POLYNOMIALS OF INTERPOLATION AND APPROXIMATION TO MEROMORPHIC FUNCTIONS[1]

BY

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1. **Introduction.** The object of the present paper is twofold: (1) to establish some general results concerning the convergence and divergence of certain sequences of polynomials which interpolate to a meromorphic function, and (2) to deduce from these results some properties of the polynomials of best uniform approximation to a meromorphic function.

Throughout the paper E shall denote a closed bounded point set in the z-plane whose complement K (with respect to the extended plane) is connected and regular in the sense that K possesses a classical Green's function $G(z)$ with pole at infinity. We let Γ_ρ ($\rho > 1$) denote generically the locus $G(z) = \log \rho$, and denote by E_ρ the closed interior of Γ_ρ. Furthermore we set $H(z) \equiv -G_x(z) + iG_y(z)$.

Suppose the function $f(z)$ is analytic on E_ρ ($\rho > 1$) except for a finite number of poles on Γ_ρ and let r be the highest order of the poles of $f(z)$ on Γ_ρ. Let $P_n(z)$ be the polynomial of degree n of best uniform approximation to $f(z)$ on E and set $E_n(f) \equiv [\max |f(z) - P_n(z)|; z \text{ on } E]$. In §2 and §3 we show that if the point set E has a smooth boundary and $f(z)$ has a pole of order r at some point of Γ_ρ which is not a critical point of the Green's function $G(z)$, then there exist constants A_1, A_2 such that

$$(1) \qquad 0 < A_1 \leqq n^{1-r}\rho^n E_n(f) \leqq A_2 < \infty, \qquad n > 0.$$

Inequalities (1) were established by Sewell [4, p. 178] for the special case where E is bounded by a single analytic Jordan curve. However, for this configuration $G(z)$ has no critical points and consequently Sewell's methods cannot be applied to the more general point sets E which we shall consider in this paper.

For the case where the function $f(z)$ has poles in critical points we show in §4 that the inequalities (1) are valid if the exponent of n is replaced by an integer λ which depends on the orders of the poles of $f(z)$ and on the multiplicities of the critical points of $G(z)$.

Furthermore, we extend to the geometric setting described above some results of P. Dienes [2, §114] on the divergence of the Taylor expansion of a meromorphic function. In particular, we deduce that the best approximating polynomials $P_n(z)$ can converge in at most a finite number of points exterior to Γ_ρ.

2. **Interpolating polynomials.** It is known [1, p. 74] that with the geometric conditions imposed on the point set E there exists a sequence of polynomials $\omega_n(z)$ which have respectively $n+1$ zeros and which satisfy for some Δ (>0)

$$
(2) \qquad \lim_{n \to \infty} |\omega_n(z)|^{1/n} = \Delta \exp G(z),
$$

uniformly for z on each closed bounded set in K. Concerning interpolation in the roots of such $\omega_n(z)$ we prove

THEOREM 1. *Let E, K, $G(z)$, and $H(z)$ be as described in §1, and suppose $\omega_n(z)$ is a sequence of polynomials having respectively $n+1$ zeros and satisfying (2). If $s_{n,r}(z)$ is the polynomial of degree n which interpolates to the function $(z-\alpha)^{-r}$ in the $n+1$ roots of $\omega_n(z)$, where r is a positive integer and α ($\neq \infty$) lies in K, then*

$$
(3) \qquad \lim_{n \to \infty} n^{-r} s_{n,r}(\alpha) = H(\alpha)^r / r!.
$$

Proof. We note that as a consequence of (2) the $\omega_n(z)$ have no finite limit point of zeros in K, and hence for n sufficiently large no point of interpolation coincides with α.

It is clear from the definition of $s_{n,r}(z)$ that

$$
(4) \qquad (z-\alpha)^{-r} - s_{n,r}(z) = \omega_n(z) p_{n,r}(z) / (z-\alpha)^r,
$$

where $p_{n,r}(z)$ is a polynomial of degree $r-1$. The equivalence of (4) with

$$
(5) \qquad s_{n,r}(z) = \omega_n(z)(\theta_n(z) - p_{n,r}(z)) / (z-\alpha)^r,
$$

where $\theta_n(z) \equiv 1/\omega_n(z)$, implies (since $s_{n,r}(z)$ has no finite pole) that $p_{n,r}(z)$ interpolates to $\theta_n(z)$ in the point α considered of multiplicity r. This property of interpolation is also possessed by the polynomial

$$
q_{n,r}(z) \equiv \sum_{\nu=0}^{r-1} \theta_n^{(\nu)}(\alpha)(z-\alpha)^\nu / \nu!,
$$

and since both $p_{n,r}(z)$ and $q_{n,r}(z)$ are of degree $r-1$ we have $p_{n,r}(z) \equiv q_{n,r}(z)$. From (5) it now follows that

$$
(6) \qquad (r!)n^{-r} s_{n,r}(\alpha) = n^{-r} \omega_n(\alpha) \theta_n^{(r)}(\alpha).
$$

To evaluate the limit as $n \to \infty$ of the right-hand member of (6) we first note that equation (2) implies

$$
(7) \qquad \lim_{n \to \infty} n^{-1} \omega_n'(z) \theta_n(z) = -H(z),
$$

uniformly on each compact subset of K.

If we assume that for some positive integer ν

$$(8) \qquad \lim_{n\to\infty} n^{-\nu}\omega_n(z)\theta_n^{(\nu)}(z) = H(z)^\nu,$$

uniformly on each compact subset of K, then the convergence of the derived sequence implies

$$\lim_{n\to\infty} n^{-1}[n^{-\nu}\omega_n(z)\theta_n^{(\nu+1)}(z) + n^{-\nu}\omega_n'(z)\theta_n^{(\nu)}(z)] = 0.$$

Since

$$\lim_{n\to\infty} (n^{-1}\omega_n'(z)\theta_n(z))(n^{-\nu}\omega_n(z)\theta_n^{(\nu)}(z)) = -H(z)^{\nu+1},$$

we thus obtain

$$\lim_{n\to\infty} n^{-\nu-1}\omega_n(z)\theta_n^{(\nu+1)}(z) = H(z)^{\nu+1},$$

uniformly on each compact subset of K.

From (7) it easily follows that equation (8) holds for $\nu = 1$. Hence, by induction, it holds for ν equal to any positive integer and so (3) is a consequence of (6) and (8).

Concerning the degree of approximation to $(z-\alpha)^{-r}$ on E we have

COROLLARY 1. *If* $|\omega_n(z)/\omega_n(\alpha)| \leq e_n$ *for* z *on* E, *then for each positive integer* r *there exists a constant* A *independent of* n *such that*

$$(9) \qquad |(z-\alpha)^{-r} - s_{n,r}(z)| \leq An^{r-1}\varepsilon_n, \qquad z \text{ on } E, \quad n > 0.$$

Proof. From equation (4) we have

$$(z-\alpha)^{-r} - s_{n,r}(z) = [\omega_n(z)/\omega_n(\alpha)] \sum_{\nu=0}^{r-1} \omega_n(\alpha)\theta_n^{(\nu)}(\alpha)(z-\alpha)^{\nu-r}/\nu!,$$

and from (8) there follows

$$(10) \qquad |\omega_n(\alpha)\theta_n^{(\nu)}(\alpha)| \leq A_0 n^{r-1}, \qquad \nu = 0, 1, \ldots, r-1.$$

Hence (9) is a consequence of (10) and the triangle inequality.

It is interesting to note from (4) that the polynomial $s_{n,r}(z)$, uniquely determined by the condition that it interpolate to $(z-\alpha)^{-r}$ in the $n+1$ roots of $\omega_n(z)$, also interpolates to $(z-\alpha)^{-r}$ in the roots (if any) of $p_{n,r}(z)$. Regarding the behavior of these additional points of interpolation we prove

COROLLARY 2. *Suppose* $H(\alpha) \neq 0$, *i.e.,* α *is not a critical point of the Green's function* $G(z)$. *Then for* n *sufficiently large each* $p_{n,r}(z)$ *has precisely* $r-1$ *zeros, and as* n *becomes infinite these zeros approach respectively the point* α.

Proof. From (8) and the representation

$$\omega_n(\alpha)p_{n,r}(z) = \sum_{\nu=0}^{r-1} \omega_n(\alpha)\theta_n^{(\nu)}(\alpha)(z-\alpha)^\nu/\nu!,$$

we deduce

$$(11) \qquad \lim_{n\to\infty} n^{1-r}\omega_n(\alpha)p_{n,r}(z) = H(\alpha)^{r-1}(z-\alpha)^{r-1}/(r-1)!,$$

uniformly for z on each compact subset of the plane. Since the right-hand member of (11) is not identically zero, Hurwitz's Theorem implies that for n sufficiently large each $p_{n,r}(z)$ has at least, and hence precisely, $r-1$ zeros; and these zeros approach the point α considered of multiplicity $r-1$.

Equation (11) is also useful in establishing the following result concerning the divergence of the $s_{n,r}(z)$ (compare [2, p. 480]):

COROLLARY 3. *If α lies on Γ_ρ, then*

$$(12) \qquad \lim_{n\to\infty} \frac{s_{n,r}(z)\omega_n(\alpha)}{n^{r-1}\omega_n(z)} = \frac{-H(\alpha)^{r-1}}{(r-1)!(z-\alpha)^r},$$

uniformly on each closed set exterior to Γ_ρ.

Here and below we interpret $H(\alpha)^0$ as unity.

Proof. Let $\rho < \tau < \infty$, and note that (2) implies $\omega_n(\alpha)/\omega_n(z) \to 0$ uniformly for z on Γ_τ. Hence from (11) and the equation

$$\frac{s_{n,r}(z)\omega_n(\alpha)}{n^{r-1}\omega_n(z)} = \frac{\omega_n(\alpha)}{\omega_n(z)n^{r-1}(z-\alpha)^r} - \frac{\omega_n(\alpha)p_{n,r}(z)}{n^{r-1}(z-\alpha)^r},$$

we deduce that (12) holds uniformly on Γ_τ. Since the functions which appear in (12) are analytic exterior to Γ_τ, even at infinity, the limit is uniform as well for z exterior to Γ_τ, and so the corollary follows from the arbitrariness of τ.

Theorem 1 easily extends to polynomials which interpolate to a meromorphic function of the form

$$f(z) = g(z) + \sum_{k=1}^{r} B_k(z-\alpha)^{-k},$$

where α lies on Γ_ρ and $g(z)$ is analytic on E_ρ. For suppose that the polynomials $\omega_n(z)$ of Theorem 1 have no limit point of zeros exterior to E, and let $L_n(f; z)$ and $L_n(g; z)$ be the polynomials of degree n which interpolate respectively to the functions $f(z)$ and $g(z)$ in the roots of $\omega_n(z)$. Then clearly

$$L_n(f; z) = L_n(g; z) + \sum_{k=1}^{r} B_k s_{n,k}(z),$$

and since the sequence $L_n(g; \alpha)$ converges [1, p. 154], we deduce from Theorem 1 that

$$\lim_{n\to\infty} n^{-r}L_n(f; \alpha) = B_r H(\alpha)^r/r!.$$

To study the case where $f(z)$ has a pole in more than one point on Γ_ρ we make the additional assumption that there exists a constant M such that

$$(13) \qquad |\omega_n(z)/\omega_n(t)| \leqq M/\rho^n, \qquad z \text{ on } E, \ t \text{ on } \Gamma_\rho.$$

It is known [3] that with certain smoothness conditions on the boundary of E, which are discussed in §3, $\omega_n(z)$ exist satisfying both (2) and (13). In particular

such $\omega_n(z)$ exist if E is a finite line segment or if E is bounded by a finite number of mutually exterior analytic Jordan curves.

THEOREM 2. *Suppose that the polynomials $\omega_n(z)$ of Theorem 1 satisfy* (13) *and have no limit point of zeros exterior to E. Let the function $f(z)$ be analytic on E_ρ ($\rho > 1$) except for a finite number of poles on Γ_ρ, and let r be the highest order of the poles of $f(z)$ on Γ_ρ. Then for the polynomials $L_n(f; z)$ of respective degrees n which interpolate to $f(z)$ in the roots of $\omega_n(z)$ we have*

(14) $$|f(z) - L_n(f; z)| \leqq An^{r-1}/\rho^n, \qquad z \text{ on } E.$$

Furthermore, if α is a point of Γ_ρ at which $f(z)$ has a pole of order r, then

(15) $$\lim_{n \to \infty} n^{-r} L_n(f; \alpha) = BH(\alpha)^r/r!,$$

where $B = \lim_{z \to \alpha} (z - \alpha)^r f(z)$.

Here and below constants A are independent of n and z and may change from one inequality to another.

Proof. Let $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_m$ be the distinct poles of $f(z)$ on Γ_ρ and write $f(z) = g(z) + \sum_1^m S_k(z)$, where $S_k(z)$ is the singular part of the pole α_k. Let $L_n(g; z)$, $L_n(S_1; z), \ldots, L_n(S_m; z)$ be the polynomials of degree n which interpolate respectively to the functions $g(z), S_1(z), \ldots, S_m(z)$ in the roots of $\omega_n(z)$. Since $g(z)$ is analytic on E_ρ we have [1, p. 154]

$$|g(z) - L_n(g; z)| \leqq A/\rho^n, \qquad z \text{ on } E,$$

and from (13) and Corollary 1 there follows

(16) $$|S_k(z) - L_n(S_k; z)| \leqq An^{r-1}/\rho^n, \qquad z \text{ on } E, k = 1, \ldots, m.$$

Hence the linearity of the interpolation operator together with the triangle inequality imply (14).

To prove the second part of the theorem we first note that by the remarks preceding the statement of Theorem 2 we have

$$\lim_{n \to \infty} n^{-r}[L_n(g; z) + L_n(S_1; \alpha)] = BH(\alpha)^r/r!.$$

It is clear from (16) that

$$|(z - \alpha_k)^r S_k(z) - (z - \alpha_k)^r L_n(S_k; z)| \leqq An^{r-1}/\rho^n, \qquad z \text{ on } E,$$

and since the function whose absolute value appears in the last inequality is a polynomial of degree $n + r$, the generalized Bernstein lemma [1, p. 77] implies that

$$|(\alpha - \alpha_k)^r S_k(\alpha) - (\alpha - \alpha_k)^r L_n(S_k; \alpha)| \leqq An^{r-1}.$$

Thus for $k = 2, \ldots, m$ we deduce $n^{-r} L_n(S_k; \alpha) \to 0$, and so (15) follows.

We remark that the second part of Theorem 2 generalizes a result of P. Dienes [2, p. 479] concerning the divergence of the Taylor expansion of a meromorphic function.

An extension of Corollary 3 is given in

THEOREM 3. *With the hypotheses of Theorem 2, let $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ be the poles of $f(z)$ on Γ_ρ which are of order r. Then the sequence $L_n(f; z)\omega_n(\alpha_1)/n^{r-1}\omega_n(z)$ forms a normal family in the exterior of Γ_ρ and each limit function of the sequence is of the form*

$$(17) \qquad \frac{-B_1 H(\alpha_1)^{r-1}}{(r-1)!(z-\alpha_1)} - \sum_{k=2}^{\nu} \frac{\lambda_k B_k H(\alpha_k)^{r-1}}{(r-1)!(z-\alpha_k)},$$

where $B_k = \lim_{z \to \alpha_k} (z-\alpha_k)^r f(z)$ and the set of possible constants λ_k is bounded.

Proof. Let $\alpha_{\nu+1}, \alpha_{\nu+2}, \ldots, \alpha_m$ be the poles of $f(z)$ on Γ_ρ which are of order less than r and write $f(z) = g(z) + \sum_{k=1}^{m} S_k(z)$, where $S_k(z)$ is the singular part of the pole α_k.

Let $\rho < \tau < \infty$, and note that since $g(z)$ is analytic on E_ρ there exists a constant μ, $\rho < \mu < \tau$, such that the interpolating polynomials $L_n(g; z)$ are uniformly bounded on Γ_μ. Thus from the fact [1, p. 67] that $\Gamma_\tau = (\Gamma_\mu)_{\tau/\mu}$ and the Bernstein lemma we have $|L_n(g; z)| \le A\tau^n/\mu^n$ for z on Γ_τ. There now follows from (2)

$$\limsup_{n \to \infty} [\max |L_n(g; z)\omega_n(\alpha_1)/n^{r-1}\omega_n(z)| ; z \text{ on } \Gamma_\tau]^{1/n} \le \rho/\mu,$$

and since $\tau \, (> \rho)$ is arbitrary we have $L_n(g; z)\omega_n(\alpha_1)/n^{r-1}\omega_n(z) \to 0$ uniformly for z on each closed set exterior to Γ_ρ.

It is clear from Corollary 3 that

$$\lim_{n \to \infty} \frac{L_n(S_k; z)\omega_n(\alpha_k)}{n^{r-1}\omega_n(z)} = \frac{-B_k H(\alpha_k)^{r-1}}{(r-1)!(z-\alpha_k)}, \qquad k = 1, 2, \ldots, \nu,$$

and since (13) implies that $|\omega_n(\alpha_1)/\omega_n(z)| \le M\rho$ for z on Γ_ρ, we also have

$$\lim_{n \to \infty} \frac{L_n(S_k; z)\omega_n(\alpha_1)}{n^{r-1}\omega_n(z)} = 0, \qquad k = \nu+1, \ldots, m;$$

the limits being uniform on each closed set exterior to Γ_ρ.

Now given any increasing sequence of positive integers we can extract a subsequence n_i such that

$$\lim_{i \to \infty} \omega_{n_i}(\alpha_1)/\omega_{n_i}(\alpha_k) = \lambda_k, \qquad k = 2, \ldots, \nu,$$

and thus

$$\lim_{i \to \infty} \frac{L_{n_i}(S_k; z)\omega_{n_i}(\alpha_1)}{n_i^{r-1}\omega_{n_i}(z)} = \frac{-\lambda_k B_k H(\alpha_k)^{r-1}}{(r-1)!(z-\alpha_k)}, \qquad k = 2, \ldots, \nu,$$

uniformly for z on each closed set exterior to Γ_ρ. Hence from the linearity of the process of interpolation it follows that each subsequence of $L_n(f; z)\omega_n(\alpha_1)/n^{r-1}\omega_n(z)$ possesses a subsequence which converges uniformly on each closed set exterior to Γ_ρ to a function of the form (17), where the constants λ_k are bounded by $M\rho$.

It is easy to see from the proof of Corollary 3 that if $r > 1$ and $|\omega_n(\alpha)/\omega_n(z)| \leq M_0$ for z on Γ_ρ, then the limit (12) also holds for each z ($\neq \alpha$) on Γ_ρ. Hence a slight modification of the proof of Theorem 3 shows that if $r > 1$, then every subsequence of $L_n(f; z)\omega_n(\alpha_1)/n^{r-1}\omega_n(z)$ possesses a subsequence that converges at each point z on or exterior to Γ_ρ, z not a pole of $f(z)$ on Γ_ρ, to a function of the form (17).

COROLLARY 4. *Suppose, in addition to the hypotheses of Theorem 2, that α_1 is the only pole of highest order r of $f(z)$ on Γ_ρ. Then*

$$\lim_{n \to \infty} \frac{L_n(f; z)\omega_n(\alpha_1)}{n^{r-1}\omega_n(z)} = \frac{-B_1 H(\alpha_1)^{r-1}}{(r-1)!(z-\alpha_1)},$$

uniformly for z on each closed set exterior to Γ_ρ.

3. **Approximating polynomials.** We now apply the results of §2 to deduce properties of certain sequences of polynomials that approximate a meromorphic function on the point set E. Such applications will be justified if we assume that the boundary of E consists of curves of the type described in the following

DEFINITION. A Jordan curve is said to be *of class* A if it can be represented parametrically in terms of arc length s by $x = x(s)$, $y = y(s)$, where $x(s)$ and $y(s)$ possess second derivatives with respect to s which satisfy a Lipschitz condition of some positive order in s.

THEOREM 4. *Let E be the closed interior of the mutually exterior Jordan curves $C_1, C_2, \ldots, C_m$, where each C_j is of class A. Suppose $f(z)$ is analytic on E_ρ ($\rho > 1$) except for a finite number of poles on Γ_ρ and let r be the highest order of the poles of $f(z)$ on Γ_ρ. Then there exists a sequence of polynomials $p_n(z)$ of respective degrees n which satisfy*

$$(18) \qquad |f(z) - p_n(z)| \leq An^{r-1}/\rho^n, \qquad z \text{ on } E, \quad n > 0.$$

Furthermore, if $\alpha \in \Gamma_\rho$ is a pole of $f(z)$ of order r and $p_n(z)$ is any sequence of polynomials of respective degrees n which satisfy (18), then

$$(19) \qquad \lim_{n \to \infty} n^{-r}p_n(\alpha) = BH(\alpha)^r/r!,$$

where $B = \lim_{z \to \alpha} (z - \alpha)^r f(z)$.

Proof. Walsh has shown [3, Theorem 1] that for such a point set E there exist polynomials $\omega_n(z)$ which have respectively $n+1$ roots and satisfy

$$(20) \qquad |\omega_n(z)| \leq M_1 e^{ng}, \qquad z \text{ on } E,$$

$$(21) \qquad |G(z) + g - n^{-1}\log|\omega_n(z)|\,| \leq M_2 n^{-1},$$

for z on each compact set exterior to E, where e^g is the capacity of E. The roots of the $\omega_n(z)$ are chosen so as to be partitioned among the C_j and equally distributed on each C_j.

It is easy to see from (20) and (21) that the $\omega_n(z)$ satisfy both (2) and (13) and hence, by Theorem 2, the polynomials $L_n(f; z)$ which interpolate to $f(z)$ in the roots of $\omega_n(z)$ satisfy (18).

To prove the second part of the theorem assume that polynomials $p_n(z)$ of respective degrees n satisfy (18). The triangle inequality implies

$$(22) \qquad |p_n(z) - L_n(f; z)| \leqq A_1 n^{r-1}/\rho^n, \qquad z \text{ on } E,$$

and so from the Bernstein lemma we obtain

$$|p_n(\alpha) - L_n(f; \alpha)| \leqq A_1 n^{r-1}.$$

Thus $n^{-r} p_n(\alpha) - n^{-r} L_n(f; \alpha) \to 0$, and so (19) follows from (15).

As a consequence of Theorem 3 we establish

THEOREM 5. *Let E, $f(z)$, and r be as in Theorem 4 and let K_ρ denote the set of points which lie on or exterior to Γ_ρ. Suppose the polynomials $p_n(z)$ of respective degrees n satisfy* (18). *If $r > 1$ and if $f(z)$ has a pole of order r at some point α_1 on Γ_ρ which is not a critical point of the Green's function $G(z)$, then*

(i) *Except for a finite number of points the inequality*

$$(23) \qquad \limsup_{n \to \infty} \frac{|p_n(z)|\rho^n}{n^{r-1} \exp(nG(z))} > 0$$

holds for each z on K_ρ.

(ii) *There exists a neighborhood Δ of α_1 such that*

$$(24) \qquad \liminf_{n \to \infty} \frac{|p_n(z)|\rho^n}{n^{r-1} \exp(nG(z))} > 0, \qquad z \text{ in } \Delta \cap K_\rho.$$

The $p_n(z)$ need not be defined for every n.

Conclusion (i) implies that the set of points z in K_ρ for which the sequence $p_n(z)$ converges is finite, and conclusion (ii) implies that $p_n(z) \to \infty$ for z in $\Delta \cap K_\rho$.

Proof. As in the proof of Theorem 4 let $L_n(f; z)$ denote the polynomials of respective degrees n which interpolate to $f(z)$ in the $n+1$ roots of the polynomials $\omega_n(z)$ which satisfy (20) and (21). For $\mu > 1$ we obtain from (20), (21), and (22)

$$|\omega_n(\alpha_1)/\omega_n(z)| \leqq M_0 \rho^n/\mu^n, \qquad z \text{ on } \Gamma_\mu,$$

$$|p_n(z) - L_n(f; z)| \leqq A_1 n^{r-1} \mu^n/\rho^n, \qquad z \text{ on } \Gamma_\mu,$$

and so for z on or exterior to Γ_μ the sequence $(p_n(z) - L_n(f; z))\omega_n(\alpha_1)/n^{r-1}\omega_n(z)$ is uniformly bounded, and therefore forms a normal family in the exterior K of E. Thus by Theorem 3 the sequence $p_n(z)\omega_n(\alpha_1)/n^{r-1}\omega_n(z)$ forms a normal family in the exterior of Γ_ρ and each limit function of the sequence is of the form

$$(25) \qquad h(z) - \frac{B_1 H(\alpha_1)^{r-1}}{(r-1)!(z-\alpha_1)} - \sum_{k=2}^{\nu} \frac{\lambda_k B_k H(\alpha_k)^{r-1}}{(r-1)!(z-\alpha_k)},$$

where $h(z)$ is analytic in K, and $\alpha_1, \alpha_2, \ldots, \alpha_v$ are the poles of $f(z)$ on Γ_ρ which are of order r.

Now consider any such limit function $\Phi(z)$. Since $B_1 H(\alpha_1) \neq 0$, $\Phi(z)$ has at most a finite number of zeros in K_ρ. Furthermore by the remarks following the proof of Theorem 3 there exists a subsequence $p_j(z)\omega_j(\alpha_1)/j^{r-1}\omega_j(z)$ which converges to $\Phi(z)$ in $K_\rho - \{\text{poles of } f(z) \text{ on } \Gamma_\rho\}$. Thus if $z \in K_\rho$ is neither a pole of $f(z)$ on Γ_ρ nor a zero of $\Phi(z)$, then

$$0 < \lim_{j \to \infty} \left| \frac{p_j(z)\omega_j(\alpha_1)}{j^{r-1}\omega_j(z)} \right| \leq \lim_{j \to \infty} \sup \frac{|p_j(z)| A\rho^j}{j^{r-1} \exp\,(jG(z))},$$

which proves (i).

It is easy to see from (25) and the proof of Theorem 3 that there exists a neighborhood Δ of α_1 and a positive constant σ such that all limit functions $\Phi(z)$ satisfy $|\Phi(z)| \geq \sigma$ for z in Δ. If we assume that α_1 is the only pole of $f(z)$ on Γ_ρ which lies in Δ, then for fixed z $(\neq \alpha_1)$ in $\Delta \cap K_\rho$ we have for n sufficiently large

$$|p_n(z)\omega_n(\alpha_1)/n^{r-1}\omega_n(z)| \geq \sigma/2 > 0,$$

which implies (24). It is obvious from equation (19) that (24) also holds for $z = \alpha_1$, and the proof is complete.

We now show that under certain conditions the degree of approximation indicated in (18) is best possible.

THEOREM 6. *Let E, $f(z)$, and r be as in Theorem 4 and let $P_n(z)$ be the polynomial of degree n which is of best uniform approximation to $f(z)$ on E. If $r > 1$ and if $f(z)$ has a pole of order r at some point of Γ_ρ which is not a critical point of $G(z)$, then*

$$(26) \qquad\qquad n^{1-r}\rho^n E_n(f) \geq A > 0, \qquad n > 0,$$

where $E_n(f) \equiv [\max |f(z) - P_n(z)| ; z \text{ on } E]$.

Proof. Assume to the contrary that there exists an increasing sequence of integers j for which $j^{1-r}\rho^j E_j(f) \to 0$. Let $q(z)$ be the monic polynomial of smallest degree such that $q(z)f(z)$ is analytic on E_ρ, and let $\pi_n(z)$ be the polynomial of degree n which is of best uniform approximation to $q(z)f(z)$ on E. From the inequality

$$\lim_{n \to \infty} \sup\,[\max |q(z)f(z) - \pi_n(z)| ; z \text{ on } E]^{1/n} < 1/\rho,$$

it is easy to see that

$$|q(z)P_j(z) - \pi_j(z)| \leq \sigma_j j^{r-1}/\rho^j, \qquad z \text{ on } E,$$

where $\sigma_j \to 0$. Hence the lemma of Bernstein implies

$$|q(z)P_j(z) - \pi_j(z)| \leq A\sigma_j j^{r-1}\tau^j/\rho^j, \qquad z \text{ on } \Gamma_\tau,$$

and so if we choose $\tau > \rho$ so that the $\pi_n(z)$ are uniformly bounded on Γ_τ, there follows

$$|P_j(z)|\rho^j/j^{r-1}\tau^j \leq A_1\sigma_j + A_2\rho^j/j^{r-1}\tau^j \,(\to 0), \qquad z \text{ on } \Gamma_\tau.$$

But by Theorem 4 the sequence $j^{1-r}\rho^j E_j(f)$ is bounded from above and so the last inequality contradicts conclusion (i) of Theorem 5.

It can be proved directly that if $r=1$, then inequality (26) holds without any restrictions on the location of the poles of $f(z)$ on Γ_ρ. We shall include this result as part of a generalization of Theorem 6 which is proved in the next section.

4. Poles in critical points. Theorem 6 suggests that the degree of approximation indicated in (18) may be improved if the poles of highest order of $f(z)$ coincide with critical points of the Green's function $G(z)$. That this is indeed the case follows from a closer look at inequality (21).

LEMMA 1. *Let the set E be as in Theorem 4 and suppose the point α in K is a v-fold critical point of $G(z)$, i.e., α is a zero of order v of $H(z)$. Suppose that $\omega_n(z)$ is a sequence of polynomials which satisfy* (21) *for z on each compact subset of K, and let $\theta_n(z) \equiv 1/\omega_n(z)$. Then for each fixed nonnegative integer t we have the inequalities*

$$(27) \qquad |\omega_n^{(t)}(\alpha)\theta_n(\alpha)| \leq An^\lambda,$$

$$(28) \qquad |\omega_n(\alpha)\theta_n^{(t)}(\alpha)| \leq An^\lambda,$$

where $\lambda = [t/(v+1)]$ is the greatest integer less than or equal to $t/(v+1)$.

Proof. In order for (21) to hold we make the tacit assumption that the $\omega_n(z)$ have no zeros in K, and hence the quantities whose absolute values appear in (27) and (28) do indeed exist.

We now proceed by induction on t. For $t=0$ inequalities (27) and (28) are obviously valid, so assume they hold for each integer t strictly less than the positive integer s.

In order to prove that (27) holds for $t=s$ we estimate the terms on the right-hand side of the identity

$$(29) \qquad \omega_n^{(s)}(\alpha)\theta_n(\alpha) = (\omega_n'(\alpha)\theta_n(\alpha))^{(s-1)} - \sum_{i=1}^{s-1} \binom{s-1}{i} \omega_n^{(s-i)}(\alpha)\theta_n^{(i)}(\alpha).$$

It is clear that in some closed disk $N(\alpha; \delta)$ with center α and radius δ (>0) the function whose absolute value appears in (21) is the real part of an analytic function $f_n(z)$ whose derivative is $-H(z)-n^{-1}\omega_n'(z)\theta_n(z)$. From (21) we obtain

$$0 < A_2 \leq |\exp(nf_n(z))| \leq A_1, \qquad z \text{ in } N(\alpha; \delta),$$

and hence from Cauchy's inequality there follows

$$A_2 n|f_n'(z)| \leq |nf_n'(z) \exp(nf_n(z))| \leq A_3, \qquad z \text{ in } N(\alpha; \delta/2).$$

The last inequalities imply

$$|H(z)+n^{-1}\omega_n'(z)\theta_n(z)| \leq A_4 n^{-1}, \qquad z \text{ in } N(\alpha; \delta/2),$$

and so for fixed $k \geq 0$ we have

$$(30) \qquad |H^{(k)}(\alpha)+n^{-1}(\omega_n'(\alpha)\theta_n(\alpha))^{(k)}| \leq An^{-1}.$$

Now since $H^{(k)}(\alpha) = 0$ for $k = 0, 1, \ldots, \nu - 1$, there follows

$$|(\omega_n'(\alpha)\theta_n(\alpha))^{(k)}| \leq M, \qquad 0 \leq k \leq \nu - 1,$$

$$|(\omega_n'(\alpha)\theta_n(\alpha))^{(k)}| \leq Mn, \qquad \nu \leq k,$$

and hence if $\lambda \equiv [s/(\nu+1)]$ we have $|(\omega_n'(\alpha)\theta_n(\alpha))^{(s-1)}| \leq Mn^\lambda$.

To estimate the remaining terms on the right-hand side of (29) we let $1 \leq i \leq s - 1$ and set $\lambda_1 \equiv [(s-i)/(\nu+1)]$, $\lambda_2 \equiv [i/(\nu+1)]$. Then by the induction hypothesis we have

$$|\omega_n^{(s-i)}(\alpha)\theta_n^{(i)}(\alpha)| = |\omega_n^{(s-i)}(\alpha)\theta_n(\alpha)| \, |\omega_n(\alpha)\theta_n^{(i)}(\alpha)| \leq An^{\lambda_1 + \lambda_2} \leq An^\lambda,$$

and so (27) follows for $t = s$.

It is obvious from the identity

$$(31) \qquad \omega_n(\alpha)\theta_n^{(s)}(\alpha) = -\omega_n^{(s)}(\alpha)\theta_n(\alpha) - \sum_{i=1}^{s-1} \binom{s}{i} \omega_n^{(s-i)}(\alpha)\theta_n^{(i)}(\alpha),$$

that (28) also holds for $t = s$ and the lemma is proved.

From inequality (28) and the methods used to prove Corollary 1 and Theorem 2 we obtain

THEOREM 7. *Let E be the closed interior of a finite number of mutually exterior Jordan curves of class A, and let the sequence of polynomials $\omega_n(z)$ which have respectively $n + 1$ roots satisfy (20) and (21). Suppose the function $f(z)$ is analytic on E_ρ except for poles in the distinct points $\alpha_1, \alpha_2, \ldots, \alpha_m$ on Γ_ρ. Let r_k be the order of the pole of $f(z)$ at α_k and suppose $G(z)$ has a ν_k (≥ 0)-fold critical point at α_k. Then the polynomials $L_n(f; z)$ of respective degrees n which interpolate to $f(z)$ in the roots of $\omega_n(z)$ satisfy*

$$|f(z) - L_n(f; z)| \leq An^\lambda/\rho^n, \qquad z \text{ on } E,$$

where $\lambda = \max\{[(r_k-1)/(\nu_k+1)]; k = 1, \ldots, m\}$.

To obtain an extension of Theorem 3 we prove

LEMMA 2. *With the hypotheses of Lemma 1, let $\mu \equiv \nu + 1$ and set $\Gamma(\alpha, \nu, k) \equiv (k\mu - 1)! \mu (H^{(\nu)}(\alpha))^k/(k-1)!(\mu!)^k$. Then for each positive integer k we have*

$$(32) \qquad \lim_{n \to \infty} n^{-k}\omega_n^{(k\mu)}(\alpha)\theta_n(\alpha) = (-1)^k \Gamma(\alpha, \nu, k),$$

$$(33) \qquad \lim_{n \to \infty} n^{-k}\omega_n(\alpha)\theta_n^{(k\mu)}(\alpha) = \Gamma(\alpha, \nu, k).$$

Proof. We proceed by induction on k. Lemma 1 implies that $|\omega_n^{(\mu-i)}(\alpha)\theta_n^{(i)}(\alpha)| \leq A$ for $i = 1, \ldots, \nu$, and from inequality (30) we have $n^{-1}(\omega_n'(\alpha)\theta_n(\alpha))^{(\nu)} \to -H^{(\nu)}(\alpha)$. Hence from the identities (29) and (31), with s replaced by μ, it follows that (32) and (33) hold for $k = 1$.

Now suppose the lemma is valid for $k=1, 2, \ldots, r-1$, and consider the identity (29) with s replaced by $r\mu$. If $i=j\mu+q$ for some integers j and q which satisfy $0 \leq j \leq r-1$, $1 \leq q \leq \nu$, then Lemma 1 implies that $|\omega_n^{(r\mu-i)}(\alpha)\theta_n^{(i)}(\alpha)| \leq An^{r-1}$, and so

$$\lim_{n \to \infty} n^{-r}\omega_n^{(r\mu-i)}(\alpha)\theta_n^{(i)}(\alpha) = 0.$$

If $i=j\mu$, where $1 \leq j \leq r-1$, it follows from the induction hypothesis that

$$\lim_{n \to \infty} n^{-r}\omega_n^{(r\mu-i)}(\alpha)\theta_n^{(i)}(\alpha) = (-1)^{r-j}\Gamma(\alpha, \nu, r-j)\Gamma(\alpha, \nu, j).$$

Therefore since inequality (30) implies that $n^{-r}(\omega_n'(\alpha)\theta_n(\alpha))^{(r\mu-1)} \to 0$, we deduce from (29)

$$(34) \qquad \lim_{n \to \infty} n^{-r}\omega_n^{(r\mu)}(\alpha)\theta_n(\alpha) = -\sum_{j=1}^{r-1} (-1)^{r-j}\binom{r\mu-1}{j\mu}\Gamma(\alpha, \nu, r-j)\Gamma(\alpha, \nu, j).$$

From the easily verified identity

$$\binom{r\mu-1}{j\mu}\Gamma(\alpha, \nu, r-j)\Gamma(\alpha, \nu, j) = \binom{r-1}{j}\Gamma(\alpha, \nu, r),$$

we see that the right-hand side of (34) reduces to $(-1)^r\Gamma(\alpha, \nu, r)$, and so (32) holds for $k=r$.

It follows from (31) that

$$\lim_{n \to \infty} n^{-r}\omega_n(\alpha)\theta_n^{(r\mu)}(\alpha) = (-1)^{r+1}\Gamma(\alpha, \nu, r)$$

$$-\sum_{j=1}^{r-1} (-1)^{r-j}\binom{r\mu}{j\mu}\Gamma(\alpha, \nu, r-j)\Gamma(\alpha, \nu, j),$$

and so from the identity

$$\binom{r\mu}{j\mu}\Gamma(\alpha, \nu, r-j)\Gamma(\alpha, \nu, j) = \binom{r}{j}\Gamma(\alpha, \nu, r)$$

we deduce that (33) also holds for $k=r$ and the lemma is proved.

THEOREM 8. *With the hypotheses of Theorem 7, let* $1, 2, \ldots, s$ *be the values of* k *for which* $\lambda=[(r_k-1)/(\nu_k+1)]$, *and write* $r_k-1=\lambda(\nu_k+1)+q_k$, *where* $0 \leq q_k \leq \nu_k$. *Then the sequence* $g_n(z) \equiv L_n(f; z)\omega_n(\alpha_1)/n^\lambda\omega_n(z)$ *forms a normal family in the exterior of* Γ_ρ, *and each limit function of the sequence is of the form*

$$(35) \qquad \frac{-B\Gamma(\alpha_1, \nu_1, \lambda)}{(\lambda\nu_1+\lambda)!(z-\alpha_1)^{q_1+1}} - \sum_{i=1}^{q_1} \frac{A_{1i}}{(z-\alpha_1)^i} - \sum_{k=2}^{s} \sum_{i=1}^{q_k+1} \frac{A_{ki}}{(z-\alpha_k)^i},$$

where $B=\lim_{z \to \alpha_1} (z-\alpha_1)^{r_1}f(z)$ *and the set of possible constants* A_{ki} *is bounded.*

Furthermore if $\lambda > 0$, *then each subsequence of* $g_n(z)$ *possesses a subsequence which converges to a function of the form* (35) *at each point* z *on or exterior to* Γ_ρ, z *not a pole of* $f(z)$ *on* Γ_ρ.

The proof of Theorem 8, which follows from Lemma 2 and the methods used in §2, is left to the reader.

Since the first term of expression (35) is not identically zero, the reasoning used to establish Theorem 5 yields

THEOREM 9. *Let E, $f(z)$, and λ be as in Theorem 7 and let D_ρ denote the exterior of Γ_ρ. If the polynomials $p_n(z)$ of respective degrees n satisfy*

$$(36) \qquad |f(z) - p_n(z)| \leqq An^\lambda/\rho^n, \qquad z \text{ on } E,$$

then

(i) *Except for a finite number of points the inequality*

$$\limsup_{n \to \infty} \frac{|p_n(z)|\rho^n}{n^\lambda \exp(nG(z))} > 0$$

holds for each z in D_ρ.

(ii) *If for the integers r_k and ν_k, which correspond to the pole α_k, we have $\lambda = [(r_k - 1)/(\nu_k + 1)]$, then there exists a neighborhood Δ of α_k such that*

$$\liminf_{n \to \infty} \frac{|p_n(z)|\rho^n}{n^\lambda \exp(nG(z))} > 0, \qquad z \text{ in } \Delta \cap D_\rho.$$

Furthermore if $\lambda > 0$, then conclusion (i) *holds with D_ρ replaced by $D_\rho + \Gamma_\rho$, and conclusion* (ii) *holds with D_ρ replaced by $D_\rho + \Gamma_\rho - \{\alpha_k\}$.*

The $p_n(z)$ need not be defined for every n.

We conclude with a result which states that the degree of approximation indicated in (36) is best possible.

THEOREM 10. *Let E, $f(z)$, and λ be as in Theorem 7, and let*

$$E_n(f) \equiv [\max |f(z) - P_n(z)| ; z \text{ on } E],$$

where $P_n(z)$ is the polynomial of degree n which is of best uniform approximation to $f(z)$ on E. Then there exist constants A_1 and A_2 such that

$$0 < A_2 \leqq n^{-\lambda}\rho^n E_n(f) \leqq A_1 < \infty, \qquad n > 0.$$

The fact that the sequence $\varepsilon_n \equiv n^{-\lambda}\rho^n E_n(f)$ is bounded from above follows immediately from Theorem 7. To establish that $\varepsilon_n \geqq A_2 > 0$ one shows that the contrary assumption would violate conclusion (i) of Theorem 9.

We remark that the results of §3 and §4, which apply to the polynomials of best approximation in the Tchebycheff norm, can be extended so as to apply to polynomials of best approximation in other norms, such as the qth power line integral norm taken over the boundary of E. These extensions are reserved for another occasion.

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