, s_2, m) in

Proposition 3.2. *Set* $x_m := \cos(\pi/2m)$ *for* $m > 0$. *Then,*

(i) $Q_{s_2, s_1, m}(x) = (-1)^{s_1+s_2+m} Q_{s_1, s_2, m}(-x)$;

(ii) $$Q_{0,0,m}(x) = 2^{-m+1} T_m(x),$$
$$E_{0,0,m} = 2^{-m+1},$$
$$\xi_j^{(0,0,m)} = \cos\left(\frac{(m-j)\pi}{m}\right), \qquad j = 0, 1, \ldots, m;$$

(iii) $$Q_{0,1,m}(x) = \frac{2}{(1 + x_{m+1})^{m+1}} T_{m+1}\left[\left(\frac{1 - x_{m+1}}{2}\right) + \left(\frac{1 + x_{m+1}}{2}\right)x\right],$$
$$E_{0,1,m} = \frac{2}{(1 + x_{m+1})^{m+1}},$$
$$\xi_j^{(0,1,m)} = \frac{1}{(1 + x_{m+1})}\left\{2\cos\left[\frac{(m-j)\pi}{m+1}\right] + x_{m+1} - 1\right\}, \qquad j = 0, 1, \ldots, m;$$

(iv) $$Q_{1,1,m}(x) = \frac{2}{(2x_{m+2})^{m+2}} T_{m+2}(x_{m+2} \cdot x),$$
$$E_{1,1,m} = \frac{2}{(2x_{m+2})^{m+2}},$$
$$\xi_j^{(1,1,m)} = \frac{1}{x_{m+2}} \cos\left[\frac{(m+1-j)\pi}{m+2}\right], \qquad j = 0, 1, \ldots, m;$$

(v) $$Q_{s_1, s_2, 0}(x) = (x-1)^{s_1}(x+1)^{s_2}, \quad s_1 + s_2 > 0,$$
$$E_{s_1, s_2, 0} = \left(\frac{2s_1}{s_1 + s_2}\right)^{s_1}\left(\frac{2s_2}{s_1 + s_2}\right)^{s_2},$$
$$\xi_0^{(s_1, s_2, 0)} = \frac{s_2 - s_1}{s_2 + s_1};$$

(vi) $$Q_{s,s,1}(x) = x(x^2 - 1)^s,$$
$$E_{s,s,1} = \left(\frac{2s}{2s+1}\right)^s \left(\frac{1}{2s+1}\right)^{1/2},$$
$$\xi_j^{(s,s,1)} = (-1)^{j+1}\left(\frac{1}{2s+1}\right)^{1/2}, \qquad j = 0, 1;$$

(vii) $Q_{s,s,2m}(x) = (-2)^{-s-m} Q_{0,s,m}(1 - 2x^2)$.

The proofs of the above statements follow easily from the fact (Proposition 3.1) that $Q_{s_1,s_2,m}(x)$ is a monic polynomial and from the existence of the required alternation set.

We now deduce a domination property for $T_{s_1,s_2,m}(x)$ (cf. [8, Proposition 4]).

Theorem 3.3. *Let* $p \in \pi(s_1, s_2, m)$, *and let*

$$M \geqq \max\{|p(\xi_j^{(s_1,s_2,m)})| : j = 0, 1, \ldots, m\}.$$

If $x \leqq \xi_0^{(s_1,s_2,m)}$ *or if* $x \geqq \xi_m^{(s_1,s_2,m)}$, *then*

$$|p(x)| \leqq M\,|T_{s_1,s_2,m}(x)|. \tag{3.5}$$

Moreover, for each positive integer k and x real,

$$|p^{(k)}(x)| \leqq M\,|T^{(k)}_{s_1,s_2,m}(x)|, \qquad \text{for} \quad |x| \geqq 1. \tag{3.6}$$

Proof. First, define the polynomial $h(x)$ by means of

$$h(x) := (x-1)^{s_1}(x+1)^{s_2}\prod_{j=0}^{m}(x - \xi_j^{(s_1,s_2,m)}).$$

Using the Lagrange interpolation formula, it follows from the definition of $h(x)$ that

$$p(x) = \sum_{j=0}^{m} \frac{h(x)}{(x-\xi_j)}\frac{p(\xi_j)}{h'(\xi_j)}, \tag{3.7}$$

and

$$T_{s_1,s_2,m}(x) = \sum_{j=0}^{m} \frac{h(x)}{(x-\xi_j)}\frac{(-1)^{s_1+m-j}}{h'(\xi_j)}, \tag{3.8}$$

where $\xi_j = \xi_j^{(s_1,s_2,m)}$, $j = 0, 1, \ldots, m$. Thus, with the hypothesis for M, (3.7) implies that

$$|p(x)| \leqq M \sum_{j=0}^{m} \frac{|h(x)|}{|x-\xi_j|\cdot|h'(\xi_j)|}. \tag{3.9}$$

Next, noting that [8, Proposition 4] establishes the case $s_1 = 0$ of this result, we may assume $s_1 > 0$, which implies $\xi_m < 1$. For $x \in (\xi_m, 1)$, we have, by definition, that $|h(x)| = (-1)^{s_1}h(x)$ and that

$$|h'(\xi_j)| = (-1)^{s_1+m-j}h'(\xi_j).$$

Consequently, the right-hand side of (3.9) is equal to $(-1)^{s_1}\cdot M\cdot T_{s_1,s_2,m}(x)$. Hence, from (3.9) we have

$$|p(x)| \leqq M\,|T_{s_1,s_2,m}(x)| \qquad \text{for} \quad x \in (\xi_m, 1).$$

Arguing similarly for $x \in (-\infty, \xi_0)$ and $x \in (1, \infty)$, the first part of this theorem is proved. The second portion can be obtained for real x, $|x| \geqq 1$, by differentiating the formulas (3.7) and (3.8) k times. ∎

We may now improve the inequality given in (2.25). Let $p \in \pi(s_1, s_2, m)$ with $m > 0$ and with $p \not\equiv 0$, and suppose that $|p(\xi)| = \|p\|_{[-1,1]}$ where $\xi \in [-1, 1]$. Then, as a consequence of Theorem 3.3,

$$\xi_0^{(s_1, s_2, m)} \leqq \xi \leqq \xi_m^{(s_1, s_2, m)}. \tag{3.10}$$

As an application, consider any $p \in \pi_n$ with $p \not\equiv 0$ and $p(-1) = p(1) = 0$, so that $p \in \pi(1, 1, n-2)$. Then, from (3.10) and Proposition 3.2 (iv), the points ξ in $[-1, +1]$, with $|p(\xi)| = \|p\|_{[-1,+1]}$ satisfy

$$|\xi| \leqq \frac{\cos(\pi/n)}{\cos(\pi/2n)}$$

(cf. Schur [10, Section 5]).

Concerning the behavior of $E_{s_1, s_2, m}$, we have the following generalization of [8, Proposition 8].

Theorem 3.4. *Let (θ_1, θ_2) be fixed in $\bar{\Omega}$ [cf. (2.14)], and let*

$$\{(s_{1,i}, s_{2,i}, m_i)\}_{i=1}^{\infty}$$

be any infinite sequence of ordered triples of nonnegative integers for which

$$\lim_{i \to \infty} n_i = \infty \qquad (n_i := s_{1,i} + s_{2,i} + m_i,\ i \geqq 1), \tag{3.11}$$

and for which

$$\lim_{i \to \infty} \frac{s_{1,i}}{n_i} = \theta_1 \qquad \textit{and} \qquad \lim_{i \to \infty} \frac{s_{2,i}}{n_i} = \theta_2. \tag{3.12}$$

Then,

$$\Delta = \Delta(\theta_1, \theta_2) := \lim_{i \to \infty} (E_{s_{1,i}, s_{2,i}, m_i})^{1/n_i}$$

$$= \tfrac{1}{2}\sqrt{(1+\mu)^{1+\mu}(1-\mu)^{1-\mu}(1+\nu)^{1+\nu}(1-\nu)^{1-\nu}}, \tag{3.13}$$

where $\mu := \theta_2 + \theta_1$ and $\nu := \theta_2 - \theta_1$.

Proof. We first introduce the sequence of modified Jacobi polynomials $\{J_i(x)\}_{i=1}^{\infty}$ defined by

$$J_i(x) := (x-1)^{s_{1,i}}(x+1)^{s_{2,i}} P_{m_i}^{(2s_{1,i}, 2s_{2,i})}(x) \bigg/ \left[2^{-m_i}\binom{2n_i}{m_i}\right], \qquad i \geqq 1. \tag{3.14}$$

We note that these polynomials are, from (2.10), monic for all $i \geqq 1$. Moreover, from the discussion in Section 2 of their properties the following inequalities are valid for each $i \geqq 1$:

$$\int_{-1}^{1} (J_i(x))^2\, dx \leqq \int_{-1}^{1} (Q_{s_{1,i}, s_{2,i}, m_i}(x))^2\, dx \leqq 2(E_{s_{1,i}, s_{2,i}, m_i})^2. \quad (3.15)$$

Next, on expanding the polynomial $J_i(x)$ in its Legendre polynomial expansion, it follows that (cf. Szegö [11, p. 182])

$$\|J_i\|^2_{[-1,+1]} \leqq \frac{(n_i+1)^2}{2} \int_{-1}^{+1} (J_i(x))^2\, dx, \qquad i \geqq 1,$$

whence, from (3.1),

$$(E_{s_{1,i}, s_{2,i}, m_i})^2 \leqq \|J_i\|^2_{[-1,+1]} \leqq \frac{(n_i+1)^2}{2} \int_{-1}^{+1} (J_i(x))^2\, dx, \qquad i \geqq 1. \quad (3.16)$$

Thus, from (3.15) and (3.16),

$$\Delta = \lim_{i \to \infty} (E_{s_{1,i}, s_{2,i}, m_i})^{1/n_i} = \lim_{i \to \infty} \left(\int_{-1}^{+1} (J_i(x))^2\, dx \right)^{1/2n_i}.$$

From Eq. (2.8) and the definition of $J_i(x)$ in (3.14), we therefore have

$$\Delta = \lim_{i \to \infty} \left[2^{2m_i} \binom{2n_i}{m_i}^{-2} h_{m_i}^{(2s_{1,i}, 2s_{2,i})} \right]^{1/2n_i}.$$

With the definition of $h_m^{(\alpha,\beta)}$ in (2.9) and an application of Stirling's formula, we obtain the value of Δ stated in (3.13). ∎

As a consequence of the inequalities (3.15) and (3.16), upper and lower bounds for $E_{s_1, s_2, m}$ are

$$2^{-1/2} L \leqq E_{s_1, s_2, m} \leqq (n+1) L/\sqrt{2}, \quad (3.17)$$

where

$$L = L(s_1, s_2, m) := 2^m \binom{2n}{m}^{-1} (h_m^{(2s_1, 2s_2)})^{1/2},$$

$n := s_1 + s_2 + m$, and where $h_m^{(2s_1, 2s_2)}$ is given in (2.9).

Theorem 3.5. *Let* (θ_1, θ_2) *and* $\{(s_{1,i}, s_{2,i}, m_i)\}_{i=1}^{\infty}$ *be as in Theorem* 3.4. *Then, for* $z \in \mathbb{C}^* \backslash [a, b]$,

$$\lim_{i \to \infty} |Q_{s_{1,i}, s_{2,i}, m_i}(z)|^{1/n_i} = \Delta(\theta_1, \theta_2) \cdot G(z; \theta_1, \theta_2), \quad (3.18)$$

where $\Delta(\theta_1, \theta_2)$ *is defined in* (3.13) *and where* $G(z; \theta_1, \theta_2)$ *is defined in* (2.16) *and* (2.17). *Furthermore, this limit holds uniformly for any compact subset not containing the interval* $[a, b]$.

Proof. As in Saff and Varga [8], we use a normal families argument. First, for each $i \geqq 1$, set

$$n_i := s_{1,i} + s_{2,i} + m_i, \qquad \theta_{1,i} := s_{1,i}/n_i, \qquad \theta_{2,i} := s_{2,i}/n_i.$$

Setting $\mu_i := \theta_{2,i} + \theta_{1,i}$ and $\nu_i := \theta_{2,i} - \theta_{1,i}$, define a_i and b_i from (2.12), and let $\varphi_i(z)$ be defined from (2.15) for each $i \geqq 1$. Furthermore, set

$$\begin{aligned}\Delta_i &:= \Delta(\theta_{1,i}, \theta_{2,i})\\ &= \tfrac{1}{2}[(1+\mu_i)^{1+\mu_i}(1-\mu_i)^{1-\mu_i}(1+\nu_i)^{1+\nu_i}(1-\nu_i)^{1-\nu_i}]^{1/2}, \qquad i \geqq 1,\end{aligned}$$

and

$$K := \mathbb{C}^*\backslash[a, b] \qquad \text{and} \qquad K_i := \mathbb{C}^*\backslash[a_i, b_i] \qquad \text{for} \quad i \geqq 1,$$

and, for each $i \geqq 1$, set

$$u_i(z) := \frac{1}{n_i} \ln|Q_{s_{1,i}, s_{2,i}, m_i}(z)|,$$

$$\begin{aligned}v_i(z) &:= \ln \Delta_i + \ln|\varphi_i(z)| + \theta_{1,i} \ln\left|\frac{\varphi_i(z) - \varphi_i(1)}{\varphi_i(1)\varphi_i(z) - 1}\right|\\ &\quad + \theta_{2,i} \ln\left|\frac{\varphi_i(z) - \varphi_i(-1)}{\varphi_i(-1)\varphi_i(z) - 1}\right|.\end{aligned}$$

We remark that both the functions $u_i(z)$ and $v_i(z)$ for i fixed are harmonic in K_i with the exception of the points $z = -1$, $z = +1$, and $z = \infty$. In a neighborhood of $z = 1$, we can write

$$u_i(z) = \theta_{1,i} \ln|z - 1| + h_{1,i}(z)$$

and

$$v_i(z) = \theta_{1,i} \ln|z - 1| + \hat{h}_{1,i}(z),$$

where it can be verified that the functions $h_{1,i}(z)$ and $\hat{h}_{1,i}(z)$ are both harmonic at $z = 1$ for all $i \geqq 1$. Furthermore, an analogous representation is true in a neighborhood of $z = -1$. Near, $z = \infty$, we have

$$u_i(z) = \ln|z| + g_i(z), \tag{3.19}$$

and

$$v_i(z) = \ln|z| + \hat{g}_i(z). \tag{3.20}$$

In (3.19) and (3.20), both $g_i(z)$ and $\hat{g}_i(z)$ are harmonic at $z = \infty$ and

$$g_i(\infty) = 0 = \hat{g}_i(\infty).$$

Next, for each $i \geqq 1$, set

$$d_i(z) := u_i(z) - v_i(z).$$

Note that $d_i(z)$ is harmonic in K_i, even for $z = -1$, $z = 1$, and $z = \infty$. As z tends to $[a_i, b_i]$ in K_i we note that $v_i(z)$ tends to $\ln \Delta_i$, whence

$$\limsup_{\substack{z \to [a_i, b_i] \\ z \in K_i}} d_i(z) \leqq \frac{1}{n_i} \ln E_{s_{1,i}, s_{2,i}, m_i} - \ln \Delta_i. \tag{3.21}$$

From (3.13) and (3.21) it follows that on any closed subset of K, the harmonic functions $d_i(z)$ are, for i sufficiently large, uniformly bounded from above. Hence, the $d_i(z)$ form a normal family on K. If $d(z)$ denotes a limit function of this family, then from (3.12) and (3.21) we have $d(z) \leqq 0$ for all z in K. However, since $d_i(\infty) = g_i(\infty) - \hat{g}_i(\infty) = 0$, $i \geqq 1$, we conclude that $d(z) \equiv 0$ in K. Since $\lim_{i\to\infty} v_i(z) := v(z)$ uniformly on any compact subset of $K \backslash \{-1, 1, \infty\}$, we have $\lim_{i\to\infty} u_i(z) = v(z)$ uniformly on a compact set of $K \backslash \{-1, 1, \infty\}$. Hence,

$$|Q_{s_{1,i}, s_{2,i}, m_i}(z)|^{1/n_i} = e^{u_i(z)} \to e^{v(z)} = \Delta(\theta_1, \theta_2) \cdot G(z; \theta_1, \theta_2),$$

as i tends to infinity, uniformly on any compact set omitting the interval $[a, b]$. ◻

In closing this section, we establish the sharpness of Theorem 2.5 in a certain limiting sense. Let $(\theta_1, \theta_2) \in \bar{\Omega}$ and let $\{(s_{1,i}, s_{2,i}, m_i)\}_{i=1}^{\infty}$ be any infinite sequence of ordered triples of nonnegative integers satisfying

$$\lim_{i\to\infty} n_i = \infty \qquad (n_i = s_{1,i} + s_{2,i} + m_i, \quad i \geqq 1),$$

$$\lim_{i\to\infty} \frac{s_{1,i}}{n_i} = \theta_1 \qquad \text{and} \qquad \lim_{i\to\infty} \frac{s_{2,i}}{n_i} = \theta_2.$$

For the normalized Chebyshev polynomial

$$\mathcal{T}_i(z) := T_{s_{1,i}, s_{2,i}, m_i}(z) \tag{3.22}$$

associated with the set $\pi(s_{1,i}, s_{2,i}, m_i)$ defined in (3.4), we apply inequality (2.23) of Theorem 2.5 to obtain

$$|\mathcal{T}_i(z)|^{1/n_i} \leqq \|\mathcal{T}_i\|_{[-1,1]}^{1/n_i} G(z; s_{1,i}/n_i, s_{2,i}/n_i) = G(z; s_{1,i}/n_i, s_{2,i}/n_i).$$

Letting i tend to infinity yields

$$\limsup_{i\to\infty} |\mathcal{T}_i(z)|^{1/n_i} \leqq G(z; \theta_1, \theta_2).$$

But, recalling the definition of Δ in Theorem 3.4, the result of Theorem 3.5 applied to the *normalized* Chebyshev polynomials gives

$$\lim_{i\to\infty} |\mathcal{T}_i(z)|^{1/m_i} = G(z; \theta_1, \theta_2), \tag{3.23}$$

for all $z \notin [a, b]$, so that the inequality (2.23) of Theorem 2.5 is sharp in this limiting sense. We also remark that the sequence $\mathcal{T}_i(z)$ satisfies the hypothesis of Corollary 2.6, and, by (3.23), $\mathcal{T}_i(z)$ diverges for z exterior to the level curve $G(z; \theta_1, \theta_2) = 1$. Thus, Corollary 2.6 gives the largest possible open set of convergence to zero.

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