TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 249, Number 1, April 1979

ON THE ZEROS OF JACOBI POLYNOMIALS $P_n^{(\alpha_n,\beta_n)}(x)$

BY

D. S. MOAK[1], E. B. SAFF[2] AND R. S. VARGA[3]

ABSTRACT. If r_n and s_n denote, respectively, the smallest and largest zeros of the Jacobi polynomial $P_n^{(\alpha_n,\beta_n)}$, where $\alpha_n > 1$, $\beta_n - 1$, and if $\lim_{n\to\infty} \alpha_n/(2n + \alpha_n + \beta_n + 1) = a$ and if $\lim_{n\to\infty} \beta_n/(2n + \alpha_n + \beta_n + 1) = b$, then the numbers $r_{a,b}$ and $s_{a,b}$ are determined where

$$\lim_{n\to\infty} r_n = r_{a,b}, \quad \lim_{n\to\infty} s_n = s_{a,b}.$$

Furthermore, the zeros of $\{P_n^{(\alpha_n,\beta_n)}(x)\}_{n=0}^{\infty}$ are dense in $[r_{a,b}, s_{a,b}]$.

While a great deal is known (see Szegö [**2**]) about the asymptotic behavior of the zeros of Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ for a fixed type (α, β), there do not appear in the literature results concerning the limiting behavior of zeros of sequences of Jacobi polynomials $\{P_n^{(\alpha_n,\beta_n)}(x)\}_{n=0}^{\infty}$ where α_n or β_n (or both) are allowed to grow with n. Results on this latter problem have application to the study of incomplete polynomials, as is discussed in *The sharpness of Lorentz's theorem on incomplete polynomials* [**1**]. The present note is used in that paper (cf. [**1**, Lemma 3.4]), and is published separately here because of its independent interest.

Because the polynomials $P_n^{(\alpha_n,\beta_n)}(x)$, $n = 0, 1, 2, \ldots$, are in general *not* orthogonal on $[-1, 1]$, our results are not as detailed as the known theorems for a fixed type (α, β). Of course, we do know that for $\alpha_n > -1$, $\beta_n > -1$ all the zeros of $P_n^{(\alpha_n,\beta_n)}(x)$ lie in the open interval $(-1, 1)$ and, using the Sturm Comparison Theory, we can easily prove

THEOREM 1. *Let r_n and s_n be, respectively, the smallest and largest zeros of the Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(x)$, where $\alpha_n > -1$, $\beta_n > -1$. Suppose that*

$$\lim_{n\to\infty} \frac{\alpha_n}{(2n + \alpha_n + \beta_n)} = a \quad \text{and} \quad \lim_{n\to\infty} \frac{\beta_n}{(2n + \alpha_n + \beta_n)} = b, \tag{1}$$

Received by the editors September 20, 1977.

AMS (MOS) subject classifications (1970). Primary 33A65; Secondary 33A70.

Key words and phrases. Jacobi polynomials, Sturm Comparison Theorem.

[1]Research supported by NSF Grant MCS 75-06687-UW 144-G244.

[2]Research supported in part by the Air Force Office of Scientific Research under Grant AFOSR-74-2688.

[3]Research supported in part by the Air Force Office of Scientific Research under Grant AFOSR-74-2729 and by the Energy Research and Development Administration (ERDA) under Grant EY-76-S-02-2075.

and set

$$r_{a,b} := b^2 - a^2 - \left[(a^2 + b^2 - 1)^2 - 4a^2b^2\right]^{1/2}, \tag{2}$$

$$s_{a,b} := b^2 - a^2 + \left[(a^2 + b^2 - 1)^2 - 4a^2b^2\right]^{1/2}. \tag{3}$$

Then,

$$\lim_{n\to\infty} r_n = r_{a,b} \quad \textit{and} \quad \lim_{n\to\infty} s_n = s_{a,b}. \tag{4}$$

Furthermore, the zeros of the sequence $\{P_n^{(\alpha_n,\beta_n)}(x)\}_{n=0}^{\infty}$ *are dense in the interval* $[r_{a,b}, s_{a,b}]$.

The proof requires the following known results related to the Sturm Comparison Theory: (cf. Szegö [**2**, pp. 19–20]):

LEMMA 2. *Let* $H(\theta)$ *be continuous on* (θ_1, θ_2) *and suppose that* $u(\theta)$ *satisfies* $u'' + H(\theta)u = 0$ *for* $\theta \in (\theta_1, \theta_2)$. *If* $H(\theta) \geqslant n > 0$ *on* (θ_1, θ_2), *then* $u(\theta)$ *has a zero in every subinterval of* (θ_1, θ_2) *of length* $\geqslant \pi/\sqrt{n}$.

LEMMA 3. *Let* $H(\theta)$ *be continuous and negative in* (θ_1, θ_2). *Then an arbitrary solution* $u(\theta)$ $(\not\equiv 0)$ *of* $u'' + H(\theta)u = 0$, *for which* $u(\theta) \to 0$ *if* $\theta \to \theta_2^-$, *cannot vanish in* $\theta_1 \leqslant \theta < \theta_2$.

PROOF OF THEOREM 1. As is known [**2**, p. 67], the function

$$u_n(\theta) := \left(\sin\frac{\theta}{2}\right)^{\alpha_n+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta_n+1/2} P_n^{(\alpha_n,\beta_n)}(\cos\theta) \tag{5}$$

satisfies the differential equation

$$d^2u/d\theta^2 + H_n(\theta)u = 0, \quad \text{for } 0 < \theta < \pi, \tag{6}$$

where

$$H_n(\theta) := \frac{1 - 4\alpha_n^2}{16\sin^2(\theta/2)} + \frac{1 - 4\beta_n^2}{16\cos^2(\theta/2)} + \left(n + \frac{\alpha_n + \beta_n + 1}{2}\right)^2. \tag{7}$$

It is convenient to rewrite $H_n(\theta)$ in the form

$$H_n(\theta) = \frac{-(2n + \alpha_n + \beta_n + 1)^2\cos^2\theta + 2(\beta_n^2 - \alpha_n^2)\cos\theta}{4(1 - \cos^2\theta)} + \frac{(2n + \alpha_n + \beta_n + 1)^2 + 1 - 2\alpha_n^2 - 2\beta_n^2}{4(1 - \cos^2\theta)}. \tag{8}$$

Notice that the numerator of $H_n(\theta)$ in (8) is, for $n \geqslant 1$, a quadratic in $x = \cos\theta$ having negative leading coefficient. The roots of this quadratic are:

$$x_n^{\pm} := \frac{\beta_n^2 - \alpha_n^2}{(2n + \alpha_n + \beta_n + 1)^2} \pm \left[1 + \frac{(\beta_n^2 - \alpha_n^2)^2}{(2n + \alpha_n + \beta_n + 1)^4} + \frac{1 - 2\alpha_n^2 - 2\beta_n^2}{(2n + \alpha_n + \beta_n + 1)^2}\right]^{1/2}. \tag{9}$$

Because $\alpha_n > -1$ and $\beta_n > -1$, then $2n + \alpha_n + \beta_n + 1 > 2n - 1 > 0$ for all $n \geqslant 1$, so that $\lim_{n\to\infty} 1/(2n + \alpha_n + \beta_n + 1)^2 = 0$. Thus, with (1), the roots of (9) approach

$$b^2 - a^2 \pm \left[1 + (b^2 - a^2)^2 - 2(a^2 + b^2)\right]^{1/2}, \tag{10}$$

which are precisely the numbers $r_{a,b}$ and $s_{a,b}$ defined in (2) and (3). Since $\alpha_n > -1$ and $\beta_n > -1$, it easily follows from (1) that $a, b \in [0, 1]$. Furthermore, from definitions (2) and (3), it can be verified that

$$-1 \leqslant r_{a,b} \leqslant s_{a,b} \leqslant 1, \tag{11}$$

and that

$$r_{a,b} = -1 \text{ iff } b = 0, \quad s_{a,b} = 1 \text{ iff } a = 0.$$

Returning to the differential equation (6), it follows from the above discussion that for each $\varepsilon > 0$ sufficiently small,

$$H_n(\theta) < 0 \text{ for } \cos\theta \in \begin{cases} (-1, r_{a,b} - \varepsilon), & \text{if } b > 0, \\ (s_{a,b} + \varepsilon, 1), & \text{if } a > 0, \end{cases} \tag{12}$$

provided that n is sufficiently large. Hence, by applying Lemma 3 to the function $u_n(\theta)$ in (5), we have $P_n^{(\alpha_n,\beta_n)}(x) \neq 0$ in $[-1, r_{a,b} - \varepsilon) \cup (s_{a,b} + \varepsilon, 1]$ for all n large. In terms of the largest and smallest zeros of $P_n^{(\alpha_n,\beta_n)}(x)$, this means that

$$r_{a,b} - \varepsilon \leqslant \liminf_{n\to\infty} r_n, \qquad \limsup_{n\to\infty} s_n \leqslant s_{a,b} + \varepsilon,$$

and letting $\varepsilon \to 0^+$ yields

$$r_{a,b} \leqslant \liminf_{n\to\infty} r_n, \qquad \limsup_{n\to\infty} s_n \leqslant s_{a,b}, \tag{13}$$

the inequalities (13) being valid even if a and/or b are zero.

Next, we consider the inequality

$$H_n(\theta) \geqslant n, \tag{14}$$

which, using (8), is equivalent to

$$(-A_n \cos^2\theta + B_n \cos\theta + C_n)/4(1 - \cos^2\theta) \geqslant 0, \tag{15}$$

where

$$A_n := (2n + \alpha_n + \beta_n + 1)^2 - 4n, \quad B_n := 2(\beta_n^2 - \alpha_n^2),$$
$$C_n := (2n + \alpha_n + \beta_n + 1)^2 + 1 - 2\alpha_n^2 - 2\beta_n^2 - 4n. \tag{16}$$

It is immediately verified from (1) that the roots of the quadratic numerator in (15) again approach the number $r_{a,b}$ and $s_{a,b}$ as $n \to \infty$. Consequently, for each $\varepsilon > 0$ sufficiently small,

$$H_n(\theta) \geqslant n \quad \text{for } \theta \in \left[\cos^{-1}(s_{a,b} - \varepsilon), \cos^{-1}(r_{a,b} + \varepsilon)\right], \tag{17}$$

provided that n is sufficiently large. Thus, by Lemma 2, the function $u_n(\theta)$ has zeros within $\pi/\sqrt{n}$ of each of the endpoints of the interval in (17), and so

$$\limsup_{n\to\infty} r_n \leqslant r_{a,b} + \varepsilon, \qquad \liminf_{n\to\infty} s_n \geqslant s_{a,b} - \varepsilon.$$

Letting $\varepsilon \to 0^+$ and using (13) we have proved (4).

The fact that the zeros of the sequence $\{P_n^{(\alpha_n,\beta_n)}(x)\}_{n=0}^{\infty}$ are dense in $[r_{a,b}, s_{a,b}]$ also follows from Lemma 2 and the previous discussion. □

As a special case of Theorem 1, we have

COROLLARY 1. *If α and β are finite such that $\lim_{n\to\infty} \alpha_n/n = \alpha$ and $\lim_{n\to\infty} \beta_n/n = \beta$, then the conclusions of Theorem 1 are valid with $a := \alpha/(2 + \alpha + \beta)$ and with $b := \beta/(2 + \alpha + \beta)$.*

We remark that Theorem 1 also includes cases where $\alpha_n/n \to +\infty$ and/or $\beta_n/n \to +\infty$.

REFERENCES

1. E. B. Saff and R. S. Varga, *The sharpness of Lorentz's theorem on incomplete polynomials*, Trans. Amer. Math. Soc. 249 (1979), 163–186.

2. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R. I., 1975.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA, TAMPA, FLORIDA 33620

DEPARTMENT OF MATHEMATICS, KENT STATE UNIVERSITY, KENT, OHIO 44242