On the Eneström-Kakeya Theorem and Its Sharpness

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Dedicated to Alston S. Householder on his seventy-fifth birthday.

Submitted by Hans Schneider

ABSTRACT

A new proof, based on the Perron-Frobenius theory of nonnegative matrices, is given of a result of Hurwitz on the sharpness of the classical Eneström-Kakeya theorem for estimating the moduli of the zeros of a polynomial with positive real coefficients. It is then shown (Theorem 2) that the zeros of a particular set of polynomials fill out the Eneström-Kakeya annulus in a precise manner, and this is illustrated by numerical results in Fig. 1.

1. INTRODUCTION

The classical theorem due to Eneström [1] and Kakeya [4] for finding bounds for the moduli of the zeros of polynomials having positive real

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coefficients is often stated as (cf. Henrici [2, p. 462], Marden [6, p. 136], and Pólya-Szegő [8, p. 107])

**Theorem A** (Eneström-Kakeya). Let \( p_n(z) = \sum_{j=0}^{\infty} a_j z^j \) be any polynomial whose coefficients satisfy

\[
a_0 > a_1 > a_2 > \cdots > a_n > 0.
\]

Then \( p_n(z) \) has no zeros in the open unit disk \( \{ z \in \mathbb{C} : |z| < 1 \} \).

An equivalent, but perhaps more useful, statement of the above theorem, due in fact to Eneström [1], is the following:

**Theorem B.** Let \( p_n(z) = \sum_{j=0}^{n} a_j z^j \), \( n \geq 1 \), be any polynomial with \( a_i > 0 \) for all \( 0 < i \leq n \). Setting

\[
\alpha = \alpha[p_n] := \min_{0 < i < n} \left\{ \frac{a_i}{a_{i+1}} \right\}, \quad \beta = \beta[p_n] := \max_{0 < i < n} \left\{ \frac{a_i}{a_{i+1}} \right\}, \quad (1)
\]

then all the zeros of \( p_n \) are contained in the annulus

\[
\alpha \leq |z| < \beta. \quad (2)
\]

An obvious question that can be asked is whether both inequalities of (2) of Theorem B are sharp, in the sense that polynomials with positive coefficients can be found having zeros either on \( |z| = \alpha \) or on \( |z| = \beta \). Hurwitz [3] answered this question affirmatively over sixty years ago, and showed moreover that such extremal polynomials have a very special characterization. What will be shown here in Sec. 2 is that this special characterization (Theorem 1) can be established by means of the Perron-Frobenius theory of non-negative matrices, thereby affording an alternative proof of this classical analysis result. It should be remarked that Theorem 1, to be given below, slightly extends (and corrects) the original result of Hurwitz, as well as more recent results of Tomič [10] and Ostrowski [7]. In Sec. 3, it will then be shown (in Theorem 2) that the zeros of a particular set of polynomials fill out the Eneström-Kakeya annulus (2) in a precise manner. The remainder of this section gives some needed notation.

For every nonnegative integer \( n \), define

\[
\sigma_n^+ := \left\{ p_n(z) = \sum_{j=0}^{n} a_j z^j : a_j > 0 \text{ for all } 0 < j < n \right\}. \quad (3)
\]
ENESTRÖM-KAKEYA THEOREM

If \( p_n \in \pi_n^+ \), then set

\[
\overline{S} = \overline{S}(p_n) = \{ j = 1, 2, \ldots, n+1: \beta a_{n+1-j} - a_{n-j} > 0 \}, \quad \text{where} \quad a_{-1} = 0,
\]

\[
\overline{S} = S(p_n) = \{ j = 1, 2, \ldots, n+1: a_{n+1-j} - a_j > 0 \}, \quad \text{where} \quad a_{n+1} = 0,
\]

where \( \alpha \) and \( \beta \) for \( p_n \) are defined in (1). Note that these sets are nonempty, since \( n+1 \) is an element of both sets. Also associated with \( p_n \in \pi_n^+ \) are the positive integers

\[
\overline{k} = \overline{k}(p_n) = \text{g.c.d.} \{ j: j \in \overline{S} \},
\]

\[
k = k(p_n) = \text{g.c.d.} \{ j: j \in S \}.
\]

2. THEOREM 1

With the notation of Sec. 1, we now establish

**Theorem 1.** For any \( p_n \in \pi_n^+ \) with \( n \geq 1 \), all the zeros of \( p_n \) lie in the annulus (cf. (1))

\[
\alpha < |z| < \beta.
\]

Moreover, \( p_n \) can vanish on \( |z| = \beta \) iff \( \overline{k} > 1 \) (cf. (5)). If \( \overline{k} > 1 \), the zeros of \( p_n \) on \( |z| = \beta \) are simple and given precisely by

\[
\{ \beta \exp 2\pi ij/\overline{k}: j = 1, 2, \ldots, \overline{k}-1 \},
\]

and \( p_n \) has the form

\[
p_n(\beta z) = \left( 1 + z + z^2 + \cdots + z^{\overline{k}-1} \right) q_m(z^\overline{k}),
\]

where \( q_m \in \pi_m^+ \). If \( m > 1 \), then all zeros of \( q_m(w) \) lie in \( |w| < 1 \), and

\[
\beta \lceil q_m \rceil < 1.
\]
Similarly, \( p_n \) can vanish on \( |z| = \alpha \) iff \( k > 1 \). If \( k > 1 \), the zeros of \( p_n \) on \( |z| = \alpha \) are simple and given precisely by
\[
\{ \alpha \exp \frac{2\pi ij}{k} : i = 1, 2, \ldots, k-1 \}, \tag{10}
\]
and \( p_n \) has the form
\[
z^n p_n (\alpha / z) = (1 + z + \cdots + z^{k-1}) r_m (z^k), \tag{11}
\]
where \( r_m \in \pi_m^+ \). If \( m > 1 \), then all the zeros of \( r_m (w) \) lie in \( |w| < 1 \) and \( \beta [r_m] < 1 \).

Proof. For any \( p_n (z) = \sum_{l=0}^{n} a_l z^l \) in \( \pi_n^+ \), it can be verified from (1) that
\[
-\frac{(1-z) p_n (\beta z)}{a_n \beta^l} = z^{n+1} - \sum_{i=1}^{n+1} c_i z^{n+1-i} = : \tilde{p}_{n+1} (z), \tag{12}
\]
where
\[
c_i = \begin{cases} 
\frac{\beta a_{n+1-i} - a_{n-i}}{a_n \beta^l} > 0 & \text{for } 1 \leq j \leq n, \\
\frac{a_0}{a_n \beta^l} > 0 & \text{for } j = n + 1 
\end{cases} \tag{13}
\]
so that \( (1-z) p_n (\beta z) \) and \( \tilde{p}_{n+1} (z) \) have the same zeros. The \( (n+1) \times (n+1) \) companion matrix \( B \) for \( \tilde{p}_{n+1} \) is given by
\[
B = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & c_{n+1} \\
1 & 0 & \cdots & 0 & 0 & c_n \\
0 & 1 & \cdots & 0 & 0 & c_{n-1} \\
\vline & \vline & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 0 & c_2 \\
0 & 0 & \cdots & 0 & 1 & c_1
\end{bmatrix}, \tag{14}
\]
and, as the coefficients \( c_i \) from (13) and (1) are all nonnegative, then \( B \) is a nonnegative matrix. Moreover, since \( c_{n+1} > 0 \) from (13), then \( B \) is irreducible (cf. [11, p. 20]). Next, because \( z = 1 \) is a zero of \( \tilde{p}_{n+1} \) from (12), it follows that the Perron eigenvalue of \( B \) is necessarily unity, so that (cf. [11, p. 30]) all the
zeros of $\tilde{p}_{n+1}$ satisfy $|z| < 1$. Because of (12), all zeros of $p_n$ then satisfy $|z| < \beta$, which establishes the second inequality of (6).

Next, suppose that $p_n$ has a zero on $|z| = \beta$, which implies that $\tilde{p}_{n+1}$ has at least two zeros on $|z| = 1$. Equivalently, $B$ has at least two eigenvalues of modulus unity. This can happen, from the Perron-Frobenius theory of nonnegative matrices (cf. [11, p. 35]) if $B$ is cyclic of some index $h > 1$. Next, using Theorem 2.9 of [11, p. 49], this cyclic index can be expressed as the greatest common divisor of the lengths of all closed paths in the directed graph for the matrix $B$ which connect the vertex $V_{n+1}$ to itself. From the structure of $B$ in (14), it follows that any $q_j > 0$ gives rise to such a closed path, through the vertex $V_{n+1}$, of length precisely $j$. In other words, $B$ is cyclic of index $h$ where

$$h = \text{g.c.d.} \{ j : q_j > 0 \}.$$  

(15)

But, from the definitions of (4) and (5), it can be verified that $h = \tilde{k} = \tilde{k}(p_n)$. Thus, $p_n$ can vanish on $|z| = \beta$ iff $\tilde{k} > 1$. Next, if $\tilde{k} > 1$, then the eigenvalues of $B$ having modulus equal to the Perron eigenvalue (which is unity) are $\tilde{k}$ in number, are simple, and are given by (cf. [11, p. 38])

$$\{ \exp 2\pi ij/\tilde{k} : j = 0, 1, 2, \ldots, \tilde{k} - 1 \}.$$  

(16)

Hence, with (12), the number of zeros of $p_n$ on $|z| = \beta$ is $\tilde{k} - 1$, and these zeros, from (16), are all simple, and are given precisely by (7). Finally, the cyclic nature of $B$ implies that its characteristic polynomial, namely $\tilde{p}_{n+1}(z)$, must satisfy (cf. [11, p. 39])

$$\tilde{p}_{n+1}(z) = (1 - z^k)\tilde{q}_m(z^k),$$

where all the zeros of $\tilde{q}_m(w)$ lie in $|w| < 1$. On dividing the above equation by $1 - z$ and on recalling (12), then

$$p_n(\beta z) = (1 + z + \cdots + z^{k-1})q_m(z^k),$$  

(17)

where $q_m(w) = -a_n\beta^m \tilde{q}_m(w)$, which establishes (8).

Next, if $q_m(w) = \sum_{i=0}^m d_i w^i$, then from (17),

$$p_n(\beta z) = d_0(1 + z + \cdots + z^{k-1}) + d_1(z^k + \cdots + z^{2k-1})$$

$$+ \cdots + d_m(z^{mk} + \cdots + z^{(m+1)k-1}).$$  

(18)
Evidently, all the coefficients \( d_i \) must be positive, since, by hypothesis, \( p_n \in \pi_n^+ \), whence \( q_m \in \pi_m^+ \). If \( m > 1 \), the maximum ratio of successive coefficients in (18) is just

\[
\max \left\{ 1; \max_{0 < i < m} \left[ \frac{d_i}{d_{i+1}} \right] \right\} = 1,
\]

the last equality following from the definition of \( \beta \) for \( p_n \). Thus,

\[
\beta[q_m(w)] := \max_{0 < i < m} \left[ \frac{d_i}{d_{i+1}} \right] < 1,
\]

which establishes (9). The remainder of the proof follows similarly upon considering the polynomial \( z^n p_n(\alpha/z) \).

The history concerning Theorem B and Theorem 1 is worth commenting on. Kakeya in [4] stated that strict inequality held throughout (2) in Theorem B, and this error was promptly pointed out by Kempner [5], who essentially deduced the sufficiency of \( k > 1 \) in Theorem 1 for \( p_n \in \pi_n^+ \) to have zeros on \( |z| = \beta \). Theorem 1 is, as previously stated, a slight extension of the result of Hurwitz [3], but it should be noted that Hurwitz incorrectly claimed (cf. [3, p. 92, line 1]) that \( p_n \in \pi_n^+ \) has zeros on, say, \( |z| = \alpha \) iff the set \( S \) of (4) consists of all multiples \( k, 2k, \ldots, n + 1 \). This same mistake appears also in Marden [6, p. 138, Exercise 10]. That this need not be the case is illustrated in the example of (19) below. Next, Tomič [10, p. 149] later independently established Hurwitz’s result, but Tomič’s theorem incorrectly has \( k \) dividing \( n - 1 \), rather than \( n + 1 \), for \( p_n \in \pi_n^+ \) to have zeros on \( |z| = \beta \). Finally, Theorem 1 also improves upon an extension found most recently in Ostrowski [7, p. 90], where a sufficient condition (viz., that \( k = 1 \)) is given for \( p_n \in \pi_n^+ \) to have no zeros on \( |z| = \beta \).

To illustrate the result of Theorem 1, consider the following polynomial:

\[
\hat{p}_\gamma(z) = 3 + 3z + 2z^2 + 2z^3 + z^4 + z^5 + z^6 + z^7,
\]

which is in \( \pi_7^+ \). For this polynomial, we find that

\[
\alpha = 1, \quad \beta = 2,
\]

\[
S = \{2, 4, 8\}, \quad \bar{S} = \{1, 2, 3, 5, 6, 7, 8\},
\]

\[
k = 2, \quad \bar{k} = 1.
\]

Thus, as a consequence of Theorem 1, \( \hat{p}_\gamma \) has a unique simple zero \( z = -1 \) on \( |z| = 1 \), with all remaining zeros lying in the open annulus \( 1 < |z| < 2 \).
In the previous example, where $0 < \alpha \leq \beta, \bar{p}_r$ had a zero on one boundary of the annulus $\alpha < |z| < \beta$, viz. on $|z| = \alpha$, but none on the other boundary. That this is in general the case is now shown in the apparently new result of

**Corollary 1.** Let $p_n \in \mathcal{P}_n^+$ with $n > 1$ be such that (cf. (4)) $0 < \alpha < \beta$. Then, it is not possible for $p_n$ to simultaneously have zeros on $|z| = \alpha$ and on $|z| = \beta$.

**Proof.** Suppose that $p_n$ has zeros on $|z| = \beta$. Then $k > 1$ from Theorem 1, and from (8) it follows that

$$p_n(\beta z) = (1 + z + \cdots + z^{k-1})q(z^k)$$

$$= \gamma_0(1 + z + \cdots + z^{k-1}) + \cdots,$$

where $q_m(0) = \gamma_0 > 0$. Writing $p_n(z) = \sum_{l=0}^n a_lz^l$, this implies that $a_0 = \gamma_0$, and that $a_1 = \gamma_0/\beta$. Since $0 < \alpha < \beta$ by hypothesis, then $a_0 - a_1 = \gamma_0(1 - \alpha/\beta) > 0$. But from (4), $S$ then contains unity, whence $k = 1$ from (5). Invoking Theorem 1, $p_n$ then has no zeros on $|z| = \alpha$. The proof supposing $p_n$ to have zeros on $|z| = \alpha$ is similar.

As a useful consequence of Theorem 2, we also have

**Corollary 2.** If $p_n \in \mathcal{P}_n^+$ with $n > 1$ satisfies $\beta a_1 - a_0 > 0$, then all zeros of $p_n$ satisfy $|z| < \beta$.

**Proof.** By hypothesis [cf. (4)], $\mathbb{S} \supset (n, n+1)$, whence $k = 1$. Then apply Theorem 1.

As an application of Corollary 2, consider $s_n(z) = \sum_{l=1}^n \alpha^l/k!$, the familiar $n$th partial sum of $e^\alpha$. For every $n > 2$, $s_n(z)$ satisfies the hypotheses of Corollary 2 with $\beta(s_n) = n$, so that all the zeros of $s_n(z)$ satisfy

$$|z| < n \quad \forall n > 2.$$  \hspace{1cm} (20)

Actually, the above inequality is quite sharp asymptotically in the sense that $s_n$ is known (cf. Saff and Varga [9]) to have a zero of the form

$$n + \sqrt{2n} \ w_n \quad \text{with} \quad \lim_{n \to \infty} w_n = t_1 = -1.354810 + i(1.991467),$$  \hspace{1cm} (21)
$g_m(z)$ in $\pi_n^+$ with $m > 2$ for which

$$\beta[ g_m ] = a_0^+/a_n, \quad \alpha[ g_m ] = a_0^-/a_n.$$  

From Lemma 1, $h(z) = g_m(\zeta^{n+1})p_n(z)$ is in $P_{\mu^r}$. But since $p_n(\zeta) = 0$, then $h(\zeta) = 0$ also, whence $\zeta \in Z(P_{\mu^r})$. $\blacksquare$

**Proof of Theorem 2.** It is clear from Theorem 1 that, for any $\beta' > 0$,

$$Z(A_{\beta', \beta}) = \{ \beta' e^{i\theta} \text{ with } 0 < \theta < 2\pi : \theta \text{ is a rational multiple of } \pi \}. \quad (30)$$

Thus, on choosing $\beta' = \gamma'$ in Corollary 2 and letting $\beta'$ run through the interval $[\beta, \gamma]$, we evidently have, from (26), (30), and Corollary 3, the desired closure result of (27). $\blacksquare$

The essence of Theorem 2 is that the zeros of the polynomials in $P_{\mu^r}$ "fill out" the closed annulus $A_{\mu^r}$. To illustrate this numerically, consider the subset of $P_{1,2}$ defined by

$$P_{1,2}^{[0,10]} = \left\{ p_n(z) = \sum_{l=0}^{n} a_l z^l \in \pi_n^+, 1 < n < 6 : \alpha[ p_n ] = 1, \beta[ p_n ] = 2, \right. \quad (31)$$

$$\left. \quad \text{and } a_l \in \{ 1, 2, 3, \ldots, 15, 16 \} \text{ for } 0 < j < n \right\}.$$

The number of distinct polynomials in $P_{1,2}^{[0,10]}$ is approximately 26,120, each of whose zeros have been plotted in Fig. 1, up to a resolution of $\frac{1}{30}$.

In a subsequent paper, we will consider the sharpness of the Eneström-Kakeya theorem in another sense. If $p_n$ is any fixed polynomial in $\pi_n$ such that $p_n$ has no zeros on the ray $[0, +\infty)$, let $Q_m$ be the (possibly empty) subset defined by [cf. (3)]

$$Q_m = \{ q_m, p_nq_m \in \pi_n^{+m} \},$$

and set [cf. (1)]

$$E_K_m(p_n) = \inf \{ \beta[ p_nq_m ] : q_m \in Q_m \} \quad \forall m > 0,$$

where $E_K_m(p_n)$, the $m$th Eneström-Kakeya functional of $p_n$, is defined to be
Fig. 1. Zeros of $P_{[9,16]}^n$ [cf. Eq. (31)].

$+\infty$ if $Q_n$ is empty. What is to be investigated is in what sense

$$\lim_{m \to \infty} EK_m(p_n) = \tilde{\rho}(p_n)$$

is valid.

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