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THE SHARPNESS OF LORENTZ'S THEOREM ON INCOMPLETE POLYNOMIALS

BY

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ABSTRACT. For any fixed θ with $0 < \theta < 1$, G. G. Lorentz recently showed that bounded sequences $\{\sum_{\theta n_i \leq k \leq n_i} a_k(i)(1+t)^k\}_{i=1}^{\infty}$ of incomplete polynomials on $[-1, +1]$ tend uniformly to zero on closed intervals of $[-1, \Delta(\theta))$, where $2\theta^2 - 1 \leq \Delta(\theta) < 2\theta - 1$. In this paper, we show that $\Delta(\theta) = 2\theta^2 - 1$ is best possible, and that the geometric convergence to zero of such sequences on closed intervals $[t_0, t_1]$ can be precisely bounded above as a function of t_j and θ. Extensions of these results to the complex plane are also included.

1. Introduction. At the *Conference on Rational Appxoximation with Emphasis on Applications of Padé Approximants*, which was held December 15–17, 1976 in Tampa, Florida, Professor G. G. Lorentz introduced new results and open questions for sequences of incomplete polynomials. In the proceedings of this conference, Lorentz proved the following result, which, for convenience, we state for the interval $[-1, +1]$.

THEOREM 1.1 (LORENTZ [3]). *For any fixed θ with $0 < \theta < 1$, there exists a $\delta = \delta(\theta)$ with $-1 < \delta(\theta) < 1$, with the following property. For any infinite sequence of complex incomplete polynomials*

$$\left\{ w_i(t) = \sum_{\theta n_i \leq k \leq n_i} a_k(i)(1+t)^k \right\}_{i=1}^{\infty}, \qquad \lim_{i\to\infty} n_i = +\infty, \tag{1.1}$$

which satisfy $|w_i(t)| \leq M$ for all $t \in [-1, +1]$ and all $i \geq 1$, then

$$w_i(t) \to 0 \text{ uniformly in } [-1, \delta]. \tag{1.2}$$

Lorentz's construction [3] of his δ even shows that any sequence $\{w_i(t)\}_{i=1}^{\infty}$ satisfying the hypotheses of Theorem 1.1 converges geometrically to zero on

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$[-1, \delta]$, i.e.,

$$\limsup_{i\to\infty} |w_i(t)|^{1/n_i} < 1 \quad \text{for each } t \text{ with } -1 \leqslant t \leqslant \delta, \tag{1.3}$$

where w_i is a polynomial of degree at most n_i (cf. (1.1)).

One of the questions which arises naturally in the discussion of such incomplete polynomials is the relationship between θ and δ. In this regard, if

$$\Delta(\theta) := \sup\{\delta\colon \text{Theorem 1.1 is valid for } \delta\}, \tag{1.4}$$

then Lorentz [3] also proved

THEOREM 1.2. *For any θ with $0 < \theta < 1$, then*

$$2\theta^2 - 1 \leqslant \Delta(\theta) < 2\theta - 1. \tag{1.5}$$

Concerning the second inequality of (1.5), we remark that C. FitzGerald and D. Wulbert were the first to show that $\Delta(\frac{1}{2}) < 0$.

The main purpose of this paper is to sharpen (1.3) and (1.5) for sequences of incomplete polynomials. In particular, in our first main result, Theorem 2.1, we show that

$$\Delta(\theta) = 2\theta^2 - 1, \tag{1.6}$$

so that Lorentz's lower bound in (1.5) is, in fact, sharp. In our second main result, Theorem 2.2, we obtain a sharp improvement of (1.3) which gives the geometric convergence to zero in (1.3) as a function of t and θ. This theorem also provides an independent proof of the first inequality of (1.5) of Theorem 1.2.

One interesting consequence of (1.6) is the following. In Lorentz [3], the set of all incomplete polynomials for a fixed θ $(0 < \theta < 1)$ is said to have the *Weierstrass property* on the interval $[a_\theta, 1]$ if, for every continuous function f on $[a_\theta, 1]$, there exists a sequence of incomplete polynomials of the form (1.1) which converges uniformly to f on $[a_\theta, 1]$. Evidently, from (1.6) and Theorem 1.1, a *necessary* condition for the set of all incomplete polynomials for a fixed θ to have the Weierstrass property on $[a_\theta, 1]$ is that $2\theta^2 - 1 \leqslant a_\theta \leqslant 1$.

The outline of this paper is as follows. In §2, we state our new results, and in §3, we give the proof of Theorem 2.1. In §4, we develop some needed estimates for Jacobi polynomials, using the method of steepest descents. In §5, we give the proof of Theorem 2.2, its corollaries, and Theorem 2.5. In §6, we study the behavior of incomplete polynomials at the endpoints of the maximal interval of convergence to zero, and in §7, we give an extension of Theorem 2.2 to the complex plane. For the remainder of this section, we give necessary background and notation.

For real numbers α and β with $\alpha > -1$ and $\beta > -1$, $P_n^{(\alpha,\beta)}(t)$ denotes the classical Jacobi polynomial of degree n. As is well known, if

$$h_n^{(\alpha,\beta)} := \int_{-1}^{+1} (1-t)^{\alpha}(1+t)^{\beta}\left(P_n^{(\alpha,\beta)}(t)\right)^2 dt, \tag{1.7}$$

then the sequence of polynomials $\{P_n^{(\alpha,\beta)}(t)/\sqrt{h_n^{(\alpha,\beta)}}\}_{n=0}^{\infty}$ is orthonormal with respect to the weight function $(1-t)^{\alpha}(1+t)^{\beta}$ on $[-1,+1]$:

$$\int_{-1}^{+1} (1-t)^{\alpha}(1+t)^{\beta}\left(P_n^{(\alpha,\beta)}(t)/\sqrt{h_n^{(\alpha,\beta)}}\right)\left(P_m^{(\alpha,\beta)}(t)/\sqrt{h_m^{(\alpha,\beta)}}\right) dt = \delta_{n,m}, \tag{1.8}$$

for all $n, m = 0, 1, \ldots$. It is also known (cf. Szegö [**6**, p. 68]) that

$$h_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}. \tag{1.9}$$

Next, for the statement of Theorem 2.2, we need the following. For $0 < \theta < 1$, the quadratic equation

$$(1+\theta)\zeta^2 - 2(t+\theta)\zeta + (1-\theta+2\theta t) = 0 \tag{1.10}$$

has roots $\zeta^{\pm}(t,\theta)$, given by

$$\zeta^{\pm}(t,\theta) = \left\{t + \theta \pm \sqrt{(1-t)(2\theta^2-1-t)}\right\}/(1+\theta). \tag{1.11}$$

For any $t < 2\theta^2 - 1$, we note that these zeros are real and distinct. With (1.11), we then define

$$G(t,\theta) := \begin{cases} \dfrac{1}{2}\left|\dfrac{(\zeta^-(t,\theta))^2-1}{\zeta^-(t,\theta)-t}\right| \cdot \left|\dfrac{(1+\zeta^-(t,\theta))(\zeta^-(t,\theta)-t)}{(1+t)(\zeta^-(t,\theta)-1)}\right|^{\theta}, & \text{for } t \leqslant 2\theta^2-1 \text{ and } t \neq -1; \\ 0, \quad \text{for } t=-1. \end{cases} \tag{1.12}$$

Next, it can be verified from (1.12) that for any t with $t < 1$, $\lim_{\theta\uparrow 1} G(t,\theta) = |1+t|/2$. Thus, we extend the definition of (1.12) by setting

$$G(t,1) := |1+t|/2, \quad \text{for } t \leqslant 1. \tag{1.12$'$}$$

With the definitions of (1.12) and (1.12′), it can be verified that $G(t,\theta)$, as a function of real t, is continuous on $(-\infty, 2\theta^2-1]$ for any θ with $0 < \theta \leqslant 1$, and that

$$G(2\theta^2-1,\theta) = 1 \quad \text{for all } \theta \text{ with } 0 < \theta \leqslant 1. \tag{1.13}$$

Similarly, it will be shown as a consequence of Lemma 4.2 that there is a unique $r(\theta) > 1$ for which

$$G(-r(\theta),\theta) = 1, \quad \text{for all } \theta \text{ with } 0 < \theta \leqslant 1. \tag{1.14}$$

Finally, as usual, π_n denotes the set of complex polynomials of degree at most n, and, for any continuous function f defined on a compact set K of the complex plane,

$$\|f\|_K := \max\{|f(t)| : t \in K\}.$$

2. Statement of new results. Our first main result is

THEOREM 2.1. *For any θ with $0 < \theta \leqslant 1$,*

$$\Delta(\theta) = 2\theta^2 - 1. \tag{2.1}$$

As previously mentioned, this shows that Lorentz's lower bound in (1.5) of Theorem 1.2 is sharp. For our second main result, we have

THEOREM 2.2. *For any fixed θ with $0 < \theta \leqslant 1$, let*

$$\left\{ w_i(t) = \sum_{k=s_i}^{n_i} a_k(i)(1+t)^k \right\}_{i=1}^{\infty}$$

be any infinite sequence of complex polynomials for which

$$\lim_{i\to\infty} n_i = +\infty \quad \text{and} \quad \liminf_{i\to\infty} (s_i/n_i) \geqslant \theta. \tag{2.2}$$

Setting

$$M_i := \int_{-1}^{+1} (1-t^2)^{-1/2} |w_i(t)|^2 \, dt \quad \text{for all } i \geqslant 1, \tag{2.3}$$

assume that

$$\limsup_{i\to\infty} M_i^{1/n_i} \leqslant 1. \tag{2.4}$$

Then,

$$\lim_{i\to\infty} w_i(t) = 0, \quad \text{uniformly on every closed subinterval of } (-r(\theta), 2\theta^2 - 1), \tag{2.5}$$

where $r(\theta) > 1$ is defined in (1.14). *More precisely if $[t_0, t_1] \subset (-r(\theta), 2\theta^2 - 1)$, then (cf.* (1.12))

$$\limsup_{i\to\infty} \left(\|w_i\|_{[t_0,t_1]} \right)^{1/n_i} \leqslant \max_{j=0,1} \left\{ G(t_j, \theta) \right\} < 1. \tag{2.6}$$

Furthermore, the conclusions are best possible.

It is evident that any sequence of incomplete polynomials $\{w_i(t)\}_{i=1}^{\infty}$ satisfying Lorentz's Theorem 1.1 necessarily satisfies the hypotheses of Theorem 2.2, but not conversely, and that (2.6) of Theorem 2.2 is a sharpened form of Lorentz's (1.3). Moreover, we shall show that the inequality in (2.6) is sharp in the following sense: for any θ with $0 < \theta \leqslant 1$, there is an infinite sequence $\{\tilde{w}_i(t)\}_{i=1}^{\infty}$ satisfying the hypotheses of Theorem 2.2 for which equality holds in (2.6) for *every* closed interval $[t_0, t_1] \subset (-r(\theta), 2\theta^2 - 1)$. Furthermore, we show in Proposition 6.1 that the sequence $\{w_i(t)\}_{i=1}^{\infty}$ need *not* tend to zero at the endpoints $t = -r(\theta)$ and $t = 2\theta^2 - 1$ of $(-r(\theta), 2\theta^2 - 1)$.

It is important to note in Theorem 2.2 that the hypothesis (2.4) is *equivalent* to the assumption

$$\limsup_{i\to\infty} \left(\|w_i\|_{[-1,+1]}\right)^{1/n_i} \leqslant 1, \tag{2.4'}$$

and that (2.4) and (2.4′) are equivalent to the assumption

$$\limsup_{i\to\infty} \left[\int_{-1}^{+1} |w_i(t)|^2 \, d\psi(t)\right]^{1/n_i} \leqslant 1 \tag{2.4''}$$

for any distribution function $d\psi(t)$ on $[-1, +1]$ which has the following property: if $\{p_k(t, \psi)\}_{k=0}^{\infty}$ are the orthonormal polynomials associated with $d\psi(t)$, then

$$\limsup_{k\to\infty} \left(\|p_k(\cdot, \psi)\|_{[-1,+1]}\right)^{1/k} \leqslant 1.$$

For a discussion of the above condition, see Geronimus [**1**].

Our next result is an easy but important consequence of Theorem 2.2, which can, in turn, be shown to be *equivalent* to Theorem 2.2.

COROLLARY 2.3. *If* $p_n(t) = \sum_{k=s}^{n} a_k(1 + t)^k$ *with* $0 < s \leqslant n$ *and with* p_n *not identically zero, then*

$$|p_n(t)| \leqslant \|p_n\|_{[-1,+1]}\{G(t, s/n)\}^n < \|p_n\|_{[-1,+1]} \tag{2.7}$$

for any t *with* $-r(s/n) < t < 2(s/n)^2 - 1$.

As another easy consequence of Theorem 2.2 concerning sequences of polynomials which interpolate a given function f, we have

COROLLARY 2.4. *Let* $f(t)$ *be analytic at* $t = -1$. *If, for a fixed* θ *with* $0 < \theta \leqslant 1$, *there exists an infinite sequence of polynomials*

$$\{p_i(t)\}_{i=1}^{\infty} \quad \textit{with } p_i \in \pi_{n_i}$$

for all i, *with* $n_1 < n_2 < \dots$, *such that*

$$p_i^{(k)}(-1) = f^{(k)}(-1) \quad \textit{for all } 0 \leqslant k \leqslant s_i, \textit{ with } s_i \leqslant n_i, \tag{2.8}$$

$$\lim_{i\to\infty} (n_{i+1}/n_i) = 1, \quad \liminf_{i\to\infty} (s_i/n_i) \geqslant \theta, \tag{2.9}$$

and

$$\limsup_{i\to\infty} \left\{\int_{-1}^{+1} (1 - t^2)^{-1/2} |p_i(t)|^2 \, dt\right\}^{1/n_i} \leqslant 1, \tag{2.10}$$

then f *is necessarily analytic at each point of the interval* $(-r(\theta), 2\theta^2 - 1)$, *and* $p_i(t) \to f(t)$ *as* $i \to \infty$, *uniformly on each closed subinterval of* $(-r(\theta), 2\theta^2 - 1)$. *More precisely, if* $[t_0, t_1] \subset (-r(\theta), 2\theta^2 - 1)$, *then*

$$\limsup_{i\to\infty} \left\{\|f - p_i\|_{[t_0,t_1]}\right\}^{1/n_i} \leqslant \max_{j=0,1} \{G(t_j, \theta)\} < 1. \tag{2.11}$$

Our next result can be regarded as a *symmetric* version of Theorem 2.2.

THEOREM 2.5. *For any fixed* θ *with* $0 < \theta \leqslant 1$, *let* $\{q_i(t) = \sum_{k=s_i}^{n_i} a_k(i)t^k\}_{i=1}^{\infty}$ *be an infinite sequence of complex polynomials for which*

$$\lim_{i\to\infty} n_i = +\infty \quad \text{and} \quad \liminf_{i\to\infty} (s_i/n_i) \geqslant \theta. \tag{2.12}$$

If

$$\limsup_{i\to\infty} \left[\int_{-1}^{+1} (1-t^2)^{-1/2}|q_i(t)|^2\,dt\right]^{1/n_i} \leqslant 1, \tag{2.13}$$

then $q_i(t) \to 0$ *uniformly on each closed subinterval of* $(-\theta, +\theta)$. *More precisely, if* $[t_0, t_1] \subset (-\theta, +\theta)$, *then*

$$\limsup_{i\to\infty} \left(\|q_i\|_{[t_0,t_1]}\right)^{1/n_i} \leqslant \max_{j=0,1} \left\{G(2t_j^2 - 1, \theta)\right\}^{1/2} < 1. \tag{2.14}$$

Furthermore, the conclusions are best possible.

As with Theorem 2.2, the result of Theorem 2.5 is sharp in the sense that $(-\theta, +\theta)$ cannot in general be replaced by any larger interval, and for any θ with $0 < \theta < 1$, there is an infinite sequence of complex polynomials satisfying the hypotheses of Theorem 2.5 for which equality holds in (2.14) for every closed subinterval $[t_0, t_1]$ of $(-\theta, +\theta)$.

It is clear that there are "symmetric" analogues to Corollaries 2.3 and 2.4, but we leave their statements to the reader.

We note that we have *not* found an explicit formula in θ for the quantity $r(\theta) > 1$ such that $G(-r(\theta), \theta) = 1$ for all $0 < \theta \leqslant 1$ (cf. (1.14)) which enters prominently into the statements of Theorem 2.2 and its corollaries. It is, however, easily shown (cf. §4) that $1 < r(\theta) \leqslant 3$, $r(0+) = 1$, $r(\frac{1}{2}) = \frac{5}{4}$, and $r(1) = 3$. For a graph of θ vs. $r(\theta)$, see Figure 7 in §4.

Finally, there are extensions of Theorems 2.2 and 2.5 from uniform and geometric convergence on *real* intervals, to uniform and geometric convergence on compact subsets in the *complex plane*. To describe such extensions, we now extend the definition of the function $G(t, \theta)$ of (1.12) to the complex t-plane, $\mathbf{C}$. First, for each fixed θ with $0 < \theta < 1$, let $\sqrt{(1-t)(2\theta^2 - 1 - t)}$ denote the branch in the t-plane of the square root which is analytic exterior to the cut

$$\mathcal{S}(\theta) := \{t: 2\theta^2 - 1 \leqslant t \leqslant 1\}, \tag{2.15}$$

and which is positive at $t = -1$. With this (cf. (1.11)),

$$\zeta^- = \zeta^-(t, \theta) := \left\{t + \theta - \sqrt{(1-t)(2\theta^2 - 1 - t)}\right\}/(1 + \theta)$$

is then defined and analytic on $\mathbf{C} \setminus \mathcal{S}(\theta)$, and from this, $G(t, \theta)$ is defined for all $t \in \mathbf{C}$ by means of

$$G(t,\theta) := \begin{cases} \dfrac{1}{2}\left|\dfrac{(\zeta^-)^2 - 1}{\zeta^- - t}\right| \cdot \left|\dfrac{(1+\zeta^-)(\zeta^- - t)}{(1+t)(\zeta^- - 1)}\right|^{\theta}, & t \notin \mathbb{S}(\theta),\ t \neq -1, \\ 0, \quad t = -1, \\ 1, \quad t \in \mathbb{S}(\theta). \end{cases} \tag{2.16}$$

Note that $G(t, \theta)$, so defined, agrees with the real definition of (1.12), and that $G(t, \theta)$ is actually continuous for all $t \in \mathbf{C}$. Next, we define the level curve

$$\Lambda(\theta) := \{t \in \mathbf{C}\colon G(t,\theta) = 1\}, \tag{2.17}$$

which is pictured below in Figure 1 for the case $\theta = \frac{1}{2}$. We denote the interior of $\Lambda(\theta)$ by

$$\mathring{\Lambda}(\theta) := \{t \in \mathbf{C}\colon G(t,\theta) < 1\}. \tag{2.18}$$

Note that $\mathring{\Lambda}(\theta)$ is a Jordan region which does not include the cut $\mathbb{S}(\theta)$, but does include (cf. (1.14)) the open interval $(-r(\theta), 2\theta^2 - 1)$.

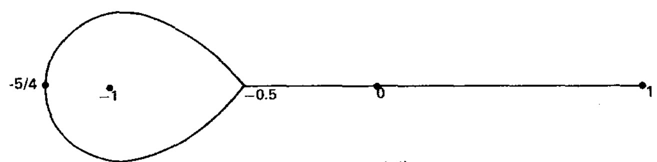

FIGURE 1. Λ(½)

We remark that the function $G(t, \theta)$ of (2.16) can also be equivalently defined in $\mathbf{C}$ as

$$G(t,\theta) = |w| \cdot \left|\frac{(1-\theta)w - (1+\theta)}{(1+\theta)w - (1-\theta)}\right|^{\theta}, \tag{2.19}$$

where

$$t = \theta^2 + (\theta^2 - 1)\left(\frac{w + w^{-1}}{2}\right) \quad \text{for all } |w| \geqslant 1. \tag{2.20}$$

This follows directly from (2.16) upon making the substitution $\zeta^- = \theta + (\theta - 1)w$. This equivalent formulation was also found by Professor G. G. Lorentz (personal communication).

We can now state the generalizations of Theorem 2.2 and Corollary 2.3 to the complex plane.

THEOREM 2.6. *Under the hypotheses of Theorem* 2.2, *then*

$$\lim_{i\to\infty} w_i(t) = 0, \quad \textit{uniformly on every closed subset of } \mathring{\Lambda}(\theta). \tag{2.21}$$

More precisely, for any closed subset K of $\mathring{\Lambda}(\theta)$, then

$$\limsup_{i\to\infty} (\|w_i\|_K)^{1/n_i} \leq \max_{t\in K} \{G(t,\theta)\} < 1. \tag{2.22}$$

Furthermore, the conclusions are best possible, in the sense that equality can hold in (2.22) for any compact subset K of $\mathring{\Lambda}(\theta)$, and that there are polynomials $\{w_i(t)\}_{i=1}^{\infty}$ satisfying the hypotheses of Theorem 2.2 which do not tend to zero for any t exterior to $\Lambda(\theta)$.

COROLLARY 2.7. *If $p_n(t) = \sum_{k=s}^{n} a_k(1+t)^k$ with $0 < s \leq n$ and with p_n not identically zero, then*

$$|p_n(t)| \leq \|p_n\|_{[-1,+1]}\{G(t,s/n)\}^n < \|p_n\|_{[-1,+1]} \tag{2.23}$$

for all $t \in \mathring{\Lambda}(s/n)$.

The extensions of Corollary 2.4 and Theorem 2.5 to the complex plane are similar, and are left to the reader.

3. Proof of Theorem 2.1. To begin, for any fixed θ with $0 < \theta < 1$, let $\{(n_i, m_i)\}_{i=1}^{\infty}$ be any infinite sequence of pairs of positive integers for which

$$\frac{m_i}{n_i} \geq \frac{\theta}{1-\theta} \quad \text{for all } i \geq 1, \quad \lim_{i\to\infty} \frac{m_i}{n_i} = \frac{\theta}{1-\theta}, \quad \lim_{i\to\infty} n_i = \lim_{i\to\infty} m_i = \infty, \tag{3.1}$$

Then, set

$$W_i(t) := (1+t)^{m_i} P_{n_i}^{(-1/2,\,2m_i-1/2)}(t) \quad \text{for all } i \geq 1. \tag{3.2}$$

Note that $W_i \in \pi_{n_i+m_i}$ and that the first condition of (3.1) implies that each $W_i(t)$ is an incomplete polynomial of the form (1.1) with n_i replaced by $n_i + m_i$.

LEMMA 3.1. *For each $i \geq 0$, $W_i(t)$ satisfies*

$$(1-t^2)W_i''(t) - tW_i'(t) + [(\lambda_i t + \tau_i)/(1+t)] \cdot W_i(t) = 0, \tag{3.3}$$

where

$$\lambda_i := (m_i+n_i)^2, \quad \tau_i := (m_i+n_i)^2 - 2m_i^2 + m_i, \qquad i \geq 1. \tag{3.4}$$

PROOF. It is known (cf. Szegö [6, p. 60]) that $y(t) = P_{n_i}^{(-1/2,\,2m_i-1/2)}(t)$ satisfies

$$(1-t^2)y'' + [2m_i - (2m_i+1)t]y' + n_i[2m_i + n_i]y = 0. \tag{3.5}$$

With (3.5) and the definition of $W_i(t)$ in (3.2), (3.3) follows immediately. □

Next, as suggested by Szegö [6, p. 165], define

$$f_i(t) := W_i^2(t) + g_i(t)\cdot(W_i'(t))^2, \tag{3.6}$$

where

$$g_i(t) := \left[(1-t^2)(1+t)\right]/(\lambda_i t + \tau_i). \tag{3.7}$$

Note that g has a pole at $-\tau_i/\lambda_i = -[(m_i+n_i)^2 - 2m_i^2 + m_i]/(m_i+n_i)^2$, which lies in $(-1, +1)$, and that, with (3.1),

$$\lim_{i\to\infty} (-\tau_i/\lambda_i) = 2\theta^2 - 1. \tag{3.8}$$

LEMMA 3.2. *$f_i'(t) \leqslant 0$ for any $t \in [-1, +1]$ with $t \neq -\tau_i/\lambda_i$.*

PROOF. From the definitions of (3.4), (3.6), and (3.7), a short calculation utilizing Lemma 3.1 shows that

$$f_i'(t) = m_i(1-2m_i)(1-t^2)(W_i'(t))^2/(\lambda_i t + \tau_i)^2, \qquad t \neq -\tau_i/\lambda_i, \tag{3.9}$$

for every $i \geqslant 1$. Thus, since $m_i \geqslant 1$, $f_i'(t) \leqslant 0$ for any $t \in [-1, +1]$ with $t \neq -\tau_i/\lambda_i$ for all $i \geqslant 1$. □

LEMMA 3.3. *Let $\xi_i := \min[\mu\colon\ W_i'(\mu) = 0$ and $\mu \neq -1]$. Then, $\xi_i \in (-\tau_i/\lambda_i, 1)$ and*

$$|W_i(\xi_i)| = \|W_i\|_{[-1,+1]}, \quad \textit{for all } i \geqslant 1. \tag{3.10}$$

PROOF. As is well known, the Jacobi polynomial $P_n^{(\alpha,\beta)}(t)$ has n simple zeros in $(-1, +1)$. Thus, by Rolle's Theorem $W_i'(t)$ of (3.2) has n_i zeros in $(-1, +1)$, and its remaining $m_i - 1$ zeros at $t = -1$. Hence, $\xi_i \in (-1, +1)$. First, suppose that $-1 < \xi_i \leqslant -\tau_i/\lambda_i$. Because $f_i'(t) \leqslant 0$ in $[-1, -\tau_i/\lambda_i)$ from Lemma 3.2, then $f_i(-1) \geqslant f_i(\xi_i)$. Now, $W_i'(\xi_i) = 0$ by definition, and hence $f_i(\xi_i) = W_i^2(\xi_i)$, even if $\xi_i = -\tau_i/\lambda_i$. As $f_i(-1) = 0$, then $f_i(-1) \geqslant f_i(\xi_i)$ implies that $0 \geqslant W_i^2(\xi_i)$, or $W_i(\xi_i) = 0$. Next, from (3.3), it follows that $W_i''(\xi_i) = 0$, and repeated differentiation of (3.3) shows that $W_i(t) \equiv 0$, a contradiction. Thus, we have that

$$-\tau_i/\lambda_i < \xi_i. \tag{3.11}$$

Now, let μ be either any other zero of $W_i'(t)$ in $(-1, +1]$, or let $\mu = 1$. By definition $\xi_i < \mu$, and thus, with (3.11),

$$-\tau_i/\lambda_i < \xi_i < \mu \leqslant 1.$$

From the monotonicity of f_i in $(-\tau_i/\lambda_i, 1]$ from Lemma 3.2, we again have

$$W_i^2(\xi_i) = f_i(\xi_i) \geqslant f_i(\mu) = W_i^2(\mu),$$

or $|W_i(\xi_i)| \geqslant |W_i(\mu)|$. Since $W_i(-1) = 0$, then $|W_i(\xi_i)| = \|W_i\|_{[-1,+1]}$, the desired result of (3.10). □

We illustrate the result of Lemma 3.3 by graphing $W_i(t)/\|W_i\|_{[-1,+1]}$ in Figure 2 below, for the case (cf. (3.2)) $m_i = 5$ and $n_i = 10$.

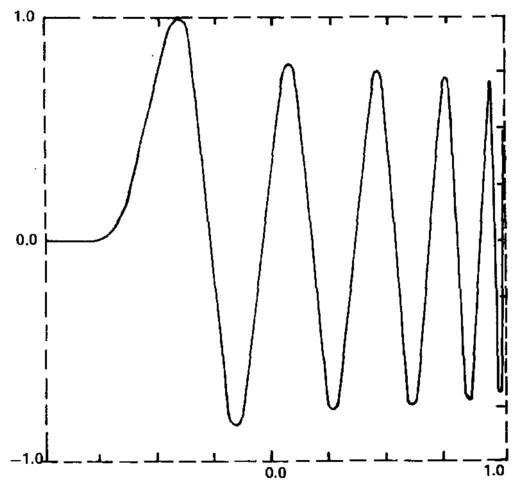

FIGURE 2. $W_i(t)/\|W_i\|_{[-1,+1]}$ for $m_i = 5, n_i = 10$

Now, let σ_i denote the smallest zero of $P_{n_i}^{(-1/2,\,2m_i-1/2)}(t)$. Then $W_i(-1) = W_i(\sigma_i) = 0$ implies $\xi_i < \sigma_i$ by Rolle's Theorem. Thus, with (3.11),

$$-\tau_i/\lambda_i < \xi_i < \sigma_i \quad \text{for all } i \geqslant 1, \tag{3.12}$$

and we are interested in the behavior of σ_i as $i \to \infty$. More generally, this raises the question of the behavior of zeros of Jacobi polynomials of the special form $P_n^{(\alpha,\beta_n)}(t)$, where $\lim_{n\to\infty} \beta_n/n = b$. Results along these lines were simultaneously obtained independently by D. S. Moak and the present authors. Because these particular results may be of independent interest, they have been gathered in the preceding note [**4**]. For our purposes here, we simply state the special case $\alpha = 0$ of [**4**, Corollary 1] as

LEMMA 3.4. *Let r_n and s_n be respectively the smallest and largest zeros of the Jacobi polynomial $P_n^{(\alpha_n,\beta_n)}(t)$, where $\alpha_n > -1$ and $\beta_n > -1$. If*

$$\lim_{n\to\infty} \alpha_n/n = 0, \ \lim_{n\to\infty} \beta_n/n = \beta, \tag{3.13}$$

then

$$\lim_{n\to\infty} r_n = \frac{2\beta^2}{(2+\beta)^2} - 1 \quad \textit{and} \quad \lim_{n\to\infty} s_n = 1. \tag{3.14}$$

Applying Lemma 3.4 to the case (cf. (3.2)) where $\alpha_{n_i} = -\frac{1}{2}$ and $\beta_{n_i} = 2m_i - \frac{1}{2}$, it follows from (3.1) that $\beta = 2\theta/(1-\theta)$ in (3.13), so that σ_i, the smallest zero of $P_{n_i}^{(-1/2,\,2m_i-1/2)}(t)$, from (3.14) satisfies

$$\lim_{i\to\infty} \sigma_i = 2\theta^2 - 1. \tag{3.15}$$

Hence, with (3.8) and (3.12), then

$$\lim_{i\to\infty} \xi_i = \lim_{i\to\infty} \sigma_i = 2\theta^2 - 1. \tag{3.16}$$

PROOF OF THEOREM 2.1. For $0 < \theta < 1$, consider now the sequence of polynomials

$$\{v_i(t) := W_i(t)/\|W_i\|_{[-1,+1]}\}_{i=1}^{\infty}, \tag{3.17}$$

where W_i is defined in (3.2). By definition, $\{v_i(t)\}_{i=1}^{\infty}$ satisfies the hypotheses of Theorem 1.1. Evidently, from (3.10) of Lemma 3.3, we have that $|v_i(\xi_i)| = 1$ for all $i \geqslant 1$, and from (3.16) that $\lim_{i\to\infty} \xi_i = 2\theta^2 - 1$. Thus, $\{v_i(t)\}_{i=1}^{\infty}$ does *not* tend uniformly to zero in $[-1, 2\theta^2 - 1 + \varepsilon]$ for any $\varepsilon > 0$, so that $\delta \leqslant 2\theta^2 - 1$ and (cf. (1.4)) $\Delta(\theta) \leqslant 2\theta^2 - 1$. But, as $\Delta(\theta) \geqslant 2\theta^2 - 1$ from (1.5) of Theorem 1.2, we have the desired result

$$\Delta(\theta) = 2\theta^2 - 1 \tag{3.18}$$

of (2.1) of Theorem 2.1 for $0 < \theta < 1$. For the remaining case $\theta = 1$, it suffices to similarly consider the sequence of polynomials $\{((1 + t)/2)^n\}_{n=1}^{\infty}$. □

It is also useful to point out the following. For any fixed $i \geqslant 1$, consider the infinite sequence $\{(v_i(t))^j := (W_i(t)/\|W_i\|_{[-1,+1]})^j\}_{j=1}^{\infty}$, where $W_i(t)$ is defined in (3.2). This sequence obviously satisfies Theorem 1.1 with $\theta_i = m_i/(n_i + m_i)$. But as $|v_i^j(\xi_i)| = 1$ for all $j \geqslant 1$, then evidently

$$\Delta(m_i/(n_i + m_i)) \leqslant \xi_i \quad \text{for all } i \geqslant 1.$$

Thus, with (3.12) and (3.18),

$$2[m_i/(n_i + m_i)]^2 - 1 < \sigma_i \quad \text{for all } i \geqslant 1, \tag{3.19}$$

where σ_i is the smallest zero of $P_{n_i}^{(-1/2, 2m_i - 1/2)}(t)$. As a consequence of this result, we have the following which may be of independent interest.

COROLLARY 3.5. *For any positive integers m and n and for all α and β with* $-1 < \alpha \leqslant -\frac{1}{2}$ *and* $\beta \geqslant -\frac{1}{2}$, *all the zeros of* $P_n^{(\alpha, 2m+\beta)}(t)$ *lie in* $(2(m/(m + n))^2 - 1, +1)$.

PROOF. All the zeros of $P_n^{(-1/2, 2m-1/2)}(t)$ lie in $(2(m/(m + n))^2 - 1, +1)$ from (3.19). However, using a result of Szegö [6, p. 121] on the monotonicity of the zeros of $P_n^{(\alpha,\beta)}(t)$ as a function of α and β, it follows that all the zeros of $P_n^{(\alpha,2m+\beta)}(t)$ also lie in the same interval if $-1 < \alpha \leqslant -\frac{1}{2}$ and if $\beta \geqslant -\frac{1}{2}$. □

4. Asymptotic estimates for Jacobi polynomials. Consider the infinite sequence $\{P_n^{(\alpha, 2\theta n/(1-\theta)+\beta)}(t)\}_{n=0}^{\infty}$ of Jacobi polynomials, where α, θ, and β are fixed real numbers satisfying $\alpha > -1$, $0 < \theta < 1$, and $\beta > -1$. As is known (cf. Szegö [6, p. 70]), $P_n^{(\alpha, 2\theta n/(1-\theta)+\beta)}(t)$ has the following integral representation:

$$P_n^{(\alpha,2\theta n/(1-\theta)+\beta)}(t) = \frac{1}{2\pi i}\int_\Gamma \left(\frac{\zeta^2-1}{2(\zeta-t)}\right)^n \left(\frac{1-\zeta}{1-t}\right)^\alpha \left(\frac{1+\zeta}{1+t}\right)^{2\theta n/(1-\theta)+\beta} \frac{d\zeta}{(\zeta-t)}, \tag{4.1}$$

where we assume that $t \neq \pm 1$, and that the integration above is about a closed contour Γ, in the positive sense, enclosing $\zeta = t$ but not the points $\zeta = \pm 1$. We write the above integral representation in the form

$$P_n^{(\alpha,2\theta n/(1-\theta)+\beta)}(t) = \int_\Gamma e^{nh(\zeta)} g(\zeta)\, d\zeta, \tag{4.2}$$

where

$$h(\zeta) = h(\zeta, t, \theta) := \ln(\zeta^2 - 1) - \ln 2 - \ln(\zeta - t) + \left(\frac{2\theta}{1-\theta}\right)[\ln(1+\zeta) - \ln(1+t)], \tag{4.3}$$

and where

$$g(\zeta) = g(\zeta, t, \theta) := \frac{1}{2\pi i}\left(\frac{1-\zeta}{1-t}\right)^\alpha \cdot \left(\frac{1+\zeta}{1+t}\right)^\beta \cdot \frac{1}{(\zeta - t)}. \tag{4.4}$$

Differentiating h with respect to ζ gives

$$h'(\zeta) = \frac{(1+\theta)\zeta^2 - 2(t+\theta)\zeta + (1-\theta+2\theta t)}{(1-\theta)(\zeta^2-1)(\zeta-t)}, \tag{4.5}$$

and

$$h''(\zeta) = -2\frac{\zeta^2+1}{(\zeta^2-1)^2} + \frac{1}{(\zeta-t)^2} - \frac{2\theta}{(1-\theta)(1+\zeta)^2}. \tag{4.6}$$

The saddle points $\zeta^\pm(t, \theta)$ for $h(\zeta)$, defined as the zeros of $h'(\zeta)$, are, from (4.5), the roots of a quadratic equation (cf. (1.10)), which are given explicitly by (cf. (1.11))

$$\zeta^\pm(t, \theta) = \left\{t + \theta \pm \sqrt{(1-t)(2\theta^2 - 1 - t)}\right\}/(1+\theta). \tag{4.7}$$

For any $t < 2\theta^2 - 1$, these two saddle points $\zeta^\pm(t, \theta)$ are, from (4.7), real and distinct, and, in addition, satisfy

$$-1 < t < \zeta^-(t, \theta) < 2\theta - 1 < \zeta^+(t, \theta) < 1 \quad \text{for } -1 < t < 2\theta^2 - 1, \tag{4.8}$$

and

$$\zeta^-(t, \theta) < t < -1 < \zeta^+(t, \theta) < 1 \quad \text{for } t < -1. \tag{4.9}$$

From (4.6), it can also be verified that

$$h''(\zeta^{\pm}(2\theta^2 - 1, \theta)) = 0. \tag{4.10}$$

With the above expressions (4.5)–(4.9), the following result can be readily established.

LEMMA 4.1. *For all $t < 2\theta^2 - 1$ and $t \neq -1$,*

$$h''(\zeta^{-}(t, \theta)) > 0 \quad and \quad h''(\zeta^{+}(t, \theta)) < 0. \tag{4.11}$$

We now apply the *steepest descent method* (cf. Henrici [2, p. 416] and Olver [5, p. 136]) to the representation (4.2) to determine the asymptotic behavior of $P_n^{(\alpha, 2\theta n/(1+\theta)+\beta)}(t)$ for fixed t with $t < 2\theta^2 - 1$, $t \neq -1$, as $n \to \infty$. First, note from (4.3) that

$$\operatorname{Re} h(\zeta) = \ln|\zeta^2 - 1| - \ln 2 - \ln|\zeta - t| + \frac{2\theta}{(1-\theta)}\left\{\ln\left|\frac{1+\zeta}{1+t}\right|\right\}. \tag{4.12}$$

This shows that $\operatorname{Re} h(\zeta) \to -\infty$ only if $\zeta \to \pm 1$, while $\operatorname{Re} h(\zeta) \to +\infty$ only if $\zeta \to t$ or if $|\zeta| \to +\infty$.

Case 1: $-1 < t < 2\theta^2 - 1$. With (4.11), the steepest descent curves (i.e., the curves in the ζ-plane where $\operatorname{Re} h(\zeta)$ decreases most rapidly) through $\zeta^{-}(t, \theta)$ are necessarily perpendicular to the real axis at $\zeta^{-}(t, \theta)$, and similarly, the steepest ascent curves through $\zeta^{+}(t, \theta)$ are perpendicular to the real axis at $\zeta^{+}(t, \theta)$. In addition, both sets of curves are symmetric about the real axis. It is next easily seen that the steepest ascent curves through $\zeta^{+}(t, \theta)$ cannot pass through $\zeta = t$, so that from the discussion following (4.12), these steepest ascent curves through $\zeta^{+}(t, \theta)$ must tend to infinity. Thus, as the steepest descent curves through $\zeta^{-}(t, \theta)$ do not intersect the steepest ascent curves through $\zeta^{+}(t, \theta)$, it follows that the steepest descent curves through $\zeta^{-}(t, \theta)$ must tend to $\zeta = -1$. The steepest descent and ascent curves through $\zeta^{\pm}(t, \theta)$ are illustrated in Figure 3, the arrows there indicating the direction for *increasing* $\operatorname{Re} h(\zeta)$. We now modify the steepest descent curve through $\zeta^{-}(t, \theta)$ by replacing the small portion of it near -1 by a small circular arc, as shown in Figure 4. The resulting contour Γ taken in the positive sense, can then be used in the integral representation of (4.1), for all $n \geqslant 0$.

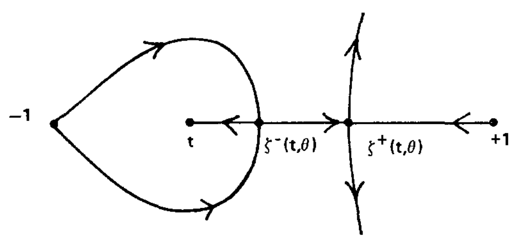

FIGURE 3. $-1 < t < 2\theta^2 - 1$

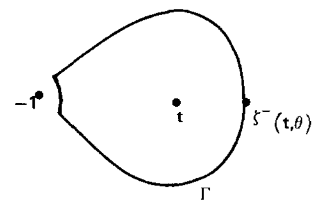

FIGURE 4. Contour for $-1 < t < 2\theta^2 - 1$

Case 2. $t < -1$. For $t < -1$, a discussion based on (4.9) and (4.12) similarly gives the behavior of the steepest descent and ascent curves through $\zeta^{\pm}(t, \theta)$, and these curves are illustrated in Figure 5. The modified contour Γ for use in (4.1) is illustrated in Figure 6.

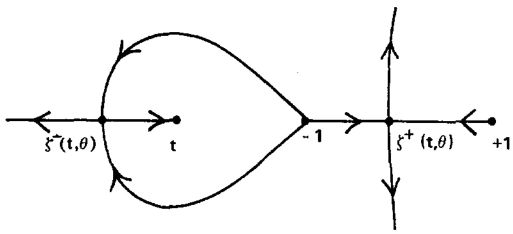

FIGURE 5. $t < -1$

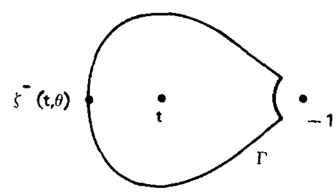

FIGURE 6. Contour for $t < -1$

Because of the above construction of the contour Γ for Case 1, $g(\zeta)$ is analytic in a neighborhood of all points of the contour Γ, and a straightforward application of the steepest descent method (cf. Henrici [2, p. 416] and Olver [5, p. 136]) to the integral of (4.2) gives, for any fixed t with $-1 < t < 2\theta^2 - 1$, that

$$P_n^{(\alpha, 2\theta n/(1-\theta)+\beta)}(t)$$

$$\sim \exp[nh(\zeta^-(t, \theta)) + i\pi/2]\, g(\zeta^-(t, \theta)) \cdot \left\{ \frac{2\pi}{nh''(\zeta^-(t, \theta))} \right\}^{1/2},$$

as $n \to \infty$. Using the definitions (4.3), (4.4), and (4.7) and taking nth roots, this implies that

$$\lim_{n\to\infty} \left|P_n^{(\alpha,2\theta n/(1-\theta)+\beta)}(t)\right|^{1/n} = \left|\frac{(\zeta^-(t,\theta))^2-1}{2(\zeta^-(t,\theta)-t)}\right| \cdot \left|\frac{1+\zeta^-(t,\theta)}{1+t}\right|^{2\theta/(1-\theta)}, \tag{4.13}$$

for any fixed t with $-1 < t < 2\theta^2 - 1$. Next, it can be verified that the steepest descent method applied to the integral of (4.2) for any fixed t with $t < -1$ gives precisely the above asymptotic behavior, whence (4.13) is valid for *any* fixed t with $t < 2\theta^2 - 1$, $t \neq -1$. From (4.13), we can further deduce that

$$\lim_{n\to\infty} \left|(1+t)^{\theta n/(1-\theta)} P_n^{(\alpha,2\theta n/(1-\theta)+\beta)}(t)\right|^{(1-\theta)/n}$$
$$= \left|\frac{(\zeta^-(t,\theta))^2-1}{2(\zeta^-(t,\theta)-t)}\right| \cdot \left|\frac{2(\zeta^-(t,\theta)-t)(1+\zeta^-(t,\theta))^2}{(\zeta^-(t,\theta)^2-1)(1+t)}\right|^{\theta}, \tag{4.14}$$

for any fixed t with $t < 2\theta^2 - 1$, $t \neq -1$.

Next, with the representation for $h_n^{(\alpha,\beta)}$ in (1.9), a routine application of Stirling's approximation for $\Gamma(\mu)$, $\mu \to +\infty$, coupled with the limit of (4.14), shows that

$$\lim_{n\to\infty} \left\{\frac{\left|(1+t)^{\theta n/(1-\theta)} P_n^{(\alpha,2\theta n/(1-\theta)+\beta)}(t)\right|}{\sqrt{h_n^{(\alpha,2\theta n/(1-\theta)+\beta)}}}\right\}^{(1-\theta)/n} = G(t,\theta) \tag{4.15}$$

for any fixed θ with $0 < \theta < 1$ and for any fixed t with $t < 2\theta^2 - 1$, where $G(t,\theta)$ is defined in (1.12). As previously mentioned, $G(t,\theta)$, as a function of t, is continuous on $(-\infty, 2\theta^2 - 1]$ for any θ with $0 < \theta < 1$.

The result of (4.15) can be extended as follows. Let $\{(m_i, n_i)\}_{i=1}^{\infty}$ be any infinite sequence of pairs of positive integers for which

$$\lim_{i\to\infty} m_i/n_i = \theta/(1-\theta) \quad \text{and} \quad \lim_{i\to\infty} n_i = +\infty, \tag{4.16}$$

where $0 < \theta < 1$. Observe that the factor $[|1+t|^{\theta n/(1-\theta)}]^{(1-\theta)/n} = |1+t|^{\theta}$ in (4.15) is unchanged in the limit if it is replaced by $[|1+t|^{m_i}]^{1/(m_i+n_i)}$, because of (4.16). Similarly, because of the explicit formula (1.9), replacing the factor $\{h_n^{(\alpha,2\theta n/(1-\theta)+\beta)}\}^{(1-\theta)/2n}$ in (4.15) by $\{h_{n_i}^{(\alpha,2m_i+\beta)}\}^{1/2(m_i+n_i)}$ makes no change in the limit in (4.15). Next, a careful study of the previous asymptotic behavior of $P_n^{(\alpha,2\theta n/(1-\theta)+\beta)}(t)$ shows that letting β grow like $o(n_i)$ again makes no change in (4.15). Thus, from (4.15),

$$\lim_{i\to\infty} \left\{\frac{\left|(1+t)^{m_i} P_{n_i}^{(\alpha,2m_i+\beta)}(t)\right|}{\sqrt{h_{n_i}^{(\alpha,2m_i+\beta)}}}\right\}^{1/(m_i+n_i)} = G(t,\theta), \tag{4.17}$$

for all t with $t < 2\theta^2 - 1$, where $\{(m_i, n_i)\}_{i=1}^{\infty}$ satisfies (4.16). It can moreover be shown that, with the complex extension of $G(t, \theta)$ of (2.16), the result of (4.17) is valid for all $t \notin \mathbb{S}(\theta)$.

We now establish additional properties for $G(t, \theta)$.

LEMMA 4.2. *For any fixed θ with $0 < \theta \leqslant 1$, $G(t, \theta)$, as a function of t, is strictly increasing on $(-1, 2\theta^2 - 1)$, and strictly decreasing on $(-\infty, -1)$.*

PROOF. From the definitions of $\zeta^-(t, \theta)$ in (1.11) and $G(t, \theta)$ in (1.12), it can be verified that

$$\frac{\partial G(t, \theta)}{\partial t} = G(t, \theta)\left\{\frac{1 + t - \theta - \theta\zeta^-(t, \theta)}{(\zeta^-(t, \theta) - t)(1 + t)}\right\}, \tag{4.18}$$

for all $t < 2\theta^2 - 1$, $t \neq -1$. Now, as $G(t, \theta)$ by definition is positive for all $t \neq -1$, it can be further verified that

$$\frac{1 + t - \theta - \theta\zeta^-(t, \theta)}{1 + t} > 0 \quad \text{for all } t \neq -1.$$

Thus, from (4.18), the sign of $\partial G(t, \theta)/\partial t$ for $t \neq -1$ is determined by the sign of $\zeta^-(t, \theta) - t$. Hence, from (4.8), $\partial G(t, \theta)/\partial t$ is positive for all $-1 < t < 2\theta^2 - 1$, while from (4.9), $\partial G(t, \theta)/\partial t$ is negative for all $t < -1$. □

Next, it can be verified that, for each fixed θ with $0 < \theta < 1$,

$$G(-\tau, \theta) \sim \frac{2\tau}{(1 + \theta)^{1+\theta}(1 - \theta)^{1-\theta}} \quad \text{as } \tau \to +\infty. \tag{4.19}$$

Hence, as $G(t, \theta) \to +\infty$ as $t \to -\infty$ from (4.19), and as $G(-1, \theta) = 0$ from (1.12), the strictly decreasing nature of $G(t, \theta)$ on $(-\infty, -1)$ from Lemma 4.2 gives us the existence of a unique $r(\theta)$ with $r(\theta) > 1$ such that (cf. (1.14))

$$G(-r(\theta), \theta) = 1 \quad \text{for each } \theta \text{ with } 0 < \theta \leqslant 1. \tag{4.20}$$

Thus, with (1.13), Lemma 4.2 further gives us that

$$G(t, \theta) < 1 \quad \text{iff} \quad t \in (-r(\theta), 2\theta^2 - 1). \tag{4.21}$$

We have graphed θ vs. $r(\theta)$ in Figure 7.

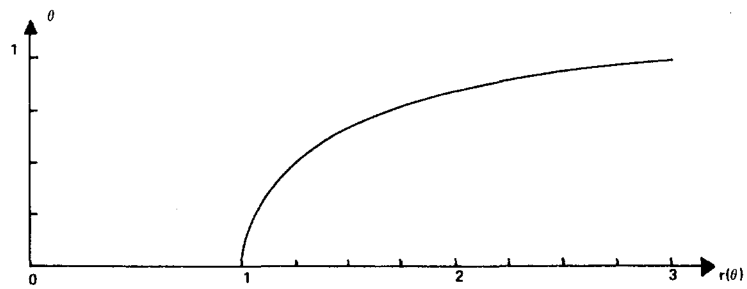

FIGURE 7. $r(\theta)$ vs. θ

We next establish

LEMMA 4.3. *For any fixed t with $t < 1$, $t \neq -1$, $G(t, \theta)$, as a function of θ, on the interval $(\sqrt{(1+t)/2}\,, 1)$ if $t > -1$ or on the interval $(0, 1)$ if $t < -1$, is strictly decreasing.*

PROOF. From the definitions of $\zeta^-(t, \theta)$ in (1.11) and $G(t, \theta)$ in (1.12), it can be verified that

$$\frac{\partial G(t,\theta)}{\partial \theta} = G(t,\theta) \cdot \ln\left|\frac{(1+\zeta^-(t,\theta))(\zeta^-(t,\theta)-t)}{(1-\zeta^-(t,\theta))(1+t)}\right| \tag{4.22}$$

for all $t < 1$, $t \neq -1$. For $t \neq -1$, it can be further shown from (1.11), (4.8), and (4.9) that

$$\left|\frac{(1+\zeta^-(t,\theta))(\zeta^-(t,\theta)-t)}{(1-\zeta^-(t,\theta))(1+t)}\right| < 1,$$

so that its logarithm is negative. But as $G(t, \theta)$ is, by definition (1.12), positive for all $t \neq -1$, then $\partial G(t, \theta)/\partial\theta$ is negative for all $t < 1$, $t \neq -1$. □

To illustrate Lemmas 4.2 and 4.3, we give, in Figure 8, $G(t, \theta)$ as a function t for several different values of θ.

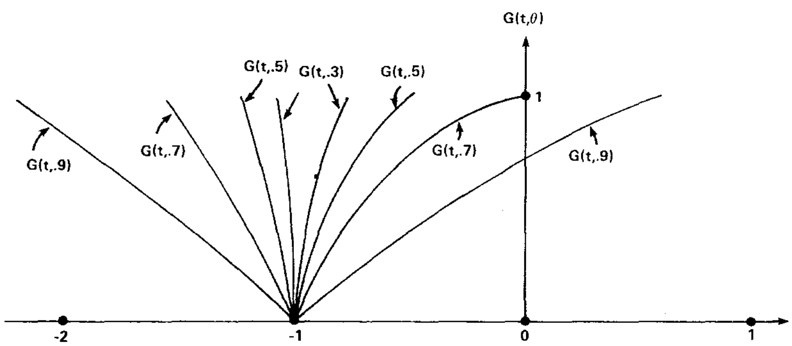

FIGURE 8. $G(t, \theta)$

As a consequence of the strictly decreasing nature of $G(t, \theta)$ as a function of θ from Lemma 4.3, it can be verified (by taking appropriate subsequences) from (4.17) that if (cf. (3.2))

$$\left\{\tilde{v}_i(t) := (1+t)^{m_i} P_{n_i}^{(\alpha, 2m_i+\beta)}(t)\Big/\sqrt{h_{n_i}^{(\alpha,2m_i+\beta)}}\,\right\}_{i=1}^{\infty}$$

is any sequence having $m_i \geqslant \theta n_i/(1-\theta)$ for all $i \geqslant 1$, then

$$\limsup_{i\to\infty} |\tilde{v}_i(t)|^{1/(m_i+n_i)} \leqslant G(t,\theta), \quad \text{for all } t < 2\theta^2 - 1.$$

5. Proofs of Theorem 2.2, its Corollaries, and Theorem 2.5. Let $\{w_i(t) = \sum_{k=s_i}^{n_i} a_k(i)(1+t)^k\}_{i=1}^{\infty}$ be any infinite sequence of complex polynomials satisfying the hypotheses of Theorem 2.2. From (2.2), it follows that, for every $\hat{\theta}$ with $0 < \hat{\theta} < \theta$, there is a finite $i_0 = i_0(\hat{\theta})$ such that

$$s_i/n_i \geqslant \hat{\theta} \quad \text{for all } i \geqslant i_0. \tag{5.1}$$

Next, on writing

$$w_i(t) = (1+t)^{s_i} q_{n_i - s_i}(t), \quad \text{where } q_{n_i-s_i} \in \pi_{n_i - s_i}, \tag{5.2}$$

it follows that the quantity M_i, defined in (2.3), can be expressed as

$$M_i = \int_{-1}^{+1} (1-t)^{-1/2}(1+t)^{2s_i - 1/2} |q_{n_i-s_i}(t)|^2 \, dt. \tag{5.3}$$

Now, recall from (1.8) that the normalized Jacobi polynomials $\{P_k^{(\alpha',\beta')}(t)/\sqrt{h_k^{(\alpha',\beta')}}\}_{k=0}^{\infty}$ are orthonormal with respect to the weight function $(1-t)^{\alpha'}(1+t)^{\beta'}$ on $[-1,+1]$. Hence, using a result of Szegö [**6**, p. 39] applied to (5.3) gives that

$$|q_{n_i-s_i}(t)|^2 \leqslant M_i \sum_{k=0}^{n_i - s_i} \left(P_k^{(-1/2, 2s_i - 1/2)}(t)\right)^2 / h_k^{(-1/2, 2s_i - 1/2)}, \qquad t \text{ real}. \tag{5.4}$$

From the kernel representation (cf. Szegö [**6**, p. 71]) for the right side of (5.4), it can be verified that (5.4) can be expressed as

$$|q_{n_i-s_i}(t)|^2 \leqslant \frac{2^{1-2s_i} M_i \Gamma(n_i - s_i + 2)\Gamma(n_i + s_i + 1)}{(2n_i+1)\Gamma\left(n_i - s_i + \frac{1}{2}\right)\Gamma\left(n_i + s_i + \frac{1}{2}\right)}$$
$$\times \left\{P'_{n_i - s_i + 1}(t) \cdot P_{n_i - s_i}(t) - P_{n_i - s_i + 1}(t) \cdot P'_{n_i - s_i}(t)\right\}, \tag{5.5}$$

where for convenience, we have suppressed the superscript $(-\frac{1}{2}, 2s_i - \frac{1}{2})$ on each of the Jacobi polynomials above. Since the derivative of a Jacobi polynomial can be expressed (cf. [**6**, p. 63]) in terms of a lower degree Jacobi polynomial, then, upon multiplying both sides of (5.5) by $|1+t|^{2s_i}$, we see with (5.2) that an upper bound for $|w_i(t)|^2$ is obtained, to which the asymptotic estimates of (4.17) can be applied. These calculations, which we omit, give rise to point-wise estimates of the form

$$\limsup_{i\to\infty} |w_i(t)|^{1/n_i} \leqslant G(t, \hat{\theta})$$

for any t with $-1 \leqslant t < 2\hat{\theta}^2 - 1$. With the above inequality and Lemma 4.2, it then easily follows for $[t_0, t_1] \subset (-r(\hat{\theta}), 2\hat{\theta}^2 - 1)$ that

$$\limsup_{i\to\infty} \left(\|w_i\|_{[t_0,t_1]}\right)^{1/n_i} \leqslant \max_{j=0,1} \left\{G\left(t_j, \hat{\theta}\right)\right\} < 1. \tag{5.6}$$

But, since $\hat{\theta}$ can be chosen arbitrarily close to θ, then (5.6) gives the desired result (2.6) of Theorem 2.2.

Finally, because of (4.17), for any θ with $0 < \theta \leq 1$, there is an infinite sequence $\{w_i(t)\}_{i=1}^{\infty}$ of incomplete polynomials satisfying the hypotheses of Theorem 2.2 for which equality holds in (2.6) for *every* subinterval $[t_0, t_1] \subset (-r(\theta), 2\theta^2 - 1)$. □

PROOF OF COROLLARY 2.3. Consider any complex polynomial $p_n(t) = \sum_{k=s}^{n} a_k(1+t)^k$ where $0 < s \leq n$, and where $p_n(t)$ is not identically zero. Forming the infinite sequence

$$\left\{ w_i(t) = \left(p_n(t)/\|p_n\|_{[-1,+1]}\right)^i \right\}_{i=1}^{\infty}, \tag{5.7}$$

this sequence then satisfies the hypotheses of Theorem 2.2 with $\theta = s/n$ and $n_i = ni$ for all $i \geq 1$. Applying (2.6) of Theorem 2.2, with $t_0 = t_1 = t$ directly gives

$$|p_n(t)| \leq \|p_n\|_{[-1,+1]}\{G(t, s/n)\}^n < \|p_n\|_{[-1,+1]} \tag{5.8}$$

for any t with $-r(s/n) < t < 2(s/n)^2 - 1$. □

PROOF OF COROLLARY 2.4. Consider the infinite sequence of complex polynomials

$$q_{i+1}(t) := p_{i+1}(t) - p_i(t), \quad n = 1, 2, \ldots, \text{ where } q_{i+1} \in \pi_{n_{i+1}}. \tag{5.9}$$

From (2.8), we see that $q_{i+1}^{(k)}(-1) = (p_{i+1}^{(k)}(-1) - f^{(k)}(-1)) - (p_i^{(k)}(-1) - f^{(k)}(-1)) = 0$ for $0 \leq k \leq \min(s_i, s_{i+1}) =: \tilde{s}_{i+1}$, and, from (2.9), that $\liminf_{i\to\infty}(\tilde{s}_i/n_i) \geq \theta$. Thus, with (2.10), $\{q_{i+1}(t)\}_{i=1}^{\infty}$ is seen to satisfy the hypotheses of Theorem 2.2. Hence, from (2.6), for any closed subinterval $[t_0, t_1]$ of $(-r(\theta), 2\theta^2 - 1)$, and for every $\varepsilon > 0$ sufficiently small, there is an i_0 sufficiently large such that

$$\|q_{i+1}(t)\|_{[t_0,t_1]} \leq \left(\max_{j=0,1} \{G(t_j, \theta)\} + \varepsilon \right)^{n_i+1} =: \gamma^{n_i+1} < 1 \qquad \text{for all } i \geq i_0. \tag{5.10}$$

This geometric convergence, as is well known (cf. Walsh [7, p. 80]), implies that

$$F(t) := p_{i_0}(t) + \sum_{j=i_0}^{\infty} \left(p_{n_{j+1}}(t) - p_{n_j}(t)\right) \tag{5.11}$$

is analytic at each point of the interval $(-r(\theta), 2\theta^2 - 1)$, and that $p_i(t) \to F(t)$ as $i \to \infty$, uniformly on each closed subinterval of $(-r(\theta), 2\theta^2 - 1)$. Thus, as $F^{(k)}(-1) = f^{(k)}(-1)$ for all $k \geq 0$, and as f is analytic at -1, we have $F(t) = f(t)$ for all $t \in (-r(\theta), 2\theta^2 - 1)$. Finally, (2.11) follows from (5.10). □

PROOF OF THEOREM 2.5. With the hypotheses of Theorem 2.5, set

$$Q_{i,0}(t) := (q_i(t) + q_i(-t))/2; \qquad Q_{i,1}(t) := (q_i(t) - Q_{i,0}(t))/t, \qquad i \geq 1, \tag{5.12}$$

so that every $Q_{i,j}$ is an even polynomial in t. Then, $q_i(t)$ can be expressed as

$$q_i(t) = Q_{i,0}(t) + tQ_{i,1}(t). \tag{5.13}$$

Setting

$$W_{i,j}(t) := Q_{i,j}\left(\sqrt{(1+t)/2}\right), \qquad j = 0, 1, i \geqslant 1, t \geqslant -1, \tag{5.14}$$

it is easily verified that the two sequences $\{W_{i,0}(t)\}_{i=1}^{\infty}$ and $\{W_{i,1}(t)\}_{i=1}^{\infty}$ each satisfy the hypotheses of Theorem 2.2. Hence, from (2.5) of Theorem 2.2, $W_{i,j}(t) \to 0$ as $i \to \infty$ for $-1 \leqslant t < 2\theta^2 - 1$, i.e., $Q_{i,j}(t) \to 0$ as $i \to \infty$ for $0 \leqslant t < \theta$. Since $Q_{i,j}(t)$ is even, then $Q_{i,j}(t) \to 0$ as $i \to \infty$ for $-\theta < t < \theta$, $j = 0, 1$, the convergence being uniform on each closed subinterval of $(-\theta, +\theta)$. Consequently, from (5.13), we have that $q_i(t) \to 0$ as $i \to \infty$ on each closed subinterval of $(-\theta, +\theta)$. Similarly, the geometric convergence rate (2.14) of Theorem 2.5 follows from (2.6) of Theorem 2.2. □

The sharpness of Theorem 2.5 follows by suitably modifying the sharpness examples of Theorem 2.2.

6. Endpoint behavior. With the hypotheses of Theorem 2.2, we have from (2.5) of Theorem 2.2 that $\lim_{i\to\infty} w_i(t) = 0$, uniformly on every closed subinterval of $(-r(\theta), 2\theta^2 - 1)$. However, there remains the possibility that pointwise convergence to zero *could* hold at the endpoints $t = -r(\theta)$ and $t = 2\theta^2 - 1$ of this interval, for every infinite sequence $\{w_i(t)\}_{i=1}^{\infty}$ satisfying the hypotheses of Theorem 2.2. By means of an explicit calculation, again using steepest descent methods, we show in Proposition 6.1 that this is *not* true in general.

First, when $t = 2\theta^2 - 1$, $0 < \theta < 1$, it follows from (4.7) that $\zeta^{\pm}(2\theta^2 - 1, \theta) = 2\theta - 1$. We know that $h'(2\theta - 1) = 0$, and from (4.10) that $h''(2\theta - 1) = 0$, but a short calculation shows, on differentiating $h''(\zeta)$ in (4.6), that

$$h^{(3)}(2\theta - 1) = -\frac{(1+\theta)}{4\theta^2(1-\theta)^3} < 0.$$

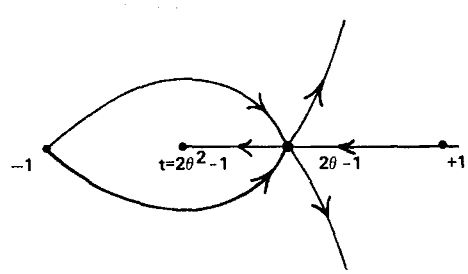

FIGURE 9. $t = 2\theta^2 - 1$

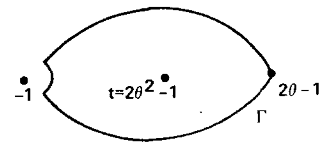

FIGURE 10. Contour for $t = 2\theta^2 - 1$

Because of this, the corresponding steepest descent and steepest ascent curves through $2\theta - 1$ are as shown in Figure 9, where again the arrows indicate the direction for increasing $\operatorname{Re} h(\zeta)$ along these curves. The modified contour Γ for use in (4.1) is illustrated in Figure 10. Note that these steepest descent and ascent curves through $\zeta = 2\theta - 1$ are separated by angles of $\pi/3$.

Again, a straightforward application of the steepest descent method gives that

$$P_n^{(\alpha,2\theta n/(1-\theta)+\beta)}(2\theta^2 - 1)$$
$$\sim \frac{g(2\theta-1)}{3} \cdot \left\{\frac{6}{n|h^{(3)}(2\theta-1)|}\right\}^{1/3} \Gamma\left(\frac{1}{3}\right) \exp\left\{nh(2\theta-1) + i\frac{4\pi}{3}\right\},$$

as $n \to \infty$, and on evaluating these coefficients from (4.3) and (4.4),

$$|P_n^{(\alpha,2\theta n/(1-\theta)+\beta)}(2\theta^2-1)| \sim \frac{M_1(\theta)}{n^{1/3}\theta^{2\theta n/(1-\theta)}}, \quad \text{as } n \to \infty,$$

where $M_1(\theta) \neq 0$ is independent of n. Next, as $(1+t)^{\theta n/(1-\theta)} = (2\theta^2)^{\theta n/(1-\theta)}$ since $t = 2\theta^2 - 1$, then

$$\left\{|(1+t)^{\theta n/(1-\theta)} P_n^{(\alpha,2\theta n/(1-\theta)+\beta)}(t)|\right\}_{t=2\theta^2-1} \sim \frac{2^{\theta n/(1-\theta)}}{n^{1/3}} M_1(\theta), \quad \text{as } n \to \infty.$$

Finally, from (1.9), we deduce that

$$h_n^{(\alpha,2\theta n/(1-\theta)+\beta)} \sim \frac{2^{2\theta n/(1-\theta)} M_2(\theta)}{n}, \quad \text{as } n \to \infty,$$

where $M_2(\theta) \neq 0$, so that

$$\frac{\left\{|(1+t)^{\theta n/(1-\theta)} P_n^{(\alpha,2\theta n/(1-\theta)+\beta)}(t)|\right\}_{t=2\theta^2-1}}{\sqrt{h_n^{(\alpha,2\theta n/(1-\theta)+\beta)}}} \sim n^{1/6} M_3(\theta), \quad \text{as } n \to \infty, \tag{6.1}$$

where $M_3(\theta) \neq 0$ is independent of n. Now, the sequence of incomplete polynomials

$$\left\{\tilde{w}_n(t) := \frac{(1+t)^{[\theta n/(1-\theta)]+1} P_n^{(-1/2,2\theta n/(1-\theta)-1/2)}(t)}{\sqrt{h_n^{(-1/2,2\theta n/(1-\theta)-1/2)}}}\right\}_{n=1}^{\infty} \tag{6.2}$$

can be shown to satisfy the hypotheses of Theorem 2.2 (where $[\mu]$ denotes the integer part of μ) for each θ with $0 < \theta < 1$, but from (6.1),

$$|\tilde{w}_n(2\theta^2 - 1)| \sim n^{1/6} M_4(\theta), \quad \text{as } n \to \infty, \tag{6.3}$$

where $M_4(\theta) \neq 0$ is independent of n. In a similar and easier fashion (because $h''(\zeta^-(r(\theta), \theta)) \neq 0$ from Lemma 4.1), it can be shown that

$$|\tilde{w}_n(-r(\theta))| \sim M_5(\theta), \quad \text{as } n \to \infty, \tag{6.4}$$

where $M_5(\theta) \neq 0$ is independent of n. This gives us

PROPOSITION 6.1. *For each θ with $0 < \theta < 1$, there is a sequence of incomplete polynomials $\{\tilde{w}_n(t)\}_{n=1}^{\infty}$ satisfying the hypotheses of Theorem 2.2 for which*

$$\lim_{n\to\infty} |\tilde{w}_n(-r(\theta))| = M_5(\theta) \neq 0; \qquad \lim_{n\to\infty} |\tilde{w}_n(2\theta^2 - 1)| = +\infty. \tag{6.5}$$

7. Complex extensions. We now sketch the complex extensions of Theorem 2.2 and Corollary 2.3, given as Theorem 2.6 and Corollary 2.7 in §2. As in §4, consider the infinite sequence $\{P_n^{(\alpha, 2\theta n/(1+\theta)+\beta)}(t)\}_{n=0}^{\infty}$ of Jacobi polynomials, where α, θ, and β are fixed real numbers satisfying $\alpha > -1$, $0 < \theta < 1$, and $\beta > -1$. With the integral representation of (4.1), now for any $-1 \neq t \in \mathbf{C} \setminus \mathbb{S}(\theta)$, we again write this integral representation as (cf. (4.2))

$$P_n^{(\alpha, 2\theta n/(1+\theta)+\beta)}(t) = \int_{\Gamma} e^{nh(\zeta)} g(\zeta)\, d\zeta, \tag{7.1}$$

and the saddle points $\zeta^{\pm}(t, \theta)$ for $h(\zeta)$ are given by (cf. (4.7))

$$\zeta^{\pm}(t, \theta) = \left\{ t + \theta \pm \sqrt{(1-t)(2\theta^2 - 1 - t)} \right\} / (1 + \theta). \tag{7.2}$$

As noted in §2, these saddle points are analytic in t in $\mathbf{C} \setminus \mathbb{S}(\theta)$.

Now, to the integral of (7.1), we again apply the steepest descent method, though in a form different from that of §4. Specifically, for any $-1 \neq t \in \mathbf{C} \setminus \mathbb{S}(\theta)$, consider the *level curves* T which separate the hills and valleys of $h(\zeta)$ with respect to $\zeta^-(t, \theta)$, i.e.,

$$T := \{\zeta \in \mathbf{C}: \operatorname{Re} h(\zeta) = \operatorname{Re} h(\zeta^-(t, \theta))\}. \tag{7.3}$$

Next, as in §4 it can be shown that $h''(\zeta^-(t, \theta)) \neq 0$ for all $-1 \neq t \in \mathbf{C} \setminus \mathbb{S}(\theta)$. Thus, these level curves intersect at right angles at $\zeta^-(t, \theta)$, and these level curves are separated by angles of $\pi/4$ at $\zeta^-(t, \theta)$ from the steepest descent and steepest ascent curves through $\zeta^-(t, \theta)$. Moreover, for any $-1 \neq t$ in $\mathring{\Lambda}(\theta)$ (cf. (2.18)), the level curves T consist of a simple loop about $\zeta = +1$, and a compound loop through $\zeta^-(t, \theta)$ which are pictured by the solid lines in Figure 11. In this case, $\operatorname{Re} h(\zeta^-(t, \theta)) < \operatorname{Re} h(\zeta^+(t, \theta))$. Note from Figure 11 that the region enclosed by the inner loop of the level curve T through $\zeta^-(t, \theta)$ contains $\zeta = t$, and is a *hill* for $h(\zeta)$. On the other hand, the region between the inner and outer loops of the level curves T through $\zeta^-(t, \theta)$ contains $\zeta = -1$, and this region is hence a *valley* for $h(\zeta)$. This further implies that descent curves through $\zeta^-(t, \theta)$ can be found in this valley, which are then used to define a contour Γ through $\zeta^-(t, \theta)$, as shown by the dotted lines in Figure 11.

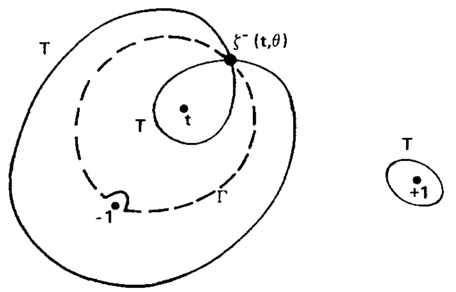

FIGURE 11. Level curves T for $-1 \neq t \in \mathring{\Lambda}(\theta)$

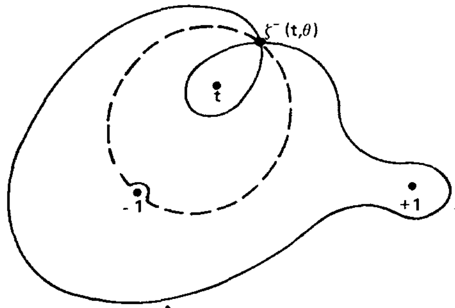

FIGURE 12. Level curves T for t outside $\Lambda(\theta)$

For t outside of $\Lambda(\theta)$ (cf. (2.17)), the level curves T are pictured in Figure 12. In this case, $\operatorname{Re} h(\zeta^-(t, \theta)) > \operatorname{Re} h(\zeta^+(t, \theta))$. Again, the region enclosed by the inner loop of the level curves T through $\zeta^-(t, \theta)$ contains $\zeta = t$ and is a *hill* for $h(\zeta)$, while the region between the inner and outer loops of the level curves T through $\zeta^-(t, \theta)$ contains $\zeta = -1$ as well as $\zeta = +1$, and is a *valley* for $h(\zeta)$. As before, there are steepest descent curves through $\zeta^-(t, \theta)$ and $\zeta = -1$ in this valley which are used to define a contour Γ through $\zeta^-(t, \theta)$, as shown in Figure 12.

Finally, making use of these newly-defined contours Γ in this complex extension, it can be similarly shown, after some calculations, that (4.17) is again valid, i.e.,

$$\lim_{i \to \infty} \left\{ \frac{\left|(1+t)^{m_i} P_{n_i}^{(\alpha, 2m_i + \beta)}(t)\right|}{\sqrt{h_{n_i}^{(\alpha, 2m_i + \beta)}}} \right\}^{1/(m_i + n_i)} = G(t, \theta), \tag{7.4}$$

for any $t \in \mathbf{C} \setminus \mathbb{S}(\theta)$, where $\{(m_i, n_i)\}_{i=1}^{\infty}$ satisfies (4.16), and where $G(t, \theta)$ is defined in (2.16). Thus, as in §5, (7.4) can be used analogously to establish the complex extensions, Theorem 2.6 and Corollary 2.7.

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Added in Proof. The question discussed in §1 concerning the Weierstrass property for incomplete polynomials of type θ has been resolved by the authors; the property holds on every closed interval $[\lambda, 1]$ with $\lambda > 2\theta^2 - 1$. (See Internat. J. Math. and Math. Sci. **1** (1978), pp. 407–420.) More recently M. v. Golitschek obtained a different proof of this fact.

The authors have also found alternate proofs of the fundamental limit property (4.17) and its complex extension (7.4). These new proofs utilize a normal families argument instead of the steepest descent method of the present paper (cf. *Numerische Methoden der Approximationentheorie*, Band 4, pp. 281–298, ISNM Vol. 42, Birkhäuser Verlag, 1978). Lorentz and Kemperman have obtained an alternate proof of Corollary 2.3 and its complex extension Corollary 2.7 by using a Bernstein type inequality.

Generalizations of the results of the present paper to incomplete polynomials vanishing at *both* endpoints of the interval have recently been found by the authors together with Mr. M. Lachance.

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