On the Eneström-Kakeya Theorem and Its Sharpness

N. Anderson*
Department of Mathematics
Kent State University
Kent, Ohio 44242

E. B. Saff†
Department of Mathematics
University of South Florida
Tampa, Florida 33620

and

R. S. Varga‡
Department of Mathematics
Kent State University
Kent, Ohio 44242

Dedicated to Alston S. Householder on his seventy-fifth birthday.

Submitted by Hans Schneider

ABSTRACT

A new proof, based on the Perron-Frobenius theory of nonnegative matrices, is given of a result of Hurwitz on the sharpness of the classical Eneström-Kakeya theorem for estimating the moduli of the zeros of a polynomial with positive real coefficients. It is then shown (Theorem 2) that the zeros of a particular set of polynomials fill out the Eneström-Kakeya annulus in a precise manner, and this is illustrated by numerical results in Fig. 1.

1. INTRODUCTION

The classical theorem due to Eneström [1] and Kakeya [4] for finding bounds for the moduli of the zeros of polynomials having positive real

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coefficients is often stated as (cf. Henrici [2, p. 462], Marden [6, p. 136], and Pólya-Szegö [8, p. 107])

THEOREM A (Eneström-Kakeya). *Let $p_n(z)=\sum_{j=0}^{n} a_j z^j$ be any polynomial whose coefficients satisfy*

$$a_0 \geqslant a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n > 0.$$

Then $p_n(z)$ has no zeros in the open unit disk $\{z\in\mathbb{C}:|z|<1\}$.

An equivalent, but perhaps more useful, statement of the above theorem, due in fact to Eneström [1], is the following:

THEOREM B. *Let $p_n(z)=\sum_{j=0}^{n} a_j z^j$, $n\geqslant 1$, be any polynomial with $a_i>0$ for all $0\leqslant i\leqslant n$. Setting*

$$\alpha = \alpha[p_n] := \min_{0\leqslant i<n}\{a_i/a_{i+1}\}, \qquad \beta = \beta[p_n] := \max_{0\leqslant i<n}\{a_i/a_{i+1}\}, \tag{1}$$

then all the zeros of p_n are contained in the annulus

$$\alpha \leqslant |z| \leqslant \beta. \tag{2}$$

An obvious question that can be asked is whether both inequalities of (2) of Theorem B are *sharp*, in the sense that polynomials with positive coefficents can be found having zeros either on $|z|=\alpha$ or on $|z|=\beta$. Hurwitz [3] answered this question affirmatively over sixty years ago, and showed moreover that such extremal polynomials have a very special characterization. What will be shown here in Sec. 2 is that this special characterization (Theorem 1) can be established by means of the Perron-Frobenius theory of nonnegative matrices, thereby affording an alternative proof of this classical analysis result. It should be remarked that Theorem 1, to be given below, slightly extends (and corrects) the original result of Hurwitz, as well as more recent results of Tomič [10] and Ostrowski [7]. In Sec. 3, it will then be shown (in Theorem 2) that the zeros of a particular set of polynomials fill out the Eneström-Kakeya annulus (2) in a precise manner. The remainder of this section gives some needed notation.

For every nonnegative integer n, define

$$\pi_n^+ := \left\{ p_n(z) = \sum_{j=0}^{n} a_j z^j : a_j > 0 \text{ for all } 0 \leqslant j \leqslant n \right\}. \tag{3}$$

If $p_n \in \pi_n^+$, then set

$$\bar{S} = \bar{S}(p_n) := \{j = 1, 2, \ldots, n+1 : \beta a_{n+1-j} - a_{n-j} > 0\}, \quad \text{where} \quad a_{-1} := 0,$$

$$\underline{S} = \underline{S}(p_n) := \{j = 1, 2, \ldots, n+1 : a_{j-1} - \alpha a_j > 0\}, \quad \text{where} \quad a_{n+1} := 0, \quad (4)$$

where α and β for p_n are defined in (1). Note that these sets are *nonempty*, since $n+1$ is an element of both sets. Also associated with $p_n \in \pi_n^+$ are the positive integers

$$\bar{k} = \bar{k}(p_n) := \text{g.c.d. } \{j : j \in \bar{S}\},$$
$$\underline{k} = \underline{k}(p_n) := \text{g.c.d. } \{j : j \in \underline{S}\}. \quad (5)$$

2. THEOREM 1

With the notation of Sec. 1, we now establish

THEOREM 1. *For any $p_n \in \pi_n^+$ with $n \geqslant 1$, all the zeros of p_n lie in the annulus* (*cf.* (1))

$$\alpha \leqslant |z| \leqslant \beta. \quad (6)$$

Moreover, p_n can vanish on $|z| = \beta$ iff $\bar{k} > 1$ (cf. (5)). If $\bar{k} > 1$, the zeros of p_n on $|z| = \beta$ are simple and given precisely by

$$\{\beta \exp 2\pi i j / \bar{k} :\ j = 1, 2, \ldots, \bar{k} - 1\}, \quad (7)$$

and p_n has the form

$$p_n(\beta z) = \{1 + z + z^2 + \cdots + z^{\bar{k}-1}\} \cdot q_m(z^{\bar{k}}), \quad (8)$$

where $q_m \in \pi_m^+$. If $m \geqslant 1$, then all zeros of $q_m(w)$ lie in $|w| < 1$, and

$$\beta[q_m] \leqslant 1. \quad (9)$$

Similarly, p_n can vanish on $|z|=\alpha$ iff $\underline{k}>1$. If $\underline{k}>1$, the zeros of p_n on $|z|=\alpha$ are simple and given precisely by

$$\{\alpha \exp 2\pi i j/\underline{k}:\ j=1,2,\ldots,\underline{k}-1\},\tag{10}$$

and p_n has the form

$$z^n p_n(\alpha/z)=\{1+z+\cdots+z^{\underline{k}-1}\}r_m(z^{\underline{k}}),\tag{11}$$

where $r_m\in\pi_m^+$. If $m\geqslant 1$, then all the zeros of $r_m(w)$ lie in $|w|<1$ and $\beta[r_m]\leqslant 1$.

Proof. For any $p_n(z)=\sum_{j=0}^n a_j z^j$ in π_n^+, it can be verified from (1) that

$$\frac{-(1-z)p_n(\beta z)}{a_n\beta^n}=z^{n+1}-\sum_{j=1}^{n+1}c_j z^{n+1-j}=:\tilde{p}_{n+1}(z),\tag{12}$$

where

$$c_j=\begin{cases}\dfrac{\beta a_{n+1-j}-a_{n-j}}{a_n\beta^j}\geqslant 0 & \text{for}\quad 1\leqslant j\leqslant n,\\[2ex] \dfrac{a_0}{a_n\beta^n}>0 & \text{for}\quad j=n+1\end{cases}\tag{13}$$

so that $(1-z)p_n(\beta z)$ and $\tilde{p}_{n+1}(z)$ have the same zeros. The $(n+1)\times(n+1)$ companion matrix B for $\tilde{p}_{n+1}$ is given by

$$B=\begin{bmatrix}0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & c_{n+1}\\ 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & c_n\\ 0 & 1 & \ddots & \cdot & \cdot & 0 & 0 & c_{n-1}\\ \vdots & \vdots & & \ddots & & & \vdots & \vdots\\ 0 & 0 & \cdot & \cdot & & \ddots & 0 & c_2\\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & c_1\end{bmatrix},\tag{14}$$

and, as the coefficients c_j from (13) and (1) are all nonnegative, then B is a nonnegative matrix. Moreover, since $c_{n+1}>0$ from (13), then B is *irreducible* (cf. [11, p. 20]). Next, because $z=1$ is a zero of $\tilde{p}_{n+1}$ from (12), it follows that the Perron eigenvalue of B is necessarily unity, so that (cf. [11, p. 30]) all the zeros of $\tilde{p}_{n+1}$ satisfy $|z| \leqslant 1$. Because of (12), all zeros of p_n then satisfy $|z| \leqslant \beta$, which establishes the second inequality of (6).

Next, suppose that p_n has a zero on $|z| = \beta$, which implies that $\tilde{p}_{n+1}$ has at least two zeros on $|z| = 1$. Equivalently, B has at least two eigenvalues of modulus unity. This can happen, from the Perron-Frobenius theory of nonnegative matrices (cf. [11, p. 35]) iff B is cyclic of some index $h > 1$. Next, using Theorem 2.9 of [11, p. 49], this cyclic index can be expressed as the greatest common divisor of the lengths of all closed paths in the directed graph for the matrix B which connect the vertex V_{n+1} to itself. From the structure of B in (14), it follows that any $c_j > 0$ gives rise to such a closed path, through the vertex V_{n+1}, of length precisely j. In other words, B is cyclic of index h where

$$h = \text{g.c.d.}\{ j : \; c_j > 0 \}. \tag{15}$$

But, from the definitions of (4) and (5), it can be verified that $h = \bar{k} = \bar{k}(p_n)$. Thus, p_n can vanish on $|z| = \beta$ iff $\bar{k} > 1$. Next, if $\bar{k} > 1$, then the eigenvalues of B having modulus equal to the Perron eigenvalue (which is unity) are $\bar{k}$ in number, are simple, and are given by (cf. [11, p. 38])

$$\{ \exp 2\pi i j / \bar{k} : \; j = 0, 1, 2, \ldots, \bar{k} - 1 \}. \tag{16}$$

Hence, with (12), the number of zeros of p_n on $|z| = \beta$ is $\bar{k} - 1$, and these zeros, from (16), are all simple, and are given precisely by (7). Finally, the cyclic nature of B implies that its characteristic polynomial, namely $\tilde{p}_{n+1}(z)$, must satisfy (cf. [11, p. 39])

$$\tilde{p}_{n+1}(z) = (1 - z^{\bar{k}}) \tilde{q}_m(z^{\bar{k}}),$$

where all the zeros of $\tilde{q}_m(w)$ lie in $|w| < 1$. On dividing the above equation by $1 - z$ and on recalling (12), then

$$p_n(\beta z) = (1 + z + \cdots + z^{\bar{k}-1}) q_m(z^{\bar{k}}), \tag{17}$$

where $q_m(w) = -a_n \beta^n \tilde{q}_m(w)$, which establishes (8).

Next, if $q_m(w) = \sum_{j=0}^{m} d_j w^j$, then from (17),

$$p_n(\beta z) = d_0(1 + z + \cdots + z^{\bar{k}-1}) + d_1(z^{\bar{k}} + \cdots + z^{2\bar{k}-1})$$
$$+ \cdots + d_m(z^{m\bar{k}} + \cdots + z^{(m+1)\bar{k}-1}). \tag{18}$$

Evidently, all the coefficients d_i must be positive, since, by hypothesis, $p_n \in \pi_n^+$, whence $q_m \in \pi_m^+$. If $m \geqslant 1$, the maximum ratio of successive coefficients in (18) is just

$$\max\left\{1; \max_{0 \leqslant i < m} [d_i/d_{i+1}]\right\} = 1,$$

the last equality following from the definition of β for p_n. Thus,

$$\beta[q_m(w)] := \max_{0 \leqslant i < m} [d_i/d_{i+1}] \leqslant 1,$$

which establishes (9). The remainder of the proof follows similarly upon considering the polynomial $z^n p_n(\alpha/z)$. ∎

The history concerning Theorem B and Theorem 1 is worth commenting on. Kakeya in [4] stated that strict inequality held throughout (2) in Theorem B, and this error was promptly pointed out by Kempner [5], who essentially deduced the sufficiency of $\bar{k} > 1$ in Theorem 1 for $p_n \in \pi_n^+$ to have zeros on $|z| = \beta$. Theorem 1 is, as previously stated, a slight extension of the result of Hurwitz [3], but it should be noted that Hurwitz incorrectly claimed (cf. [3, p. 92, line 1]) that $p_n \in \pi_n^+$ has zeros on, say, $|z| = \alpha$ iff the set $\underline{S}$ of (4) consists of *all* multiples $\underline{k}$, $2\underline{k}, \ldots, n+1$. This same mistake appears also in Marden [6, p. 138, Exercise 10]. That this need not be the case is illustrated in the example of (19) below. Next, Tomič [10, p. 149] later independently established Hurwitz's result, but Tomič's theorem incorrectly has $\bar{k}$ dividing $n-1$, rather than $n+1$, for $p_n \in \pi_n^+$ to have zeros on $|z| = \beta$. Finally, Theorem 1 also improves upon an extension found most recently in Ostrowski [7, p. 90], where a sufficient condition (viz., that $\bar{k} = 1$) is given for $p_n \in \pi_n^+$ to have no zeros on $|z| = \beta$.

To illustrate the result of Theorem 1, consider the following polynomial:

$$\hat{p}_7(z) = 3 + 3z + 2z^2 + 2z^3 + z^4 + z^5 + z^6 + z^7, \tag{19}$$

which is in π_7^+. For this polynomial, we find that

$$\begin{aligned} \alpha &= 1, & \beta &= 2, \\ \underline{S} &= \{2,4,8\}, & \bar{S} &= \{1,2,3,5,6,7,8\}, \\ \underline{k} &= 2, & \bar{k} &= 1. \end{aligned}$$

Thus, as a consequence of Theorem 1, $\hat{p}_7$ has a unique simple zero $z = -1$ on $|z| = 1$, with all remaining zeros lying in the open annulus $1 < |z| < 2$.

In the previous example, where $0<\alpha<\beta$, $\hat{p}_7$ had a zero on one boundary of the annulus $\alpha \leqslant |z| \leqslant \beta$, viz. on $|z|=\alpha$, but none on the other boundary. That this is in general the case is now shown in the apparently new result of

COROLLARY 1. *Let $p_n \in \pi_n^+$ with $n \geqslant 1$ be such that* (*cf.* (4)) $0<\alpha<\beta$. *Then, it is not possible for p_n to simultaneously have zeros on $|z|=\alpha$ and on $|z|=\beta$.*

Proof. Suppose that p_n has zeros on $|z|=\beta$. Then $\bar{k}>1$ from Theorem 1, and from (8) it follows that

$$p_n(\beta z) = \left(1+z+\cdots+z^{\bar{k}-1}\right)q_m\left(z^{\bar{k}}\right)$$

$$= \gamma_0\left(1+z+\cdots+z^{\bar{k}-1}\right)+\cdots,$$

where $q_m(0):=\gamma_0>0$. Writing $p_n(z)=\sum_{j=0}^{n} a_j z^j$, this implies that $a_0=\gamma_0$, and that $a_1=\gamma_0/\beta$. Since $0<\alpha<\beta$ by hypothesis, then $a_0-\alpha a_1=\gamma_0\{1-\alpha/\beta\}>0$. But from (4), $\underline{S}$ then contains unity, whence $\underline{k}=1$ from (5). Invoking Theorem 1, p_n then has no zeros on $|z|=\alpha$. The proof supposing p_n to have zeros on $|z|=\alpha$ is similar. ■

As a useful consequence of Theorem 2, we also have

COROLLARY 2. *If $p_n \in \pi_n^+$ with $n \geqslant 1$ satisfies $\beta a_1 - a_0>0$, then all zeros of p_n satisfy $|z|<\beta$.*

Proof. By hypothesis [cf. (4)], $\bar{S}\supset\{n,n+1\}$, whence $\bar{k}=1$. Then apply Theorem 1. ■

As an application of Corollary 2, consider $s_n(z):=\sum_{k=0}^{n} z^k/k!$, the familiar nth partial sum of e^z. For every $n \geqslant 2$, $s_n(z)$ satisfies the hypotheses of Corollary 2 with $\beta(s_n)=n$, so that all the zeros of $s_n(z)$ satisfy

$$|z| < n \qquad \forall n \geqslant 2. \tag{20}$$

Actually, the above inequality is quite sharp asymptotically in the sense that s_n is known (cf. Saff and Varga [9]) to have a zero of the form

$$n+\sqrt{2n}\, w_n \quad \text{with} \quad \lim_{n\to\infty} w_n = t_1 \doteqdot -1.354810+i(1.991467), \tag{21}$$

$g_m(z)$ in π_m^+ with $m \geq 2$ for which

$$\beta[g_m] = a_0\nu/a_n, \qquad \alpha[g_m] = a_0\mu/a_n.$$

From Lemma 1, $h(z) := g_m(z^{n+1})\cdot p_n(z)$ is in $P_{\mu,\nu}$. But since $p_n(\zeta)=0$, then $h(\zeta)=0$ also, whence $\zeta \in Z(P_{\mu,\nu})$. ∎

Proof of Theorem 2. It is clear from Theorem 1 that, for any $\beta' > 0$,

$$Z(A_{\beta',\beta'}) = \{\beta' e^{i\theta} \text{ with } 0<\theta<2\pi : \theta \text{ is a rational multiple of } \pi\}. \quad (30)$$

Thus, on choosing $\beta'=\gamma'$ in Corollary 2 and letting β' run through the interval $[\beta,\gamma]$, we evidently have, from (26), (30), and Corollary 3, the desired closure result of (27). ∎

The essence of Theorem 2 is that the zeros of the polynomials in $P_{\mu,\nu}$ "fill out" the closed annulus $A_{\mu,\nu}$. To illustrate this numerically, consider the subset of $P_{1,2}$ defined by

$$P_{1,2}^{(6,16)} := \left\{ p_n(z) = \sum_{j=0}^{n} a_j z^j \in \pi_n^+,\ 1 \leq n \leq 6 : \alpha[p_n]=1, \beta[p_n]=2, \right.$$
$$\left. \text{and } a_j \in \{1,2,3,\ldots,15,16\} \text{ for } 0 \leq j \leq n \right\}. \quad (31)$$

The number of distinct polynomials in $P_{1,2}^{(6,16)}$ is approximately 26,120, each of whose zeros have been plotted in Fig. 1, up to a resolution of $\frac{1}{20}$.

In a subsequent paper, we will consider the sharpness of the Eneström-Kakeya theorem in another sense. If p_n is any fixed polynomial in π_n such that p_n has no zeros on the ray $[0, +\infty)$, let Q_m be the (possibly empty) subset defined by [cf. (3)]

$$Q_m := \{q_m \in \pi_m : p_n \cdot q_m \in \pi_{n+m}^+\},$$

and set [cf. (1)]

$$\mathrm{EK}_m(p_n) := \inf\{\beta[p_n \cdot q_m] : q_m \in Q_m\} \qquad \forall m \geq 0,$$

where $\mathrm{EK}_m(p_n)$, the *mth Eneström-Kakeya functional of* p_n, is defined to be

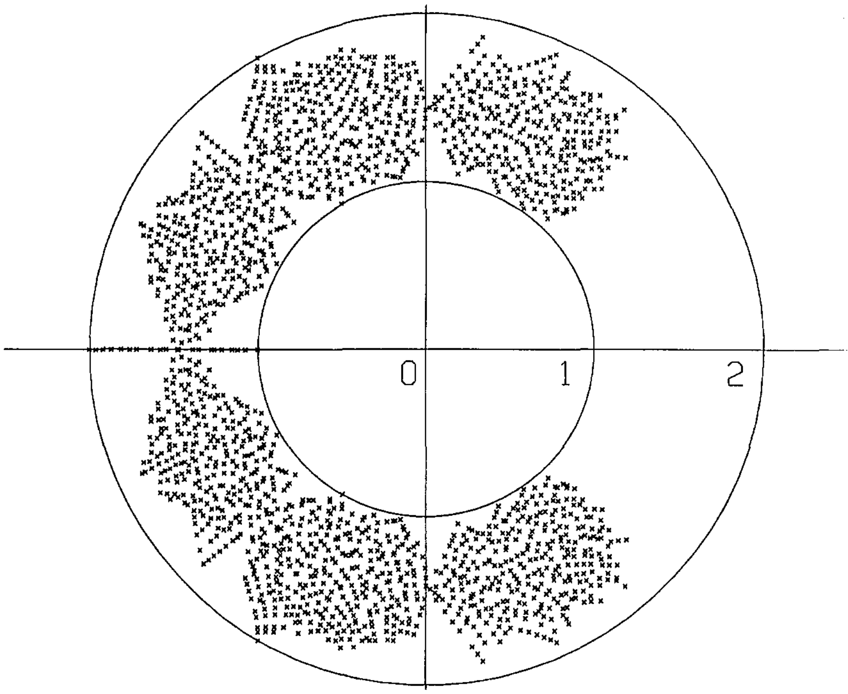

FIG. 1. Zeros of $P_{1,2}^{(6,16)}$ [cf. Eq. (31)].

$+\infty$ if Q_m is empty. What is to be investigated is in what sense

$$\lim_{m\to\infty} \mathrm{EK}_m(p_n) = \bar{\rho}(p_n)$$

is valid.

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